Category theory and diagrammatic reasoning

13th February 2019
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3 Universal properties, limits and colimits

A *division problem* is a question of the following form:

Given a and b, does there exist x such that a composed with x is equal to b?
If it exists, is it unique?

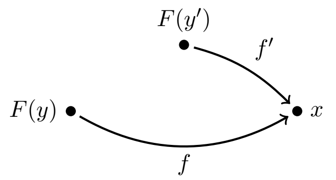
Such questions are ubiquitious in mathematics, from the solvability of systems of linear equations, to the existence of sections of fibre bundles. To make them precise, one needs additional information:

- What types of objects are a and b?
- Where can I look for x?
- How do I compose a and x?

Since category theory is, largely, a theory of composition, it also offers a unifying framework for the statement and classification of division problems. A fundamental notion in category theory is that of a *universal property*: roughly, a universal property of a states that for all b of a suitable form, certain division problems with a and b as parameters have a (possibly unique) solution.

Let us start from universal properties of morphisms in a category. Consider the following division problem.

Problem 1. Let $F : Y \to X$ be a functor, x an object of X. Given a pair of morphisms



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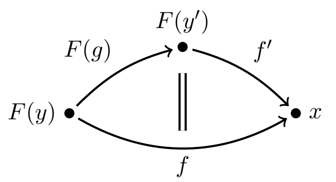
does there exist a morphism $g : y \to y'$ in Y such that



?

If it exists, is it unique?

This has the form of a division problem where a and b are arbitrary morphisms in X (which need to have the same target), x is constrained to be in the image of a functor F, and composition is composition of morphisms.

Definition 2. A morphism $f' : F(y') \to x$ is *weakly universal* from F to x if Problem 1 has a solution for each $f : F(y) \to x$. It is *universal* from F to x if the solution is always unique.

Example 3. Let id_X be the identity functor on X. A weakly universal morphism from id_X to x is a morphism $p : y \to x$ in X such that, for all $f : y' \to x$, there exists a factorisation $f = s_f; p$ of f through a morphism $s_f : y' \to y$.

In particular, taking $f := \mathrm{id}_x : x \to x$, we find $s : x \to y$ such that $s; p = \mathrm{id}_x$, that is, s is a one-sided inverse (a *section*) for p. A morphism p with the property that $s; p = \mathrm{id}_x$ for some s is called a *split epimorphism.* Conversely, if p is a split epimorphism, then for all $f : y' \to x$ we have $f = f; \mathrm{id}_x = f; s; p$, and we have a factorisation of f through p. Hence, weakly universal morphisms from id_X to x are the same as split epimorphisms with target x.

Example 4. Suppose that $p : y \to x$ is universal from id_X to x. Factorising p through itself, we have

$$p = \mathrm{id}_y; p = p; \mathrm{id}_x = p; (s; p) = (p; s); p,$$

and from the uniqueness of factorisations, $\mathrm{id}_y = p; s$. Therefore, s is a two-sided inverse for p, and p is in fact an isomorphism. The converse, that if p is an isomorphism then it is universal, is easy to check. Hence, universal morphisms from id_X to x are the same as isomorphisms with target x.

Exercise 5. A morphism $p : y \to x$ is an *epimorphism* if it satisfies the following "cancellability" property:

if $f, g : x \to z$ are two morphisms such that $p; f = p; g$, then $f = g$.

1. Prove that every split epimorphism is an epimorphism.

2. Show that the epimorphisms in **Set** are the surjective functions.

3. The *axiom of choice* is the statement that, if $\{X_i\}_{i \in I}$ is a family of inhabited (that is, non-empty) sets, there is a set $\{x_i\}_{i \in I}$ where $x_i \in X_i$ for each $i \in I$. Prove that the axiom of choice is equivalent to the statement that all epimorphisms in **Set** are split epimorphisms.

4. Give an example of a category containing an epimorphism which is not split.

Example 6. Let $\imath : \mathbb{Z} \to \mathbb{R}$ be the inclusion of the integers into the real numbers; this is a functor between partial orders, seen as categories. Given an arbitrary real number r, let $\lfloor r \rfloor$ be the *floor* of r, that is, the largest integer smaller or equal than r. The defining property of the floor means that, for all integers k, if $k \leq r$, then $k \leq \lfloor r \rfloor$. Rephrased in the language of categories, this says precisely that the unique morphism $\lfloor r \rfloor = \imath(\lfloor r \rfloor) \to r$ is universal from $\imath$ to r.

