# A Complexity Approach to Tree Algebras: the Polynomial Case


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Let $\Sigma$ be a ranked alphabet and $\mathcal{V}$ be a countably infinite set of variables. The free tree algebra has as carrier sets the $(T_X)_{X \subseteq \mathcal{V} \text{ finite}}$.

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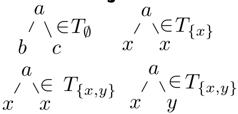

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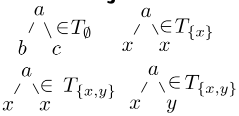


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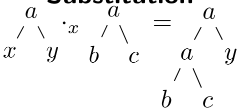

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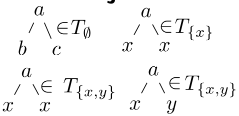

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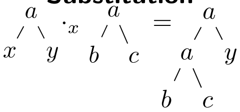

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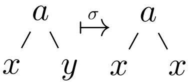

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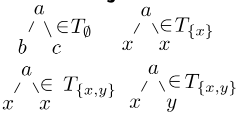


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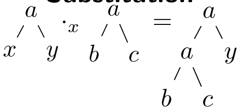


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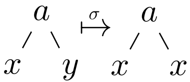


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- **Constants.** $a(x_0, \ldots, x_{n-1})^{\mathcal{A}} \in A_{\{x_0, \ldots, x_{n-1}\}}$ for all $a \in \Sigma_n$ and variables $x_i$,
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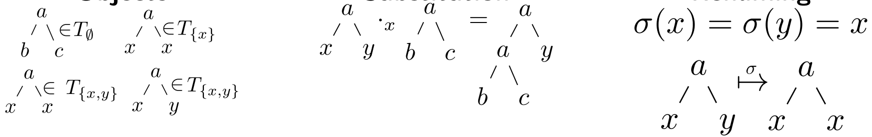


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**Identities?** $a(x, y) \cdot_y b$ $\quad$ $a(x, z) \cdot_z b$

We also define morphisms, congruences...

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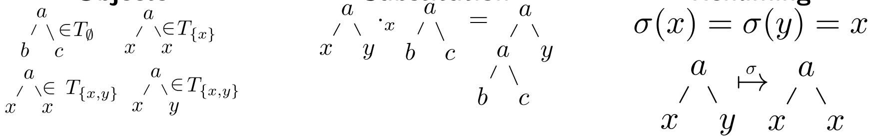


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Given a finite tree algebra $\mathcal{A}$, there is a unique morphism from the free algebra to $\mathcal{A}$. It is called the evaluation morphism of $\mathcal{A}$.

# Languages and the size of the algebra

## Definition (Language recognized by an algebra)

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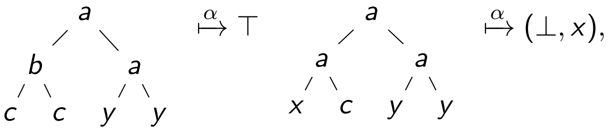

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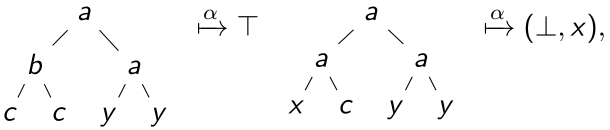


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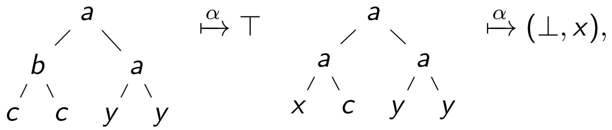


$A_X = \{\top, \bot\} \uplus (\{\top, \bot\} \times X)$ $\qquad |A_X| = 2 + 2|X|$ is linear in $|X|$.

This algebra has linear complexity.

