Diagrammatic sets: weak higher categories for rewriting

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TallCat Seminar
11 June 2020

arXiv:1909.07639

There is a paper. But I'm reworking it heavily.
Read at your own risk.

Higher categories for all

Higher categories for a **homotopy theorist/algebraic geometer/etc**:

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$\rightsquigarrow$ complete Segal spaces, complicial sets... pick your favourite.

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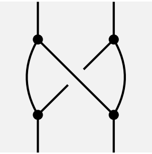
- *(While in dimension 2...)* Oh, this diagrammatic proof is justified, because bla bla Mac Lane coherence bla bla Joyal Street bla bla
- *(Then higher dimensions appear)* *panic*

2014

Bialgebra equation



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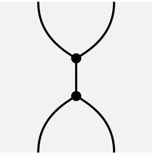

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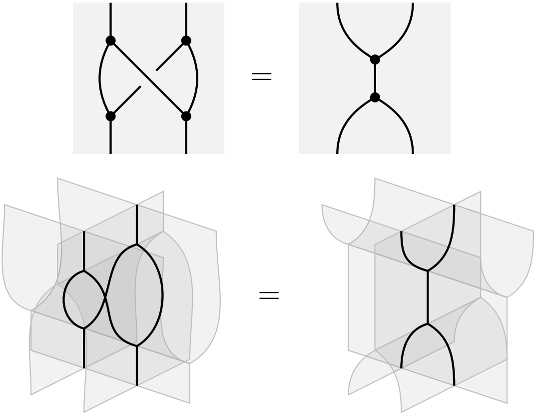

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An interaction of *planar* (2d) diagrams,

producing a transformation of 3d diagrams

(hence a 4d transformation)

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How do we interpret this?

Pasting theorem

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There is a lack of pasting theorems
for mainstream models of weak higher categories.

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The golden age of strict ω-categories

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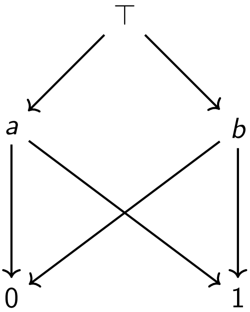
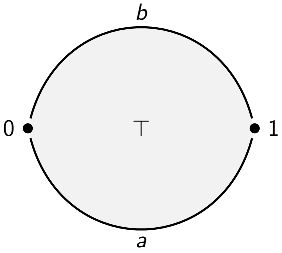
Then several works on the combinatorics of *pasting diagrams* and their *pasting theorems* in strict n-categories:

- **1988**: John Power
- **1989**: Michael Johnson
- **1991**: Ross Street, John Power
- **1993**: Richard Steiner

Steiner's directed complexes

We can associate to a cell complex its face poset...





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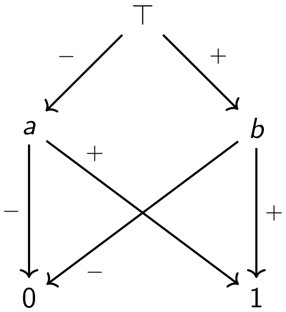

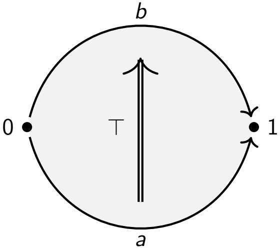

and to a pasting diagram its oriented face poset.

Steiner's directed complexes

Steiner 1993, *The algebra of directed complexes*, gives sufficient conditions for

- an oriented poset to be the oriented face poset of a pasting diagram, and
- the pasting diagram to be **reconstructed** from its oriented face poset.

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- an oriented poset to be the oriented face poset of a pasting diagram, and
- the pasting diagram to be **reconstructed** from its oriented face poset.

Many oriented posets present ω-categories —
fewer present polygraphs,
that is,
ω-categories that are freely generated by some of their cells.

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A variant of this was used to define
the lax Gray product of ω-categories
(Steiner 2004, Ara-Maltsiniotis 2017)

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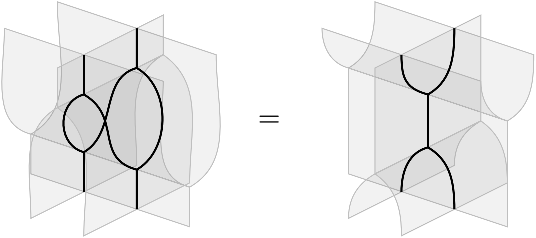
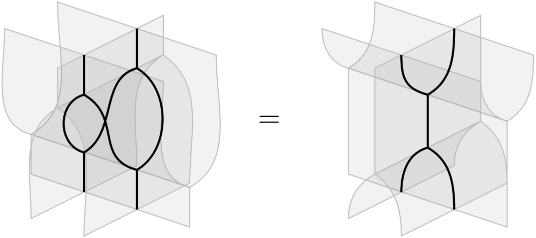
Lax Gray products and diagrammatic algebra



2d + 2d = 4d

Around this time, I start seeing lax Gray products everywhere

Lax Gray products and diagrammatic algebra



$$2d + 2d = 4d$$

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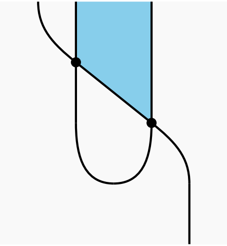
(I'm not the only one)

Lax Gray products and diagrammatic algebra

Example: Biunitary equations

Used by Jamie Vicary and Mike Stay to unify quantum and encrypted communication protocols. They are models of a lax Gray product of 2-categories.



