# Category theory and diagrammatic reasoning

20th February 2019

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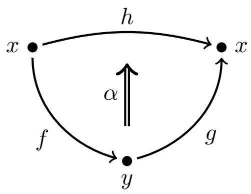
## 4 String diagrams and algebraic theories

String diagrams are an alternative representation for 2-cells and their compositions in a bicategory. A 2-cell with two 1-cells in its source and one 1-cell in its target has the following representation as a pasting diagram:

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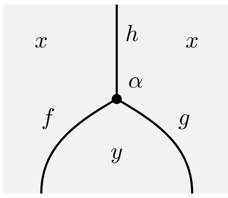
and the following as a string diagram:

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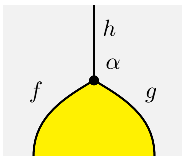
In their appearance, string diagrams are topological graphs embedded into a square, with the endpoint of some edges bound to the top or bottom side of the square (a *framing* of the graph) and some additional labelling.

The basic recipe is the following:

- 2-cells become *nodes* in the string diagram;
- 1-cells in the source of a 2-cell become *incoming wires*, and 1-cells in the target become *outgoing wires* (the "flow" goes from bottom to top);
- 0-cells are *regions of the plane* bounded by the wires.

*Remark* 1. In the literature, you may also find 2-cells pictured as *boxes* with their label inside, instead of nodes.

When there are few 0-cells involved, we may *colour-code* the regions instead of labelling them, with one colour corresponding to each 0-cell: for example, with $y$ colour-coded as yellow, the diagram above becomes

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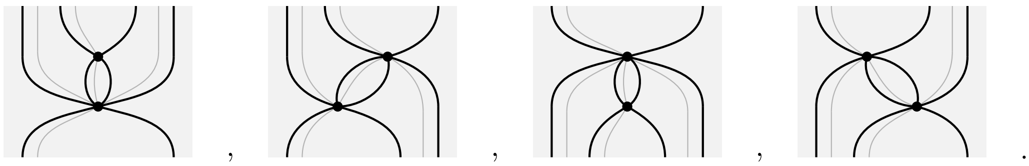
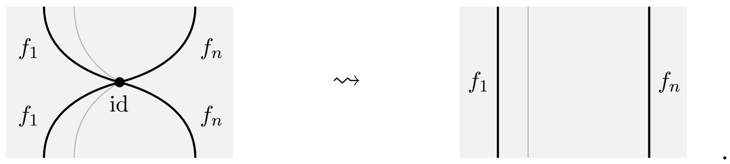
Corresponding to the "dotted edge" notation for an indefinite number of sources and targets, we draw wires with a lighter shade, indicating a repeated pattern:


In string diagrams, we have the following ways of composing 2-cells:



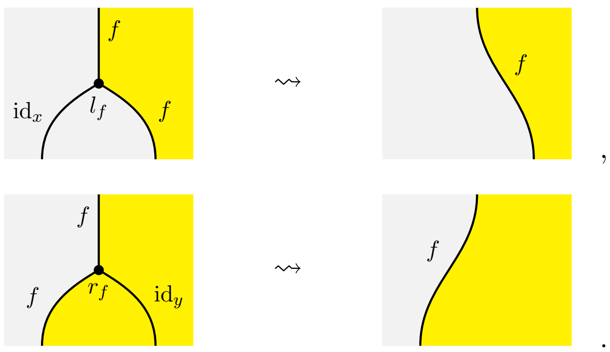
Moreover, there are two conventions generally adopted.

1. Identity 2-cells are drawn as sequences of parallel wires, with no nodes:




   This is justified by the fact that composing with an identity "changes nothing", which in the diagrammatic language is the same as "lengthening wires", since the length and curvature of wires in a string diagram have no meaning.

2. Identity 1-cells are *not* drawn, and the left and right unitor 2-cells are drawn as identities:




   Because identity 1-cells can always be introduced and eliminated freely to the left and right of other 1-cells, using the unitors and their inverses, this notational choice is inessential when dealing with 2-cells with at least one non-identity 1-cell in the source and in the target: it is the same as "agreeing to always eliminate as many identity 1-cells as possible".