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**The objective is to identify new classes of languages and to gain a better understanding of tree algebras.**

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$L =$ trees whose leftmost branch ends with $a(c, c)$, where
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```
d  ↦α d          a           ↦α a(x, z)     a        ↦α a(x, ∗)
               /   \                        / \
              a     a                      x   a
             / \   / \                        / \
            x   z y   y                      y   y
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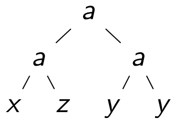


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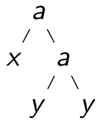


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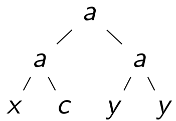


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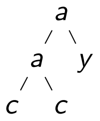


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**Orbits:** $c$, $d$, $a(x,y)$, $a(x,x)$, $a(x,c)$, $a(c,x)$, $a(x,*)$, $a(*,x)$, $a(c,c)$, $a(*,*)$

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This algebra has quadratic complexity and bounded orbit complexity.

# Orbit complexity

Let $|A_X/\mathbf{Sym}(X)|$ be the number of orbits of $A_X$ under the action of $\mathbf{Sym}(X)$ induced by renamings.

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## Another bounded hierarchy of classes

All regular languages are recognized by algebras of doubly-exponential orbit complexity.

# What complexity means

Complexity is a tool to quantify what the algebra remembers about the variables:

**Bounded complexity**
The algebra does not remember anything about the variables.
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## Equivalence theorem

For a regular language of finite trees, the following properties are equivalent:

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c. Being described by a coding automaton.

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- $X$ (resp. $f$) is called equivariant if it is supported by the empty set.

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- $x \in X$ is called finitely supported if there exists $S \subseteq \mathcal{V}$ finite such that, for every $\sigma \in \mathbf{Sym}(\mathcal{V})$, $\sigma(x) = x$ whenever $\sigma(s) = s$ for every $s \in S$.
- $X$ is called nominal if its elements are finitely supported.
- $f\colon X \to Y$ is supported by $S \subseteq \mathcal{V}$ if $f(\sigma(x)) = \sigma(f(x))$, for all $x \in X$, $\sigma \in \mathbf{Sym}(\mathcal{V} \setminus S)$.
- $X$ (resp. $f$) is called equivariant if it is supported by the empty set.

A deterministic orbit-finite nominal automaton is given by

- an orbit-finite nominal set $A$ (the alphabet),
- an orbit-finite nominal set $Q$ (the states),
- equivariant subsets $\{q_0\}$ and $F$ of $Q$ (the initial state and the final states),
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**Example:** a deterministic register automaton can be seen as a deterministic orbit-finite nominal automaton.

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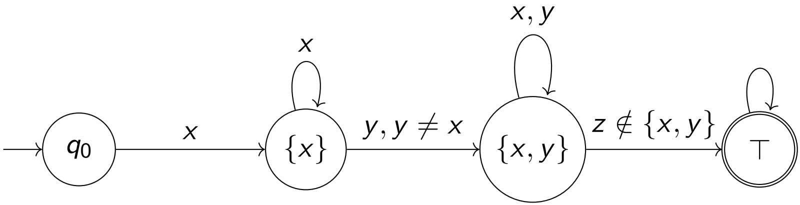
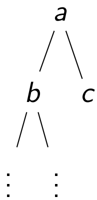


# Coding of trees

How to build the following tree ?



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$x$ $\qquad$ $[x]$

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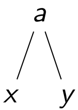


$[x]$

$[\cdot_x a(x, y)]$

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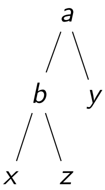


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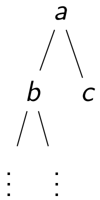


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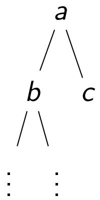


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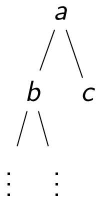


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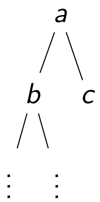


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A language $L$ of codings describes a language $K \subseteq T_\emptyset$ of trees if, for every coding $c$ such that $T(c) \in T_\emptyset$, $c \in L$ if and only if $T(c) \in K$.

