Diagrammatic sets and higher categories

Amar Hadzihasanovic

Formerly: RIMS, Kyoto University
Currently: IRIF, Université de Paris

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   - We care about **presented** higher categories
   - Pasting theorem, diagrammatic reasoning

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Diagrammatic sets: let's have it both ways

RDSs: a model of weak higher categories

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2. Conditional to a conjecture (later in the talk), strict ω-categories **embed** as a full subcategory (if one takes morphisms that preserve a choice of weak composites)
3. There are n-**truncated** RDSs corresponding to weak n-categories. 2-truncated RDSs are equivalent to bicategories

Dimensions in rewriting

- Dimension 0: (labelled) abstract rewrite system

$$\bullet$$

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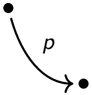

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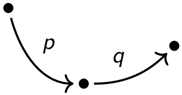

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$$\bullet \xrightarrow{p} \bullet \xrightarrow{q} \bullet \xrightarrow{r} \bullet$$

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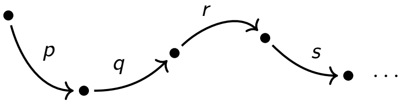

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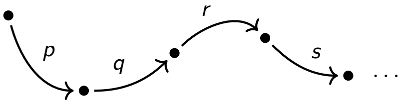

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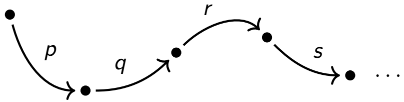

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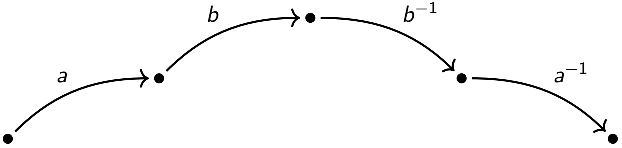

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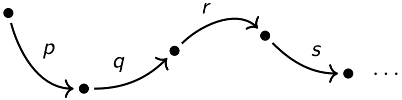

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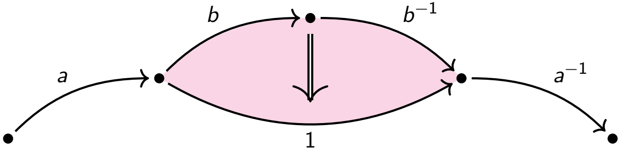

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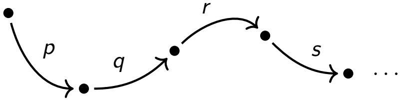

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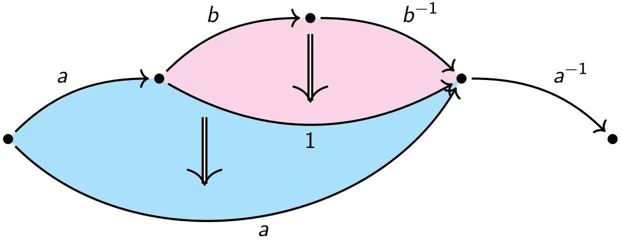

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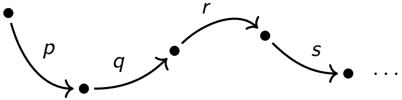

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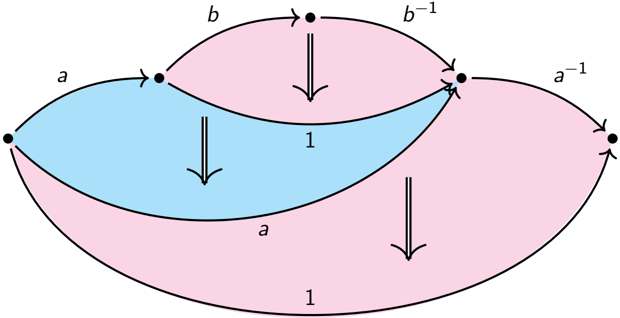
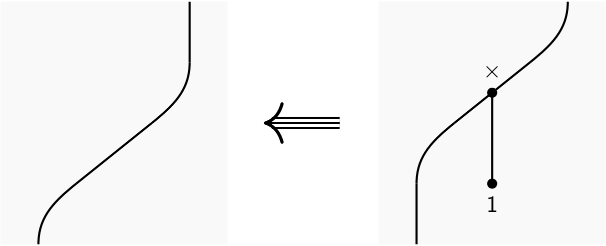