The only real effect of this convention is that we can draw diagrams with *no* incoming or outgoing wires, such as

and that they can be "moved around" in the way that one intuitively expects.

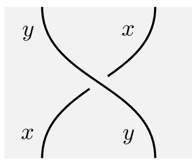
You may ask: if string diagrams and pasting diagrams convey the exact same information, why have two languages instead of one? The answer is that the intuitive usage of the two languages, hence their *heuristic* value, is different.

- Pasting diagrams are close both to commutative diagrams in categories, and to pasting diagrams of topological cells. Consequently, they are ideally suited to the study of bicategories both as generalisations of categories, and of 2-dimensional topological spaces.

- On the other hand, string diagrams look like graphs, and have a different "natural" set of topological manipulations, such as bending wires, or making one wire cross another. It turns out that important classes of bicategories and monoidal categories are *characterised* by the fact that certain topological moves can always be performed on their 2-cells, pictured in string diagrams.

While the language of string diagrams is surprisingly useful in general bicategories (as we will see later in the course), in this lecture we will focus on monoidal categories. String diagrams are particularly intuitive in this context: because there is no "colour-coding" of the background, the way that the plane is partitioned by the diagram is inessential, and we can think of going beyond planar embeddings of graphs.

**Definition 2.** A *braided monoidal category* is a monoidal category $X$ together with a family of invertible 2-cells $\{b_{x,y} : (x, y) \Rightarrow (y, x)\}$, called the *braidings*, parametrised by pairs $x, y$ of 1-cells, with the following property. Picture $b_{x,y}$ as the following string diagram:

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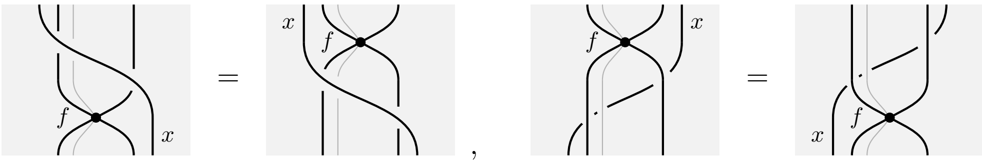
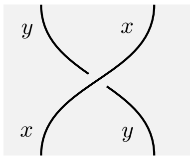
Then, for all 2-cells $f$ and 1-cells $x$ in $X$, the following hold, with the only compatible labelling of braidings:




, . (1)

The way that the braiding 2-cells are drawn is suggestive of their use: braidings are a way of having wires "cross over" other wires, in such a way that one can "slide" any other diagram under or over it.

*Remark* 3. We draw the inverse of $b_{x,y}$ in the following way:

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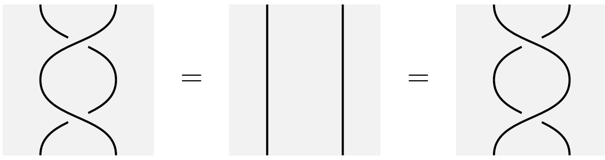
that is, with the opposite crossing. Then the fact that the $b_{x,y}$ are invertible is expressed in string diagrams as



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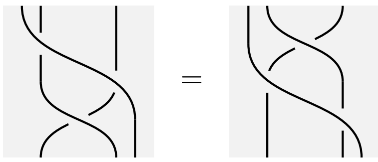
and the special case of (1) where $f$ is itself a braiding is



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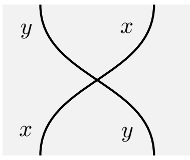
If you have encountered the theory of knots, you may recognise the second and third *Reidemeister move*: thus, any 1-cell in any braided monoidal category gives an interpretation of braid equations.

Apart from areas related to knot theory, braided monoidal categories are frequently encountered in quantum algebra; outside of it, the following "degenerate" case is more common.

**Definition 4.** A *symmetric monoidal category* is a braided monoidal category $X$ with the property that $b_{x,y} = b^{-1}_{y,x}$ for each pair of 1-cells $x, y$.