In general, for functors $F : P \to Q$ between partial orders (that is, order-preserving maps), universal morphisms from F to x capture the idea of a *best approximation from below* of x in P.

An important feature of universal properties, if not *the most* important, is that they can be used to specify objects uniquely up to isomorphism, by their property only, that is, in a "structure-less" way: see the following result.

Lemma 7. *Suppose $f : F(y) \to x$ and $f' : F(y') \to x$ are two universal morphisms from F to x. Then there exists a unique isomorphism $e : y \to y'$ such that $f = F(e); f'$.*

Proof. By the universal property of f' we can factor f as $f = F(e); f'$ in a unique way, and by the universal property of f we can factor f' as $f' = F(e'); f$ in a unique way. Therefore

$$f = F(e); F(e'); f = F(e; e'); f, \qquad f' = F(e'); F(e); f',$$

but also

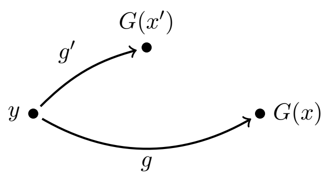
$$f = \mathrm{id}_{F(y)}; f = F(\mathrm{id}_y); f, \qquad f' = \mathrm{id}_{F(y')}; f' = F(\mathrm{id}_{y'}); f'.$$

By the uniqueness of the factorisations of f and f' through themselves, we conclude that $e; e' = \mathrm{id}_y$ and $e'; e = \mathrm{id}_{y'}$, that is, e and e' are each other's inverses. □

In the practice of mathematics, proving that something satisfies a universal property can be an important sanity check: it separates what may be a syntactic artefact, dependant on the particular way we have constructed an object, from what is stable under isomorphisms, hence dependant only on the relation between the object and the category to which it belongs. For example, we may have many ways of defining real numbers — by Cauchy sequences, Dedekind cuts, and so on — hence different ways of embedding the integers; yet the floor of a real number is going to be invariant under translation between these encodings, because it is defined by a universal property.

By the principle of duality, the notion of universal morphism from F to x has a dual, corresponding to the same notion instantiated in the opposite categories.

Problem 8. Let $G : X \to Y$ be a functor, y an object of Y. Given a pair of morphisms



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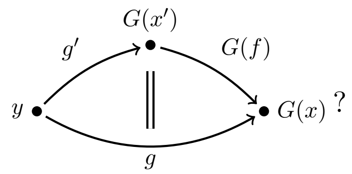
does there exist a morphism $f : x' \to x$ in X such that



If it exists, is it unique?

Definition 9. A morphism $g' : y \to G(x')$ is *weakly universal* from y to G if Problem 8 has a solution for each $g : y \to G(x)$. It is *universal* from y to G if the solution is always unique.

Example 10. A morphism $i : x \to y$ is weakly universal from x to id_X if and only if it is a *split monomorphism*, that is, there exists a morphism $r : y \to x$ (a *reflection*) such that $i; r = \mathrm{id}_x$. This is strictly stronger than the notion of *monomorphism*, which is a morphism $i : x \to y$ such that

$$\text{if } f, g : z \to x \text{ are two morphisms such that } f; i = g; i, \text{ then } f = g.$$

A universal morphism $i : x \to y$ from x to id_X is the same as an isomorphism with source x.

Exercise 11. Show that functors preserve split epimorphisms and split monomorphisms. Do they preserve epimorphisms and monomorphisms?

Example 12. For the inclusion $\imath : \mathbb{Z} \to \mathbb{R}$, the universal morphism from a real number r to $\imath$ is the unique morphism corresponding to the inequality $r \leq \lceil r \rceil$, where $\lceil r \rceil$ is the *ceiling* of r.

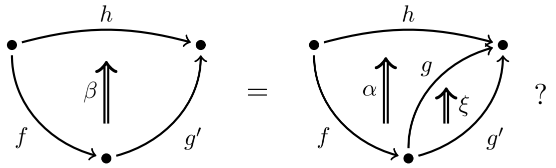
In general, for functors $F : P \to Q$ between partial orders, universal morphisms from x to F capture the idea of a *best approximation from above* of x in P.

Of course, the dual of Lemma 7 holds.