Dimensions in rewriting

- Dimension 2: algebraic theories / monoidal categories



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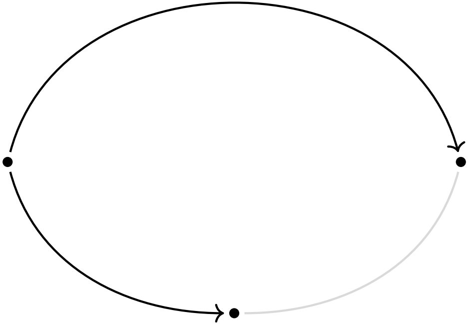

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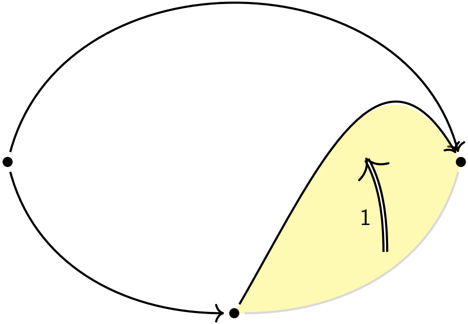

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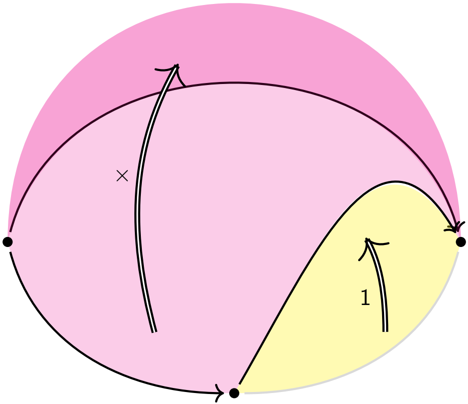

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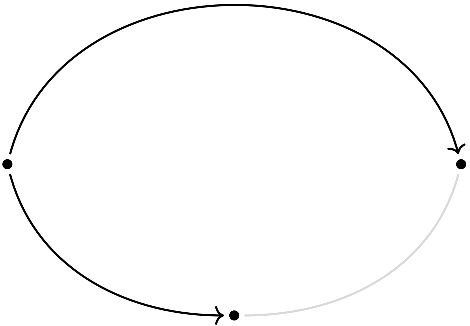

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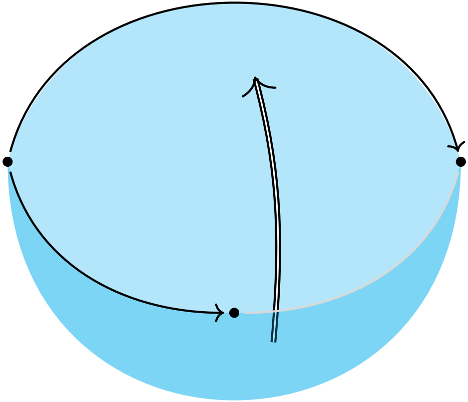

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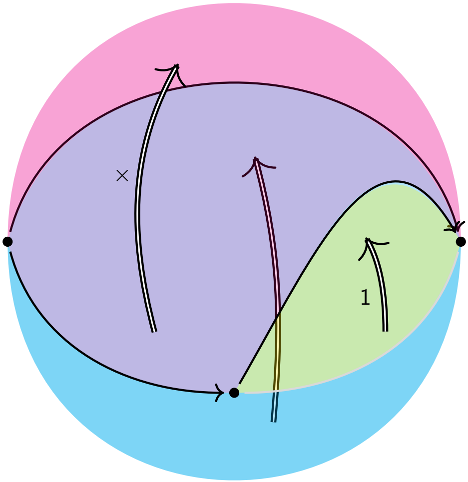