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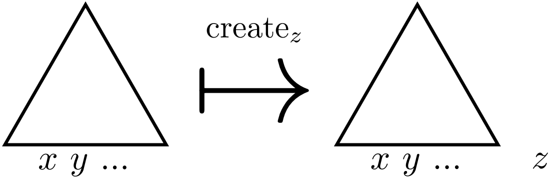

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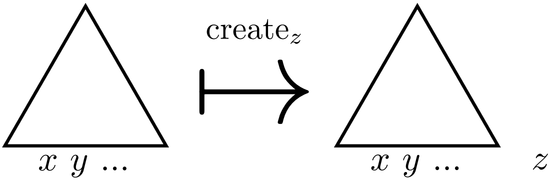


$$T(c) = \mathrm{create}_z(a(x,y)) \cdot_z c = a(x,y)$$

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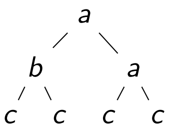
We assume that there is no transition toward the initial state $q_0$.

# Language described by a coding automaton 1/2

$K =$ "trees with a $b$ on the leftmost branch"



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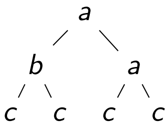


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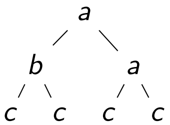


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$$Q = \{q_0, \top, \bot\} \cup \{x \mid x \in \mathcal{V}\}$$

$$q_0 \xrightarrow{[x]} x \xrightarrow{[\cdot_x a(x,y)]} x \xrightarrow{[\cdot_y a(z,z)]} x \xrightarrow{[\cdot_x b(z,z)]} \top \xrightarrow{[\cdot_z c]} \top$$

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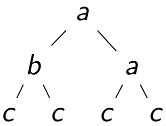


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**Remark** we should also consider $a(c, *)$ and $a(*, c)$.

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|---|---|---|---|---|---|---|---|---|---|---|
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$T([x][\cdot_x a(x,x)][\cdot_x c]) \in K$ and $T([x][\cdot_x a(x,y)][\cdot_x c]) \notin K$ even though $\delta(q_0,[x][\cdot_x a(x,x)][\cdot_x c]) = \delta(q_0,[x][\cdot_x a(x,y)][\cdot_x c])$.

# Minimizing coding automata

**Myhill-Nerode relation** of a tree language $L$. Let $c, c' \in C_{\mathcal{V}} C^*_{\mathcal{V},\Sigma}$ be tree codings. $c \equiv_L c'$ if

$$T(cv) \in L \Leftrightarrow T(c'v) \in L \text{ for all } v \in C^*_{\mathcal{V},\Sigma} \text{ such that } T(cv) \in T_\emptyset \text{ and } T(c'v) \in T_\emptyset.$$

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**The minimal automaton** $\mathrm{Min}_L$ of $L$ is defined as follows:

- the set of states is $Q = \{q_0\} \uplus C_{\mathcal{V}} C^*_{\mathcal{V},\Sigma} / \equiv_L$,
- $[c]_{\equiv_L}$ is accepting if $[c]_{\equiv_L} \subseteq L$,
- $\delta(q_0, [x]) = [\ [x]\ ]_{\equiv_L}, \qquad \delta([c]_{\equiv_L}, v) = [cv]_{\equiv_L}$.

# Minimizing coding automata

**Myhill-Nerode relation** of a tree language $L$. Let $c, c' \in C_{\mathcal{V}} C^*_{\mathcal{V},\Sigma}$ be tree codings. $c \equiv_L c'$ if

$$T(cv) \in L \Leftrightarrow T(c'v) \in L \text{ for all } v \in C^*_{\mathcal{V},\Sigma} \text{ such that } T(cv) \in T_\emptyset \text{ and } T(c'v) \in T_\emptyset.$$

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## Minimal automaton

For $L$ a tree language described by a coding automaton, $\mathrm{Min}_L$ is a coding automaton which describes $L$.