Because in a symmetric monoidal category there is no need to distinguish a braiding from its inverse, in string diagrams we denote them both by

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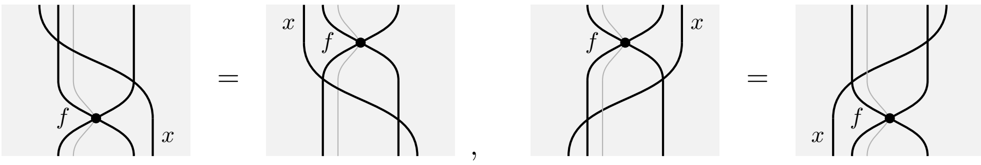
This is sometimes called a *swap*. The axioms (1) become

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Thus in a symmetric monoidal category we can rearrange nodes of a string diagram without being constrained by the planarity of the embedding.

**Example 5.** The monoidal category $\mathbf{Set}_\times$ is a symmetric monoidal category with the braidings $b_{S,T} : (S, T) \Rightarrow (T, S)$ given by $(x, y) \mapsto (y, x)$ for all $x \in S$ and $y \in T$.

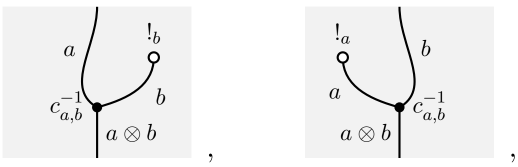
**Example 6.** A meet-semilattice $P$ with greatest element, seen as a monoidal category, is a symmetric monoidal category whose braiding $b_{x,y}$ is the unique morphism corresponding to the relation $x \wedge y \leq y \wedge x$.

In the previous lecture, we saw that, in both these examples, identity 1-cells and composites of 1-cells also satisfy certain universal properties. Recall that for each pair of 0-cells $x, y$ of a bicategory $X$, there is a category $\mathrm{Hom}_X(x, y)$ whose objects are 1-cells $x \to y$, and morphisms are 2-cells between them.

*Remark* 7. In what follows, if $X$ is a generic monoidal category, we denote by $i$ the identity on the unique 0-cell $*$ of $X$, and by $a \otimes b$ the composite of two 1-cells $a, b$.

**Definition 8.** Let $X$ be a monoidal category. We say that $X$ is a *cartesian monoidal category* if

1. $i$ is a terminal object of $\mathrm{Hom}_X(*, *)$, and
2. the following pair of 2-cells forms a limit cone in $\mathrm{Hom}_X(*, *)$, exhibiting $a \otimes b$ as the product of $a$ and $b$:




(2)

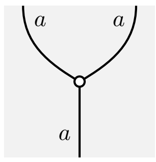
where $!_a$ and $!_b$ are the unique morphisms into the terminal object.

**Exercise 9.** Prove that every cartesian monoidal category becomes a symmetric monoidal category in a canonical way.

In fact, it is possible to give an entirely algebraic definition of cartesian monoidal categories, as the following result shows.

**Theorem 10.** *Let $X$ be a symmetric monoidal category. The following conditions are equivalent:*

1. *$X$ is a cartesian monoidal category;*
2. *for all 1-cells $a$, there exist 2-cells*

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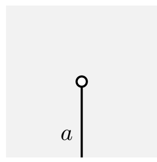
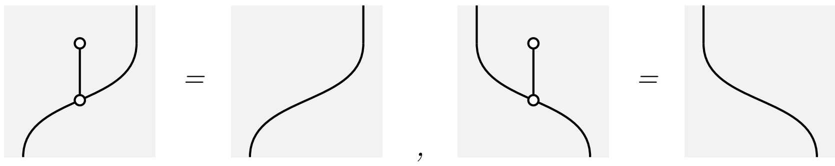



,

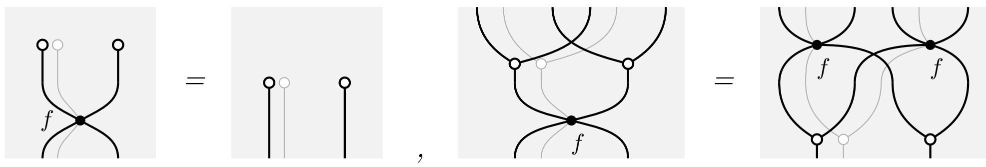
*(copy)* *(discard)*

*satisfying the following equations:*



(3)

*and, for all 2-cells $f$,*




(4)