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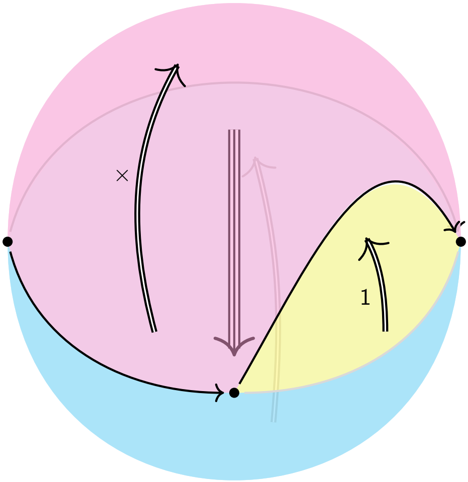

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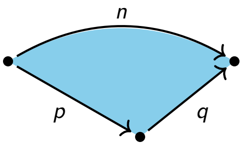

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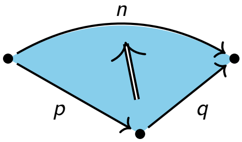

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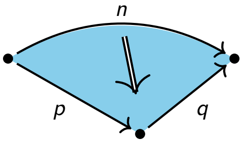

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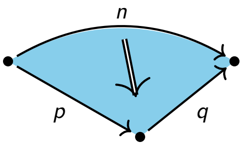

"Point-set" cells not available — need a combinatorial notion of

1. **directed cells**
2. their **pasting**

Polygraphs

Standard framework for higher-dimensional rewriting:

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...plus other technical issues

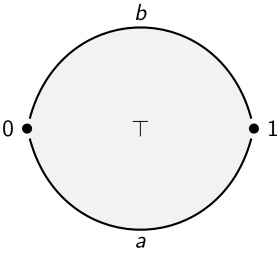
Towards diagrammatic sets

Let the CW complex interpretation
guide us

Face posets

We can associate to a CW complex its face poset...

$$
\begin{tikzcd}
 & \top \arrow[dl] \arrow[dr] & \\
a \arrow[d] \arrow[drr] & & b \arrow[d] \arrow[dll] \\
0 & & 1
\end{tikzcd}
$$



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\begin{tikzcd}
& \top \arrow[dl, "-"'] \arrow[dr, "+"] & \\
a \arrow[d, "-"'] \arrow[drr, "+" near start] & & b \arrow[d, "+"] \arrow[dll, "-" near end] \\
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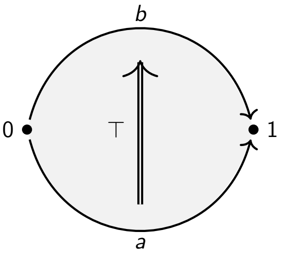

and to a pasting diagram its oriented face poset.

Face posets

Regular CW complex X: pasting maps are homeomorphisms with their image

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Conjecture

A regular pasting diagram is specified up to cellular isomorphism by its oriented face poset

Diagrammatic sets

- Directed n-cells are modelled by regular directed complexes

  (which are oriented face posets of regular pasting diagrams)

  with a greatest element of rank n

  (so the underlying poset is the face poset of a regular CW n-ball)

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- Pasting is given by maps of posets that are compatible functorially with both realisations

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- surjections, giving **units** and **degeneracy** operations
  $\to$ "nullary" operations in universal algebra

Enough for higher-dimensional rewriting?