# Reminder

## Equivalence theorem

For a regular language of finite trees, the following properties are equivalent:

a. Being recognized by a finite tree algebra of polynomial complexity.

b. Being recognized by a finite tree algebra of bounded orbit complexity.

c. Being described by a coding automaton.

Let us prove c. $\Rightarrow$ a. and b.

# From coding automata to tree algebras 1/2

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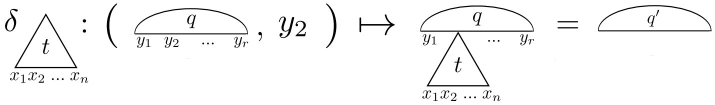
Fix a tree $t$ with variables $x_1, \ldots, x_n$, we define a function $\delta_t$ as

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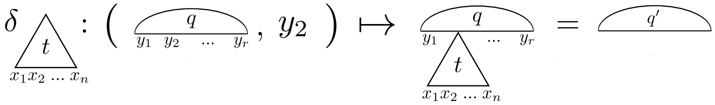


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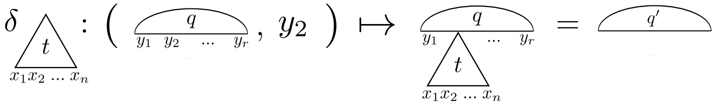


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## $\delta_t$ is well defined

The definition of $\delta_t$ does not depend on a particular choice of coding.
Let $\mathrm{Trans}(\mathrm{Min}_L)$ be the set of all functions $\delta_t$.

# From coding automata to tree algebras 2/2

We define the tree as algebra $\mathcal{A}$ as

$$A_X = \{\delta_t \in \mathrm{Trans}(\mathrm{Min}_L) \mid \delta_t \text{ is supported by } X\} \ .$$

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**Support of $\delta_t$**

The size of the supports of the $\delta_t$'s is bounded by an integer $K$.

Let $A$ and $B$ be orbit-finite nominal sets. The set of all functions from $A$ to $B$ with support of size at most $K$ is orbit-finite.

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$\mathcal{A}$ **has bounded orbit complexity.** $\mathrm{Trans}(\mathrm{Min}_L)$ has finitely many orbits. $f, g \in A_X$ are on the same **Sym**$(X)$-orbit if and only if they are on the same **Sym**$(\mathcal{V})$-orbit.

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$\mathcal{A}$ **has bounded orbit complexity.** $\mathrm{Trans}(\mathrm{Min}_L)$ has finitely many orbits. $f, g \in A_X$ are on the same $\mathbf{Sym}(X)$-orbit if and only if they are on the same $\mathbf{Sym}(\mathcal{V})$-orbit.

$\mathcal{A}$ **has polynomial complexity.** $A_X$ has boundedly many orbits. On any orbit, there are at most $\frac{|X|!}{(|X|-k)!}$ elements under the action of $\mathbf{Sym}(X)$.

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3. Thus, only the elements in sorts $A_X$ where $|X| \leq K$ matter. Let

$$Q = \bigcup_{|X| \leq K} A_X \ .$$

This is used to define a coding automaton that describes $L$.

# Decidability

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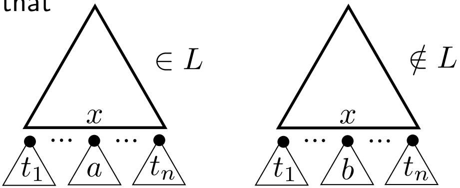
Fix $L$. A tree $t \in T_{\{\bullet\}}$ is $L$-sensitive to a leaf $x$ if there exist trees $a, b, t_1, \ldots, t_n$ such that



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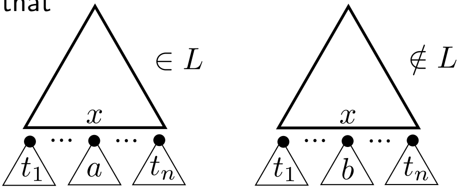


## Lemma

A regular language of trees $L$ is described by a coding automaton if and only if there is a bound on the number of $L$-sensitive leaves in trees.