Lemma 13. *Suppose $g : y \to F(x)$ and $g' : y \to F(x')$ are two universal morphisms from y to G. Then there exists a unique isomorphism $e : x \to x'$ such that $g' = g; F(e)$.*

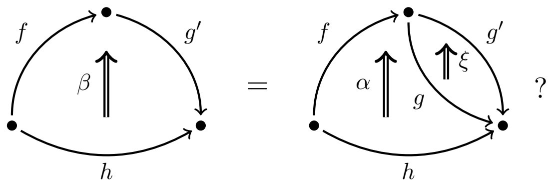
In a category, the possible configurations of composition are limited: either you pre-compose, or you post-compose a morphism. In a bicategory, there are many more possibilities. Two kinds of bicategorical division problems, dual to each other via $(-)^{\mathrm{co}}$, have a particularly important role.

Problem 14. Given two 2-cells $\alpha : (f, g) \Rightarrow (h)$ and $\beta : (f, g') \Rightarrow (h)$ in a bicategory X, does there exist a 2-cell $\xi : (g') \Rightarrow (g)$ such that



If it exists, is it unique?

Problem 15. Given two 2-cells $\alpha : (h) \Rightarrow (f, g)$ and $\beta : (h) \Rightarrow (f, g')$ in a bicategory X, does there exist a 2-cell $\xi : (g) \Rightarrow (g')$ such that



If it exists, is it unique?

Definition 16. A 2-cell $\alpha : (f, g) \Rightarrow (h)$ is a *right Kan extension* of h along f if Problem 14 has a unique solution for each $\beta : (f, g') \Rightarrow (h)$.

Dually, a 2-cell $\alpha : (h) \Rightarrow (f, g)$ is a *left Kan extension* of h along f if Problem 15 has a unique solution for each $\beta : (h) \Rightarrow (f, g')$.

Similarly to how an object is specified up to isomorphism by the universality property of a morphism, a 1-cell is specified up to isomorphism by the universality property of a 2-cell. The proof is similar to that of Lemma 7 and we leave it as an exercise.

Exercise 17. Let $\alpha : (f, g) \Rightarrow (h)$ and $\alpha' : (f, g') \Rightarrow (h)$ be two right Kan extensions of h along f. Show that there is a unique isomorphism $(g) \Rightarrow (g')$ through which α factors.

Dualise to obtain an analogous result on left Kan extensions.

Remark 18. In category theory, it is common to use the definite article ("the") to speak of something which is unique up to unique isomorphism. Thus, you will hear about "*the* right Kan extension of h along f", and you will see g be given a specific label, for example $\mathrm{Ran}_f h$.

Remember that unless some kind of algebraic structure is present, such a label does not uniquely specify a 1-cell in the bicategory, but only an isomorphism class thereof.

Example 19. The monoidal category $\mathbf{Set}_\times$ of [Lecture 2, Example 12] has right Kan extensions of every 1-cell along every other 1-cell. Given two sets S, T, let T^S be the set of functions from S to T. There is a 2-cell $\mathrm{ev} : (S, T^S) \Rightarrow (T)$ corresponding to the *evaluation* function

$$(x, f) \mapsto f(x)$$

for each $x \in S$ and $f : S \to T$. We claim that this is a right Kan extension of T along S.

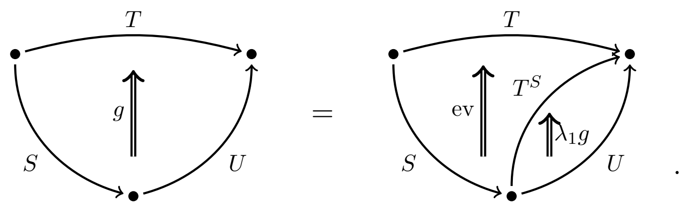
To prove it, consider any other 2-cell $g : (S, U) \Rightarrow (T)$. This is a function sending pairs (x, y) of $x \in S$ and $y \in U$ to elements $g(x, y)$ of T. Let $\lambda_1 g : U \to T^S$ be the function defined by

$$y \quad \mapsto \quad g(-, y) : S \to T,$$

for each $y \in U$. This corresponds to a 2-cell $(U) \Rightarrow (T^S)$, and g factors as

$$(x, y) \mapsto (x, g(-, y)) \mapsto g(x, y),$$

which is to say,



Suppose that there is another $h : U \to T^S$ through wich g factors. Then $\mathrm{ev}(x, h(y)) = h(y)(x) = g(x, y) = \lambda_1 g(y)(x)$ for all $x \in S$ and $y \in U$; this implies that $h(y)$ is equal to $\lambda_1 g(y)$ for all $y \in U$, so $\lambda_1 g = h$, and we have proved uniqueness.