*with the only compatible labelling of edges.*

*Proof.* Suppose $X$ is cartesian monoidal. For each 1-cell $a$, we define the discard 2-cell to be the unique 2-cell $d_a : (a) \Rightarrow (i)$. Moreover, by the universal property of $a \otimes a$, there is a unique 2-cell $(a) \Rightarrow (a \otimes a)$ corresponding to the cone $\{\mathrm{id}_a : (a) \Rightarrow (a), \mathrm{id}_a : (a) \Rightarrow (a)\}$ in $\mathrm{Hom}_X(*, *)$. Post-composing it with the inverse of the compositor $c_{a,a} : (a, a) \Rightarrow (a \otimes a)$, we obtain a 2-cell $c_a : (a) \Rightarrow (a, a)$: this will be our copy 2-cell.

**Exercise 11.** Prove that $d_a$ and $c_a$ so defined satisfy equations (3) and (4).

Conversely, suppose we are given 2-cells $c_a : (a) \Rightarrow (a, a)$ and $d_a : (a) \Rightarrow (i)$ satisfying the equations. First, let us show that $i$ is terminal in $\mathrm{Hom}_X(*, *)$; we will use the same names for 2-cells in $X$ and morphisms in $\mathrm{Hom}_X(*, *)$.

For all 1-cells $a$, the 2-cell $d_a$ is a morphism $d_a : a \to i$ in $\mathrm{Hom}_X(*, *)$; we need to show that it is unique. Consider any other morphism $f : a \to i$. We can post-compose it with $d_i : i \to i$; by the leftmost equation of (4),

$$f; d_i = d_a,$$

so it suffices to show that $d_i = \mathrm{id}_i$. The copy 2-cell $c_i : (i) \Rightarrow (i, i)$, post-composed with a unitor, gives a 2-cell $\tilde{c}_i : (i) \Rightarrow (i)$, corresponding to a morphism $\tilde{c}_i : i \to i$. The leftmost equation of (3) implies that

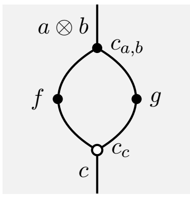
$$\tilde{c}_i; d_i = \mathrm{id}_i.$$

Moreover, by the leftmost equation of (4), we have that $d_i; d_i = d_i$, so

$$d_i = \mathrm{id}_i; d_i = \tilde{c}_i; d_i; d_i = \tilde{c}_i; d_i = \mathrm{id}_i.$$

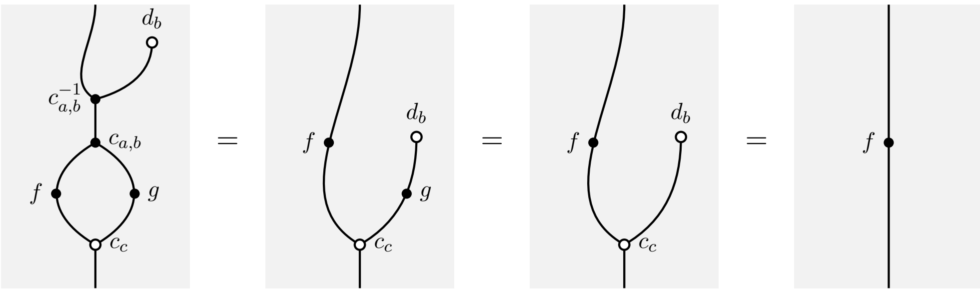
This proves that $i$ is the terminal object.

Now, consider any span $\{f : c \to a, g : c \to b\}$ in $\mathrm{Hom}_X(*, *)$. We define $\langle f, g\rangle : c \to a \otimes b$ to be the composite



. (5)

We have



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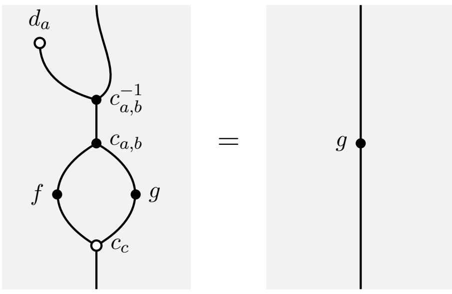
and similarly