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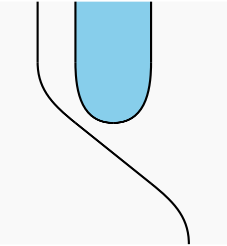
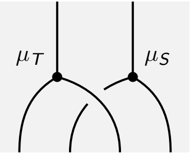


Lax Gray products and diagrammatic algebra

Example: Distributive laws of monads

They are models in **Cat** of a lax Gray product of 2-categories.



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These are naturally pointed objects in $\omega\mathbf{Cat}$.
With pointed objects, it is natural to take smash products $\wedge$.

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PRO $\wedge$ **PRO** $\rightsquigarrow$ 4-cat with one 0-cell, one 2-generator

Morally this should be a *braided monoidal category*.
But in strict ω-categories, it is a *commutative monoidal category*.
This breaks everything.

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Voevodsky's non-proof...

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The core of the argument relies on the fact that "doubly monoidal" degenerates to "commutative" in strict 3-categories (strict Eckmann-Hilton).

...still contained some good ideas

Good takeaway #1 from Kapranov-Voevodsky:

homotopy types may have **semi***strict algebraic models*
with weak units

- **2006**: André Joyal and Joachim Kock in dim 3

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- **2018**: Henry proves the homotopy hypothesis for "regular ω-groupoids".

Diagrams with spherical boundary

Regularity: only n-diagrams **with spherical boundary** have a composite

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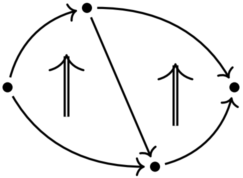
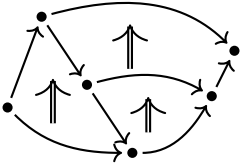
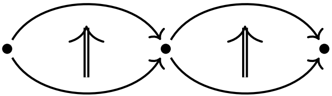
$\sim$ "are homeomorphic to n-balls"

Diagrams with spherical boundary





but not



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Diagrammatic sets

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Diagrammatic sets

Kapranov-Voevodsky pass from spaces to ω-categories
through an intermediate notion
of "spaces locally modelled on combinatorial pasting diagrams",

they call diagrammatic sets.

Diagrammatic sets

- **2019**: Kapranov-Voevodsky's equivalence of "Kan diagrammatic sets" and spaces is "morally correct"

...except they chose the wrong class of combinatorial diagrams, not closed under most of the operations they perform.

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(Work in progress: axiomatic approach relative to "nice classes of diagrams")

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This is a model where we can interpret *every regular diagram* and compose *every diagram with spherical boundary*.

Just "stuff" any diagram with units and it will become regular!

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This started a slowly rising French school of rewriting with polygraphs (Yves Lafont, Philippe Malbos, Yves Guiraud, Samuel Mimram...)
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which is why I am in Paris now

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rest on an analogy between

polygraphs and *CW complexes*,
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This analogy is limited by
the fact that strict ω-categories do not model all spaces.

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Everything that can be done with polygraphs
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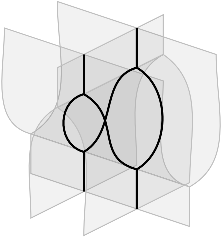
5. Lax Gray products, *joins* are easily defined and computed

Rewriting in diagrammatic sets



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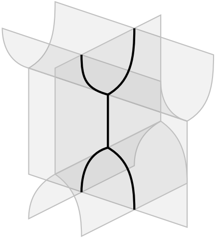

The smash product of diagrammatic sets produces this equation, the way it should.

Rewriting in diagrammatic sets

Higher-dimensional rewriting is packed with notions suggestive of a *directed homotopy theory*.

Rewriting in diagrammatic sets

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Work in progress:
a model of computation in diagrammatic sets
based on a "directed homotopy extension property".

⋮

2014

2015

2016

2017

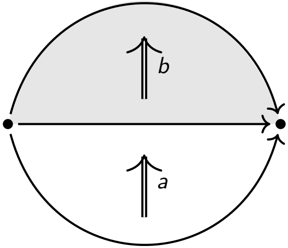
2018

2019

2020

Thanks for listening!

Eckmann-Hilton in diagrammatic sets



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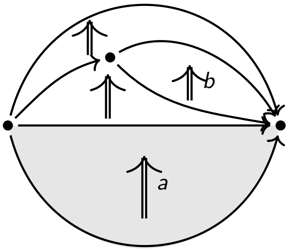

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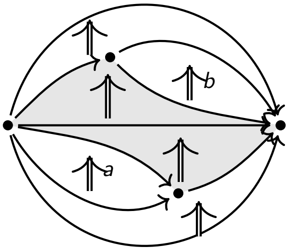

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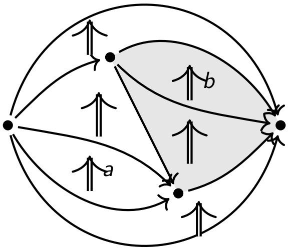

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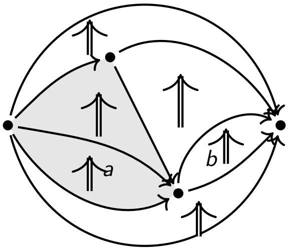

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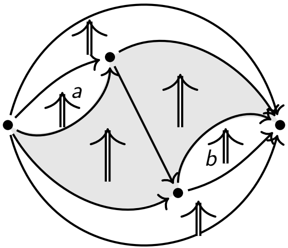

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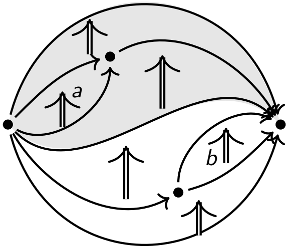

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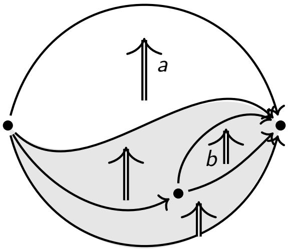

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