In general, a monoidal category with the property that right Kan extensions exist for all pairs of 1-cells is called a *monoidal right closed* category. There is a dual notion of *monoidal left closed*, via $(-)^{\mathrm{op}}$, and $\mathbf{Set}_\times$ is both left and right closed, with universal 2-cells related by permutation of variables (we will see later that this follows from the fact that $\mathbf{Set}_\times$ is a *symmetric* monoidal category).

Example 20. Let P be a meet-semilattice with a greatest element. We say that P has *implications* if, for all elements $x, y \in P$, there exists an element $x \to y \in P$ with the property that, for all $z \in P$,

$$x \wedge z \leq y \qquad \text{if and only if} \qquad z \leq x \to z.$$

This is a typical property of algebraic models of propositional logic, such as Boolean algebras or Heyting algebras: implication models logical implication, and its defining property corresponds to the rules

$$\frac{A \wedge B \vdash C}{B \vdash A \to C}\ , \qquad \frac{B \vdash A \to C}{A \wedge B \vdash C}\ ,$$

called *importation* and *exportation*, which are valid both in classical and in intuitionistic logic.

Because $x \to y \leq x \to y$, it follows that $x \wedge (x \to y) \leq y$ for all $x, y \in P$. If we see P as a monoidal category, as in [Lecture 2, Example 13], this corresponds to a 2-cell $(x, x \to y) \Rightarrow (y)$.

Moreover, by the defining property of implications, any 2-cell $(x, z) \Rightarrow (y)$ factors through a 2-cell $(z) \Rightarrow (x \to y)$: that is, the 2-cell $(x, x \to y) \Rightarrow (y)$ is a right Kan extension of x along y.

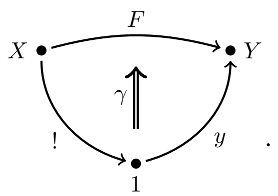
Now, we focus on Kan extensions in the bicategory **Cat**. These subsume an enormous variety of constructions in category theory, as suggested by the iconic title of a chapter in Mac Lane's *Categories for the working mathematician*:

All concepts are Kan extensions.

For now, we shall focus on a special, but far-reaching case. First of all, observe that

1. for each category X, there is a unique functor $! : X \to 1$;
2. thus, for each object y of a category Y, corresponding to a functor $y : 1 \to X$, there is a functor $!; y : X \to Y$, the *constant functor* at y.

Definition 21. Let $F : X \to Y$ be a functor between categories. A *cone over F* is a natural transformation

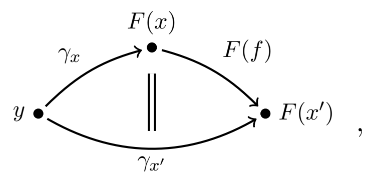


The object y of Y is called the *vertex* of the cone.

The cone γ is a *limit cone* if it is a right Kan extension of F along ! in **Cat**. In this case, we say that the vertex y is a *limit* of F.

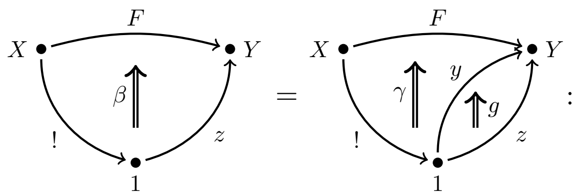
We say that F *has a limit* if there exists a limit cone over F. We say that Y *has X-limits* if all functors $F : X \to Y$ have a limit.

Let us try to understand this definition. In components, a cone over F is given by a family of morphisms $\gamma_x : y \to F(x)$, all with source y, indexed by the objects of X. Because the only morphism in the image of $!; y$ is the identity on y, the naturality squares "collapse" to commutative triangles



in Y, one for each $f : x \to x'$ in X. The idea is that the image of F is the "base" of a cone, and these triangles form the "surface" connecting the vertex to the base.

If γ is a limit cone, it means that any other cone β factors as



but a natural transformation $g : z \Rightarrow y$ is just a morphism $g : z \to y$ in Y [Lecture 2, Exercise 19].