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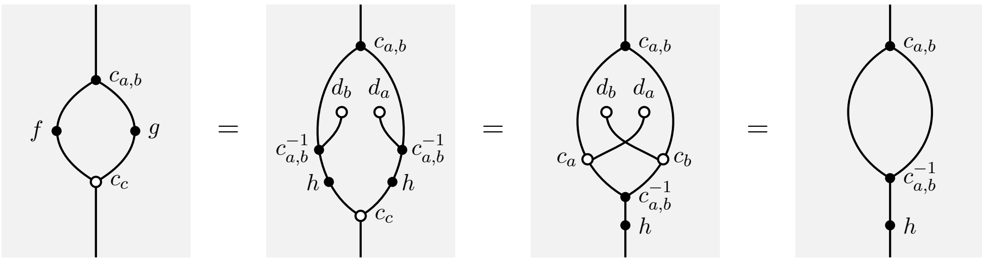
This proves that the cone $\{f : c \to a, g : c \to b\}$ factors through the cone (2) via the morphism (5).

Suppose that there is another morphism $h : c \to a \otimes b$ with the same property. Then




and this is equal to $h$. This proves uniqueness of the factorisation, and completes the proof that $a \otimes b$ is a product of $a$ and $b$. $\square$

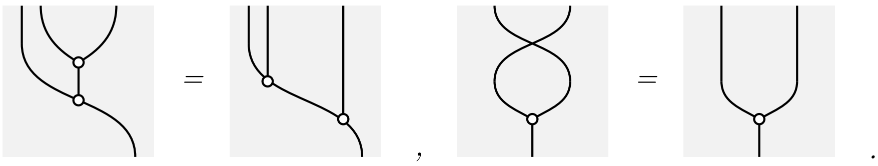
**Example 12.** In $\mathbf{Set}_\times$, the copy and discard 2-cells for a set $S$ are given, respectively, by

1. the *diagonal* function defined by $x \mapsto (x, x)$ for each $x \in S$, and
2. the unique function $x \mapsto *$ from $S$ to the one-element set.

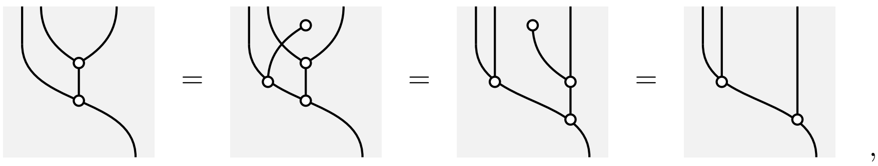
**Example 13.** In a meet-semilattice $P$ with greatest element $\top$, copy and discard for an element $x$ correspond to the relations $x \leq x \wedge x$ and $x \leq \top$, valid for all elements of $P$.

As an exercise in diagrammatic reasoning, let us prove a couple of derived properties of the copy 2-cells in a cartesian monoidal category.

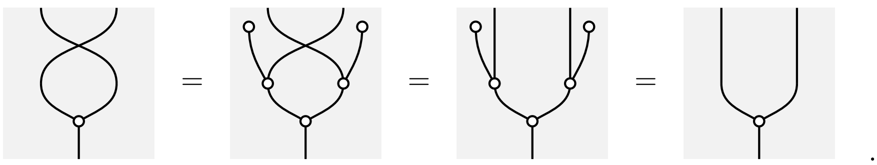
**Proposition 14.** *The following equations hold for copy 2-cells in all cartesian monoidal categories:*




*Proof.* For the first one




and for the second one




$\square$

A far-reaching observation, due to Lawvere, is that the structure of a cartesian monoidal category is exactly what is needed to define a model of an *algebraic theory*.

**Definition 15.** Let *Var* be a fixed, countably infinite set of variables, and let $F$ be a set of *functional symbols* together with a function $F \to \mathbb{N}$ assigning to each symbol its *arity*. A functional symbol with arity 0 is called a *constant*.

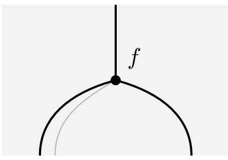
The set of *terms* on $F$, together with a set $\mathsf{free}(t)$ of *free variables* of each term $t$, are defined inductively as follows:

- each $x \in \mathit{Var}$ is a term, and $\mathsf{free}(x) := \{x\}$;
- each constant $c \in F$ is a term, and $\mathsf{free}(c) := \emptyset$;
- if $f \in F$ has arity $n > 0$ and $t_1, \ldots, t_n$ are terms, then $f(t_1, \ldots, t_n)$ is a term, and $\mathsf{free}(f(t_1, \ldots, t_n)) := \bigcup_{i=1}^{n} \mathsf{free}(t_i)$.