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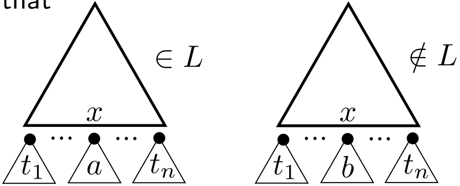


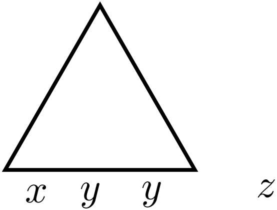

## Lemma

A regular language of trees $L$ is described by a coding automaton if and only if there is a bound on the number of $L$-sensitive leaves in trees.

The existence of such a bound can be encoded into cost-MSO. Thus, it is decidable.

# Different types of tree algebras

Unrestrained tree algebras



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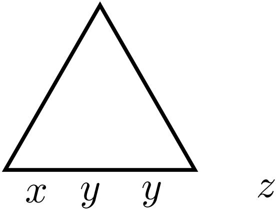
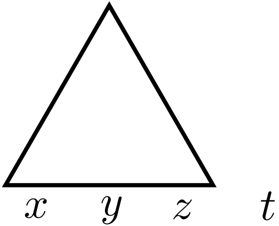



Sublinear tree algebras



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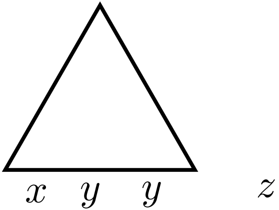


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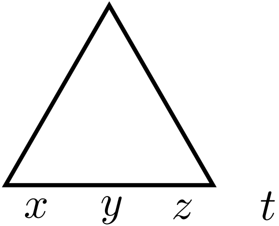


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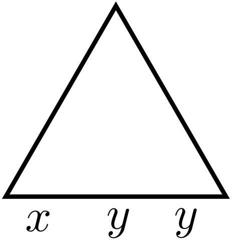

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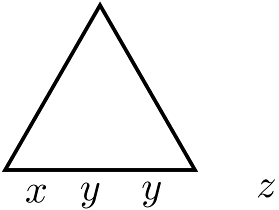


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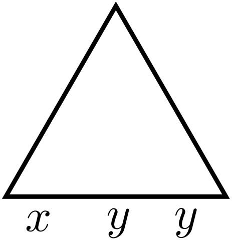


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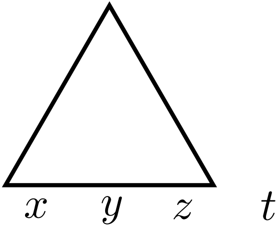


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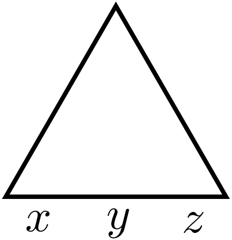

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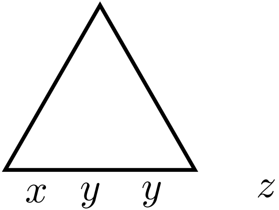


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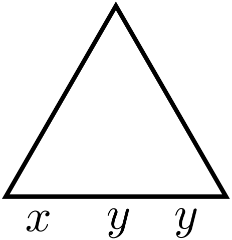


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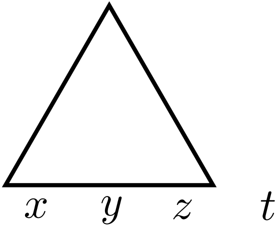


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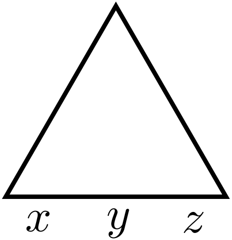

# Conclusion

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