This means that, if we have a limit cone γ over F with vertex y, we can reconstruct *any* other cone over F from the data of a single morphism: or in other words, single morphisms with target y in Y "classify", or "are in bijection with", or "encode as much information" as cones over F, which have as many components as there are objects of X.

By Exercise 17, limits are unique up to a unique isomorphism, so we can speak of "the limit" of a functor.

Let us look at some important examples of limits. Let J be an "indexing" set; we can see J as a *discrete* category with only identity morphisms, that is, the free category on the discrete graph with J as set of vertices.

Definition 22. A J-indexed *product* in a category X is the limit of a functor $F : J \to X$.

We single out a special case. If J is the empty set $\emptyset$, there is a unique functor $\emptyset \to X$.

Definition 23. A *terminal object* in X is the limit of the unique functor $\emptyset \to X$.

A limit cone over $F : J \to X$ is given by an object of X, that we denote by $\prod_{j \in J} F(j)$, together with a morphism $\pi_i : \prod_{j \in J} F(j) \to F(i)$ for each $i \in J$; the π_i are called *projections*. Because J has no non-identity morphisms, there are no non-trivial naturality conditions.

The universal property means that, for any other cone over F, given by a family of morphisms $\{\beta_i : x \to F(i)\}_{i \in J}$, there is a *unique* morphism $\langle \beta \rangle : x \to \prod_{j \in J} F(j)$ such that $\beta_i = \langle \beta \rangle ; \pi_i$ for all $i \in I$.

In the special case $J = \emptyset$, a limit cone is just given by an object of X, that we denote by 1, and the universal property means that for any other cone, that is, any other object x of X, there is a *unique* morphism $! : x \to 1$.

The case $J = 1$ is trivial: the limit cone of any functor $1 \to X$, corresponding to an object x of X, is given by x and the identity morphism on x.

Example 24. Suppose J has two or more elements, and is "small" in the sense that it corresponds to an object of **Set**. A functor $S : J \to \mathbf{Set}$ is the same as a J-indexed family of sets $\{S(j)\}_{j\in J}$. We claim that the cartesian product of the $S(j)$, that is, the set

$$\prod_{j\in J} S(j) := \{(x_j)_{j\in J} \,|\, x_j \in S(j)\}$$

of J-indexed sequences of elements $x_j \in S(j)$, together with the *projection* functions

$$\pi_i : (x_j)_{j\in J} \mapsto x_i,$$

is a limit cone over S in **Set**.

To show this, consider another cone over S, that is, a set T together with a family of functions $\{f_i : T \to S(i)\}_{i\in J}$. We define a function $\langle f\rangle : T \to \prod_{j\in J} S(j)$ by

$$\langle f\rangle(y) := (f_j(y))_{j\in J}.$$

Then $\pi_i(\langle f\rangle(y)) = f_i(y)$ for all $i \in J$.

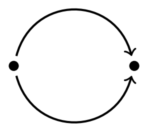
If $g : T \to \prod_{j\in J} S(j)$ is another function with the same property, the fact that $\pi_i(g(y)) = f_i(y)$ implies that $g(y)$ is a sequence with $f_i(y)$ in the i-th position, for each $i \in J$; hence $g(y) = \langle f\rangle(y)$ for all $y \in T$, implying $g = \langle f\rangle$.

The category **Set** also has a terminal object: it is the set $\{*\}$ of a single element. This proves that **Set** has all small products, that is, products indexed by a small set.

Exercise 25. What is a product in a poset?

Exercise 26. Show that if a category has a terminal object and all binary products (that is, all products indexed by the 2-element set), then it has all finite products, that is, all products indexed by a finite set.

Recall that ∂O^2 is the free category on the graph with two vertices and two parallel edges,



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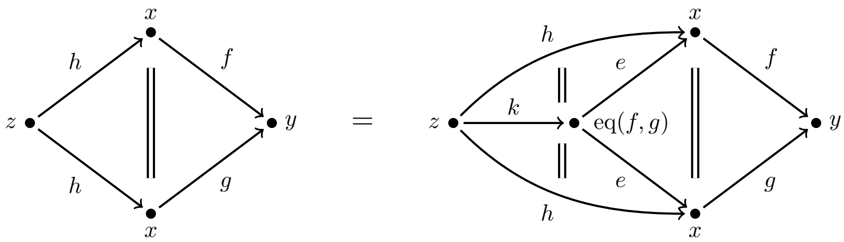
Definition 27. An *equaliser* in a category X is the limit of a functor $F : \partial O^2 \to X$.