An *algebraic theory* $T$ on $F$ is given by a set $E$ of pairs $(t_1, t_2)$ of terms on $F$, to be read as equations $t_1 = t_2$.

A (set-theoretic) *model* of $T$ is given by a set $A$, together with a function $f : A^n \to A$ for each functional symbol $f \in F$ with arity $n$, such that, for each pair $(t_1, t_2) \in E$, the equation $t_1 = t_2$ holds under all instantiations of free variables in $\mathsf{free}(t_1) \cup \mathsf{free}(t_2)$ as elements of $A$.

**Construction 16.** We can rephrase the definition of an algebraic theory in terms of diagrams in a cartesian monoidal category. We represent an $n$-ary function symbol $f$ as a diagram




. (6)

with $n$ incoming wires, all with the same label (left implicit). We call this a *generator* of the theory.

Next, we will represent terms on $F$ as diagrams, in such a way that the free variables of a term $t$ are in bijection with the incoming wires of the diagram that represents it.

We interpret a variable $x$ and a constant $c$ as



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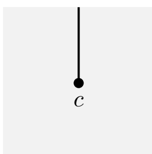


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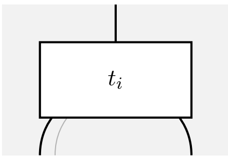
respectively. For the inductive step, suppose we have fixed a total order on *Var*, and suppose we have the interpretation of the terms $t_1, \ldots, t_n$ as certain diagrams

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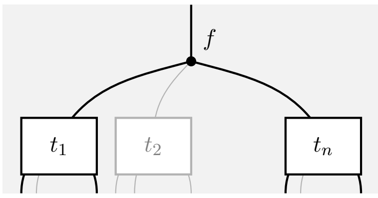
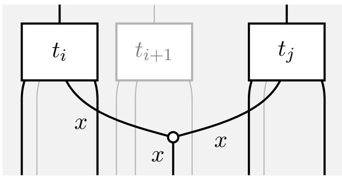
then, first we form the composite




. (7)

Each incoming wire of $t_i$ corresponds to a unique free variable of $t_i$; let us temporarily label each incoming wire of (7) with that variable. If two wires have the same label $x$, we plug the two outgoing wires of a copy 2-cell into them, and label the incoming wire $x$:

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We keep plugging copy 2-cells until there are no incoming wires with the same label; it follows from Proposition 14 that the order in which this is done does not matter.

Finally, we use braidings to permute the incoming wires so that the leftmost wires have labels that come before in the total ordering on *Var*.

This gives an interpretation of all terms $t$. Now, to interpret an equation $t_1 = t_2$ as an equation of diagrams, we need to make sure that

1. the two diagrams have the same number of incoming wires, and that
2. the $i$-th incoming wire of one diagram corresponds to the same free variable as the $i$-th incoming wire of the other diagram.

For that, we take the interpretation of $t_1$ as a diagram; then, for each free variable of $t_2$ which is *not* a free variable to $t_2$, we add a discard 2-cell to the diagram, whose incoming wire is in the correct position with respect to the fixed ordering on *Var*. Then, we do the same with the interpretation of $t_2$ as a diagram.

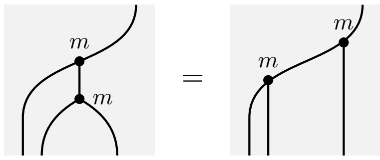
**Definition 17.** The algebraic theory of *semigroups* has a binary functional symbol $m$ and a unique equation

$$m(x, m(y, z)) = m(m(x, y), z).$$

The algebraic theory of *monoids* extends it with a constant $e$, and equations

$$m(e, x) = x, \qquad m(x, e) = e.$$

In diagrams (and with the ordering of variables $x < y < z$) these become

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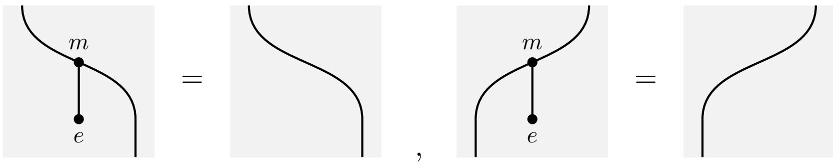