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Diagrammatic sets have *some* features of a model (pasting, units)

Idea: higher categories $\to$ diagrammatic sets with an **internal** notion of *weak composition*

(in the spirit of categorical semantics:
syntax and semantics in the same universe)

Equivalences and weak composition

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$\downarrow$

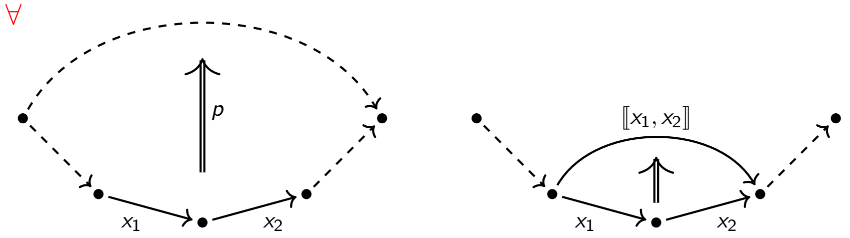
There are special equivalence cells $x \Rightarrow y$, $y \Rightarrow x$, which mediate between all cells containing x and all cells containing y in their boundary

Equivalences and weak composition

$\forall$

$$
\begin{tikzcd}
\bullet \arrow[rrrr, dashed, bend left] \arrow[dr, dashed] & & & & \bullet \\
& \bullet \arrow[r, "x_1"'] & \bullet \arrow[r, "x_2"'] \arrow[u, phantom, "\Uparrow p"] & \bullet \arrow[ur, dashed] &
\end{tikzcd}
$$

Equivalences and weak composition



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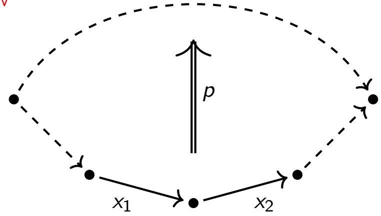
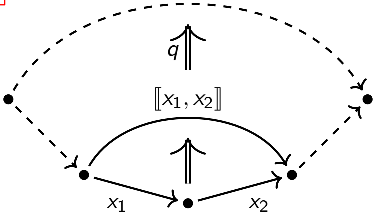


$\exists$



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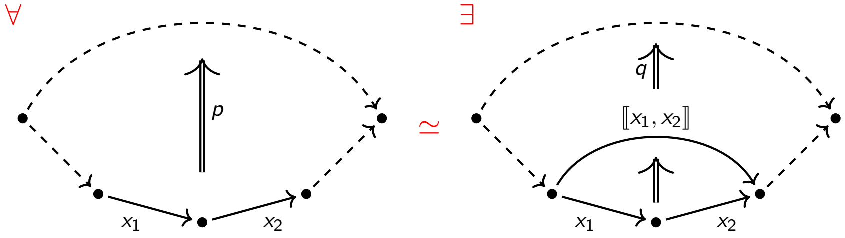

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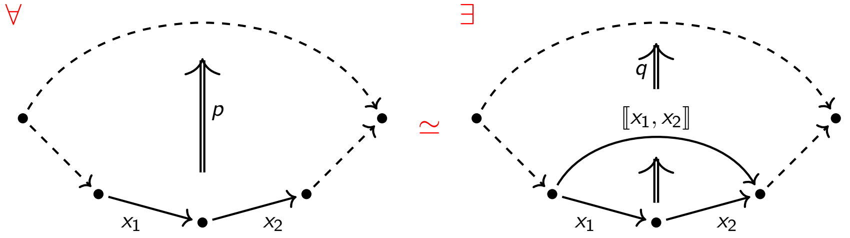

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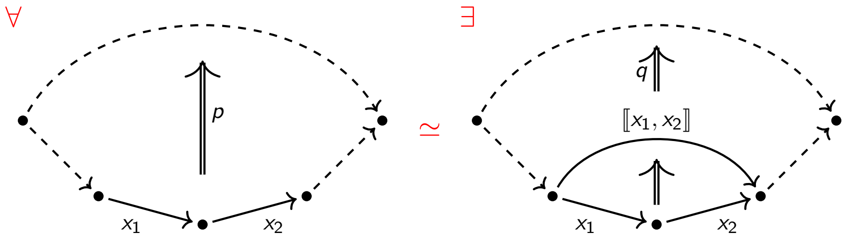
Equivalences and weak composition



And this equivalence should be witnessed by **3-dimensional equivalence cells**...

whose definition involves 4-dimensional equivalence cells, etc

Equivalences and weak composition

This is a **coinductive** definition.