The functor F picks a parallel pair of morphisms $f, g : x \to y$ in X. A cone over F is completely specified by its vertex z, together with a morphism $h : z \to x$ satisfying

$$h; f = h; g.$$

The universal property of a limit cone over F, given by a vertex $\mathrm{eq}(f, g)$ and a morphism $e : \mathrm{eq}(f, g) \to x$, requires that any such h factors uniquely through e: in diagrams,



for a unique $k : z \to \mathrm{eq}(f, g)$.

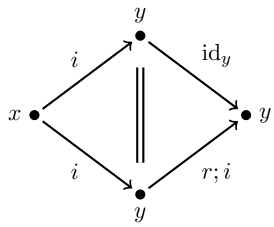
The idea is that an equaliser is a structural definition of the solution set of an equation: the instantiation of this notion in **Set** gives this intuition.

Example 28. Consider a functor $\partial O^2 \to \mathbf{Set}$, that is, a pair of functions $f, g : S \to T$. We claim that the equaliser of this pair is the set

$$\mathrm{eq}(f, g) := \{x \in S \mid f(x) = g(x)\}$$

together with its inclusion as a subset of S. To show this, observe that any function $h : U \to S$ with the property that $h; f = h; g$, that is, $f(h(y)) = g(h(y))$ for all $y \in U$, actually has its image contained in $\mathrm{eq}(f, g)$, hence factors through its inclusion in S. The fact that injective functions are monomorphisms in **Set** implies that this factorisation is unique.

Example 29. Suppose $i : x \to y$ is a split monomorphism in a category X, with reflection $r : y \to x$ such that $i; r = \mathrm{id}_x$. The following is a commutative diagram in X:



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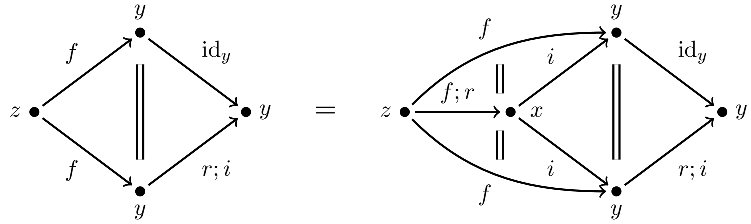
we claim that it exhibits x as the equaliser of the pair $r; i$ and $\mathrm{id}_y : y \to y$.

To show this, consider any other morphism $f : z \to y$ such that $f; \mathrm{id}_y = f = f; r; i$. Then



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and the factorisation is unique: if $f = g; i$ for some other $g : z \to x$, then $f; r = g; i; r = g$.

Products and equalisers are central in the theory of limits, because in their presence we can build any other limit, in the following sense.

Theorem 30. *Suppose that a category Y has equalisers and J-indexed products for each set J of cardinality $|J| < \kappa$. Then Y has X-limits for all categories X with $|X_1| < \kappa$.*

Proof. Let X be a category with $|X_1| < \kappa$, and $F : X \to Y$ an arbitrary functor. Because $|X_0| < |X_1|$ in a reflexive graph, we can form the two products

$$\prod_{x \in X_0} F(x) \qquad \text{and} \qquad \prod_{f \in X_1} F(t(f))$$

in Y, respectively X_0 and X_1-indexed. Now, for all $f \in X_1$, we have a projection

$$\pi_{t(f)} : \prod_{x \in X_0} F(x) \to F(t(f)),$$

and also a projection

$$\pi_{s(f)} : \prod_{x \in X_0} F(x) \to F(s(f)),$$

which can be composed with $F(f) : F(s(f)) \to F(t(f))$ to give another morphism $\pi_{s(f)}; F(f) : \prod_{x \in X_0} F(x) \to F(t(f))$.