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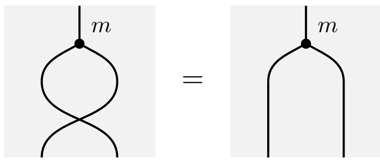
The theory of *commutative monoids* has the additional equation $m(y, x) = m(x, y)$; in string diagrams,

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**Definition 18.** The algebraic theory of *groups* extends the theory of monoids with a unary functional symbol $s$ and the equations

$$m(s(x), x) = e, \qquad m(x, s(x)) = e.$$

In string diagrams, these are


. (8)

Recalling the interpretation of copy and discard in $\mathbf{Set}_\times$, you should be able to convince yourself that a set-theoretic model of $T$ is the same as an interpretation of the diagrams (6) in $\mathbf{Set}_\times$ such that all the diagrammatic equations hold.

Because all that is needed to make sense of the diagrams is the structure of a cartesian monoidal category, we can generalise the notion of model.

**Definition 19.** Let $T$ be an algebraic theory, and consider its diagrammatic presentation as in Construction 16. Let $X$ be a cartesian monoidal category. A *model of* $T$ *in* $X$ is an interpretation of the generators of $T$ in $X$ such that all the diagrammatic equations hold.

When $T$ is the theory of "something", it is common to call a model of $T$ in $X$ an "internal something" in $X$. For example, a model of the theory of monoids in $X$ is an *internal monoid* in $X$.

**Example 20.** An internal model of $T$ in $\mathbf{Set}_\times$ is the same as a set-theoretic model of $T$. In particular, internal monoids in $\mathbf{Set}_\times$ are monoids, internal groups are groups, and so on.

**Example 21.** Consider the cartesian monoidal category $\mathbf{Top}_\times$, defined in a similar way to $\mathbf{Set}_\times$, whose 1-cells are topological spaces, and 2-cells $(X_1, \ldots, X_n) \Rightarrow (Y_1, \ldots, Y_m)$ correspond to maps $X_1 \times \ldots \times X_n \to Y_1 \times \ldots \times Y_m$ that are continuous with respect to the product topology.

Internal groups in $\mathbf{Top}_\times$ are *topological groups*: that is, group structures on topological spaces whose multiplication and inverse operations are all continuous maps.

**Exercise 22.** Let $P$ be a meet-semilattice with greatest element. What is an internal semigroup in $P$? What is an internal monoid in $P$?

Notice that for some algebraic theories, the entire structure of a cartesian monoidal category is not needed to make sense of the diagrammatic equations, because they do not use braidings, copy, or discard: for example,

- the equations of monoids can be interpreted in any monoidal category;
- the equations of commutative monoids in any *symmetric* monoidal category.

This means that we can make sense of internal monoids in any monoidal category, and of internal commutative monoids in any symmetric monoidal category.

**Example 23.** There is a symmetric monoidal category $\mathbf{Ab}_\otimes$ whose 1-cells are abelian groups, and 2-cells $(X_1, \ldots, X_n) \Rightarrow (Y_1, \ldots, Y_m)$ are homomorphisms $X_1 \otimes \ldots \otimes X_n \to Y_1 \otimes \ldots \otimes Y_m$ between tensor products of abelian groups.

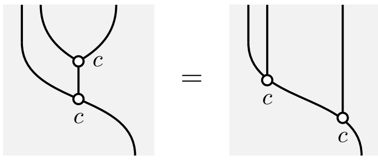
An internal monoid in $\mathbf{Ab}_\otimes$ is the same as a *ring*; an internal commutative monoid in $\mathbf{Ab}_\otimes$ is a commutative ring.