Let X be a diagrammatic set. For all subsets $A \subseteq \mathrm{Cell}(X)$, define

$$\begin{aligned}\mathcal{F}(A) := \{x : U \to X \mid\ & \text{for all } \alpha \in \{+,-\} \text{ and} \\ & (\Lambda \hookrightarrow W, \lambda : \Lambda \to X) \in \mathcal{D}iv(x, \partial^\alpha U), \\ & \text{there exists } (h : W \to X) \in A \text{ such that } h|_\Lambda = \lambda\};\end{aligned}$$

Then $\mathcal{F}$ is an order-preserving map on $\mathcal{P}(\mathrm{Cell}(X))$. Its **greatest fixed point** is the set $\mathcal{E}qX$ of equivalence cells of X.

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Proof method: if $A \subseteq \mathcal{F}(A)$, then $A \subseteq \mathcal{E}qX$.

Equivalences and weak composition

Representable diagrammatic set

A diagrammatic set where, for all diagrams x, there exist cells $[\![x]\!], [\![x]\!]'$ and equivalence cells $x \Rightarrow [\![x]\!], [\![x]\!]' \Rightarrow x$.

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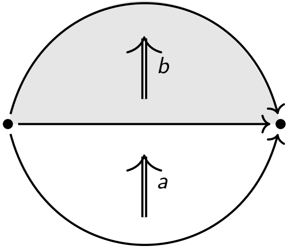
Theorem

In a representable diagrammatic set,

1. all degenerate cells are equivalence cells, and
2. $\simeq$ is an equivalence relation.

Eckmann-Hilton in diagrammatic sets



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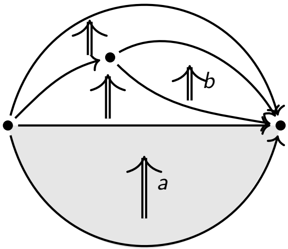

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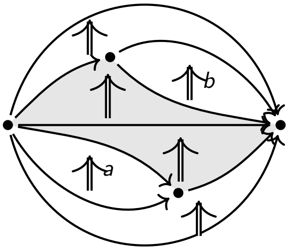

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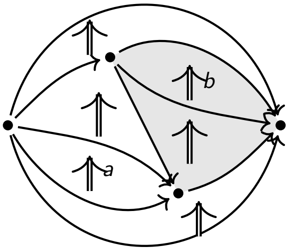

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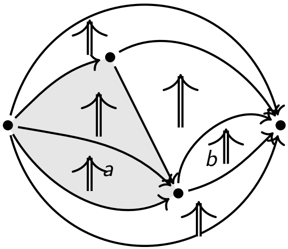
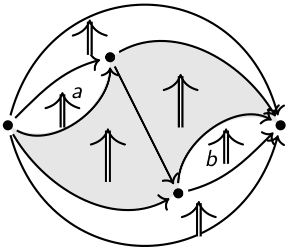
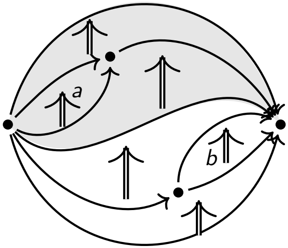


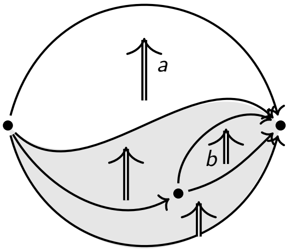
Eckmann-Hilton in diagrammatic sets



Eckmann-Hilton in diagrammatic sets



Eckmann-Hilton in diagrammatic sets



Eckmann-Hilton in diagrammatic sets