By the universal property of products, these two X_1-indexed families of morphisms induce a unique pair of parallel morphisms

$$\langle \pi_t \rangle \text{ and } \langle \pi_s; F \rangle : \prod_{x \in X_0} F(x) \to \prod_{f \in X_1} F(t(f)). \tag{1}$$

Let $e : \lim F \to \prod_{x \in X_0} F(x)$ be the equaliser of this pair of morphisms. For all $x \in X_0$, let $\gamma_x := e; \pi_x : \lim F \to F(x)$. Then,

- the γ_x form a cone over F: for all $f : x \to y$, we have

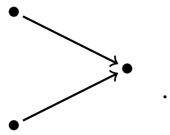
$$\gamma_x; F(f) = e; \pi_x; F(f) = e; \pi_{s(f)}; F(f) = e; \pi_{t(f)} = \gamma_y;$$

- any other cone $\{\beta_x : y \to F(x)\}_{x \in X_0}$ induces a unique morphism $\langle \beta \rangle : y \to \prod_{x \in X_0} F(x)$, and naturality implies that this morphism equalises the pair (1). It follows that $\langle \beta \rangle$ factors uniquely through e, which in turn implies that the β_x factor uniquely through the γ_x.

Thus, the γ_x form a limit cone over F, which proves that F has a limit. □

There are other, less general constructions of limits from other limits. The following exercise gives a small sample.

Exercise 31. Let V be the free category on the graph

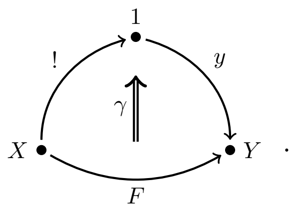


Diagrams $F : V \to X$ are called *cospans* in X. A *pullback* is the limit of a cospan.

1. Suppose that X has binary products and equalisers. Prove that X has pullbacks.
2. Suppose that X has a terminal object and pullbacks. Prove that X has binary products and equalisers.

The dual notion of a limit is a *colimit*. A colimit in X is the same as a limit in X^{op}: this statement already contains everything you need. Nevertheless, it is useful to spell out some definition.

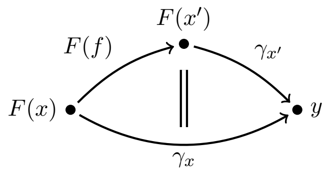
Definition 32. Let $F : X \to Y$ be a functor. A *cone under* F is a natural transformation



The cone γ is a *colimit cone* if it is a left Kan extension of F along ! in **Cat**. In this case, we say that the vertex y is a *colimit* of F.

We say that F *has a colimit* if there exists a colimit cone under F. We say that Y *has* X*-colimits* if all functors $F : X \to Y$ have a colimit.

In components, a cone under F is given by a family of morphisms $\gamma_x : F(x) \to y$, all with target y, indexed by the objects of X, and such that



for each morphism $f : x \to y$ of X. It is a colimit cone if any other cone β under F with vertex z is obtained by post-composing the γ_x with a unique morphism $[\beta] : y \to z$, that is, morphisms with source y "classify", or "are in bijection with", or "encode as much information" as cones under F.

Definition 33. For J a discrete category, a J-indexed *coproduct* in a category X is the colimit of a functor $F : J \to X$. If $J = \emptyset$, this is called an *initial object*.

A *coequaliser* in X is the colimit of a diagram $\partial O^2 \to X$.

Example 34. Given a functor $S : J \to \mathbf{Set}$, that is, a J-indexed family of sets, their coproduct is the disjoint union

$$\coprod_{j \in J} S(j) := \{(j, x) \mid j \in J, x \in S(j)\}$$

of the sets $S(j)$, together with the injections

$$\imath_i : S(i) \to \coprod_{j \in J} S(j), \qquad x \mapsto (i, x).$$

Given a functor $\partial O^2 \to \mathbf{Set}$, that is, a pair of morphisms $f, g : S \to T$, let $\sim$ be the smallest equivalence relation on T containing the pairs $(f(x), g(x))$ for all $x \in S$. The coequaliser of f and g is the quotient

$$\mathrm{coeq}(f, g) := T/\sim$$

together with the quotient map $q : T \to T/\sim$.

Of course, Theorem 30 has a dual.

Theorem 35. *Suppose that a category Y has coequalisers and J-indexed coproducts for each set J of cardinality $|J| < \kappa$. Then Y has X-colimits for all categories X with $|X_1| < \kappa$.*

Exercise 36. Let V be the category of Exercise 31. Functors $V^{\mathrm{op}} \to X$ are called *spans* in X. A *pushout* is the colimit of a span. Now dualise all the statements of Exercise 31.