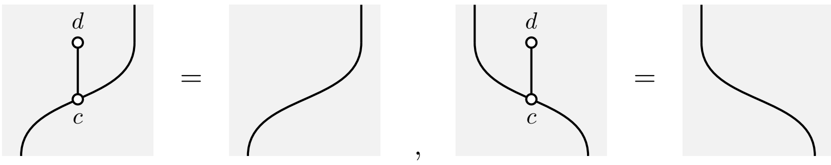
Then one can think of completely dropping the "logical" framework, and defining "generalised algebraic theories" directly by generators and relations in the language of string diagrams. What one obtains in the cartesian monoidal case is called a *Lawvere theory*, in the symmetric monoidal case a *PROP*, and in the monoidal case a *PRO*.

In the cartesian monoidal case, generators with multiple outputs do not introduce anything new, because they are entirely characterised by their single-output projections. In general, however, *coalgebraic* theories involving single-input, many-outputs operations are as rich as their algebraic counterparts.

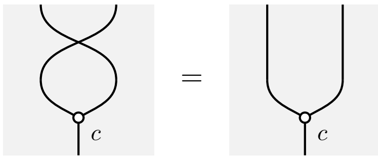
**Definition 24.** The theory of *comonoids* is presented by the generators


called *comultiplication* and *counit*, together with the equations




The theory of *cocommutative comonoids* has the additional equation




**Example 25.** There is a symmetric monoidal category $\mathbf{Vec}_{k,\otimes}$ whose 1-cells are vector spaces over a field $k$, and 2-cells $(V_1, \ldots, V_n) \Rightarrow (W_1, \ldots, W_m)$ are linear maps $V_1 \otimes \ldots \otimes V_n \to W_1 \otimes \ldots \otimes W_m$ between tensor products of vector spaces.

Any choice of a basis $\{e_i\}$ on $V$ gives an internal cocommutative comonoid in $\mathbf{Vec}_{k,\otimes}$, with comultiplication $c : V \to V \otimes V$ and counit $d : V \to k$ defined by

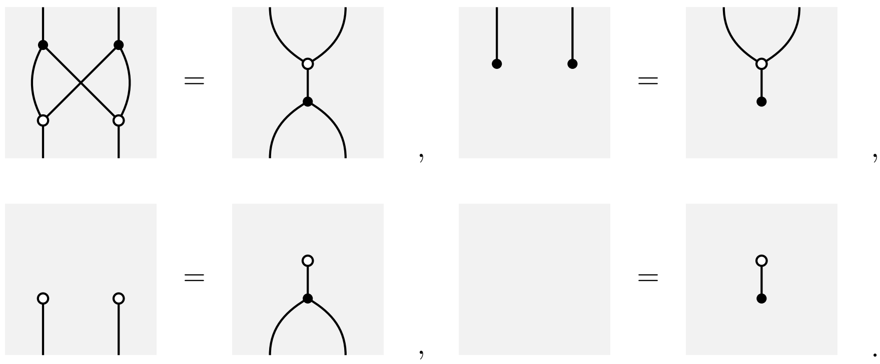
$$c : e_i \mapsto e_i \otimes e_i, \qquad d : e_i \mapsto 1$$

on basis elements.

By Theorem 10 and Proposition 14, every 1-cell in a cartesian monoidal category has the structure of an internal cocommutative comonoid, with copy as comultiplication and discard as counit; this is, in fact, the only possibility.

Also by Theorem 10, cartesian monoidal structure can be presented equationally: even in an algebraic theory whose equations make use of "copy" and "discard", we can treat the latter simply as additional generators with their own equations, rather than structure. Applying this idea to the algebraic theory of groups, we obtain the following.

**Definition 26.** The theory of *bialgebras* is presented by the generators of the theories of monoids and comonoids, together with the equations of the theories of monoids and comonoids, and the following additional equations:



The theory of *Hopf algebras* contains an additional generator $s$ and equations (8) as in the theory of groups.

In a cartesian monoidal category $X$, the comonoid is interpreted as the (only possible) copy-discard comonoid, and the additional equations of bialgebras are all special cases of (4). Thus an internal bialgebra in $X$ is the same as an internal monoid in $X$, and an internal Hopf algebra in $X$ is the same as an internal group in $X$.

In general symmetric monoidal categories, however, there can be internal monoids that are not part of a bialgebra structure. Internal Hopf algebras in various symmetric monoidal categories have a rich theory, with important applications in algebraic topology, representation theory, and mathematical physics.