A Complexity Approach to Tree Algebras: the Bounded Case

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joint work with Thomas Colcombet

IRIF, CNRS, Université de Paris

Automata Seminar | April 16, 2021

Algebras and classes of languages

Algebras are used to characterize classes of languages

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Finite words

Monoids, semigroups

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Objective: characterize classes that can be naturally defined using infinitely sorted algebras

Infinitely sorted tree algebras: FT_Σ-algebras

Let Σ be a ranked alphabet. The free FT_Σ-algebra has as carrier $(T_X)_{X \text{ finite}}$ where the X's are finite sets of variables.

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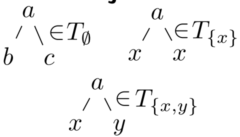

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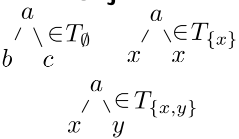

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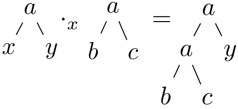

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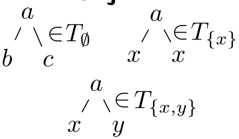

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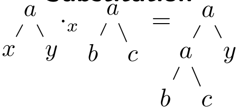
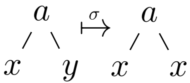


Renaming

$$\sigma(x) = \sigma(y) = x$$



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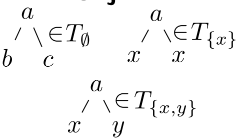

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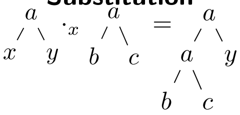

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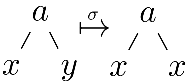

Definition (Finite Tree algebras)

A finite FT_Σ-algebra $\mathcal{A}$ consists of an infinite series of finite carrier sets A_X indexed by finite sets of variables X, together with operations:

Constants. $a(x_0, \ldots, x_{n-1})^{\mathcal{A}} \in A_{\{x_0,\ldots,x_{n-1}\}}$ for all $a \in \Sigma_n$ and variables x_i,

Substitution. $\cdot_x^{\mathcal{A}} \colon A_X \times A_Y \to A_{X\setminus\{x\}\cup Y}$ for all finite X, Y and $x \in X$,

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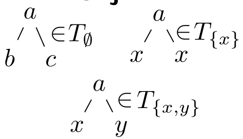

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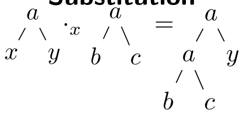

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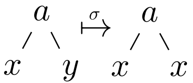

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Identities? $\quad a(x,y) \cdot_y b \qquad a(x,z) \cdot_z b$

We also define morphisms, congruences...

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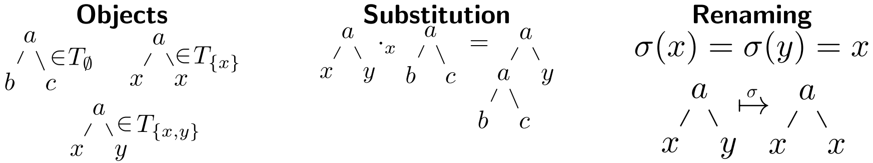

Objects
$a(b,c) \in T_\emptyset$
$a(x,x) \in T_{\{x\}}$
$a(x,y) \in T_{\{x,y\}}$
Substitution
$a(x,y) \cdot_x a(b,c) = a(a(b,c),y)$
Renaming
$\sigma(x) = \sigma(y) = x$
$a(x,y) \overset{\sigma}{\mapsto} a(x,x)$

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Given a finite FT_Σ-algebra $\mathcal{A}$, there is a unique morphism from the free algebra to $\mathcal{A}$. It is called the evaluation morphism of $\mathcal{A}$.

Languages and complexity

Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite FT_Σ-algebra $\mathcal{A}$ if there is a set $P \subseteq A_\emptyset$ such that $L = \alpha^{-1}(P)$ in which α is the evaluation morphism of $\mathcal{A}$.

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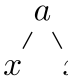

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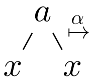

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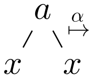

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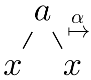

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Long term objective

Characterize the languages recognized by algebras of

- Bounded complexity **(This talk)**
- Polynomial complexity
- Exponential complexity
- ...

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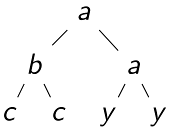
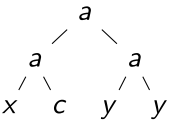


$\overset{\alpha}{\mapsto} \{a, b, c\}$



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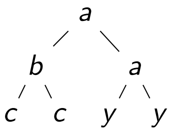

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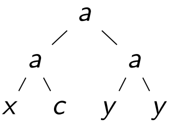

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$$A_X = 2^{\Sigma} \uplus (2^{\Sigma} \times X)$$

$$\mathrm{lb}(t) = \{a \in \Sigma \mid a \text{ occurs in the leftmost branch of } t\}$$

$$\alpha(t) = \begin{cases} \mathrm{lb}(t) & \text{if there is no variable on the leftmost branch of } t \\ (\mathrm{lb}(t), x) & \text{if } x \text{ is the variable on the leftmost branch of } t \end{cases}$$

This algebra has linear complexity.

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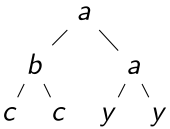

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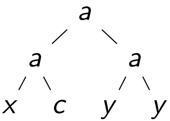

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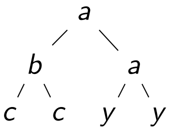

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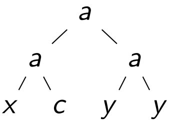

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Better algebra: $A_X = X \uplus \{\bot, \top\}$(it is the syntactic algebra of L)

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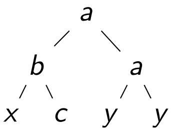

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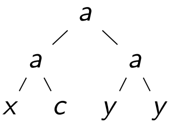

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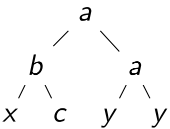

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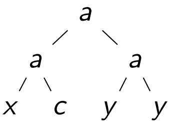

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$$\mathrm{vw}_b(t) = \{x \in X \mid x \text{ occurs on a branch that has no } b\text{'s}\}$$

$$\alpha(t) = \begin{cases} \bot & \text{if there is branch without a } b \text{ that ends with a constant} \\ \mathrm{vw}_b(t) & \text{otherwise} \end{cases}$$

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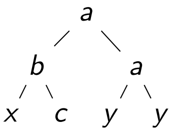

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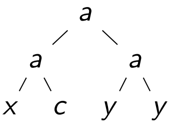

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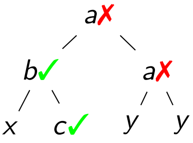

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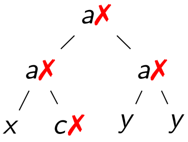

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Example: The language of all trees with at least a b on every branch

$L =$ trees with at least a b on every branch

Languages recognized by top-down deterministic automata

All languages recognized by top-down deterministic automata are recognized by FT_Σ-algebras of exponential complexity.

$$A_X = 2^X \uplus \{\bot\}$$

$$\mathrm{vw}_b(t) = \{x \in X \mid x \text{ occurs on a branch that has no } b\text{'s}\}$$

$$\alpha(t) = \begin{cases} \bot & \text{if there is branch without a } b \text{ that ends with a constant} \\ \mathrm{vw}_b(t) & \text{otherwise} \end{cases}$$

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Regular languages

A top-down nondeterministic automaton can be transformed into a FT_Σ-algebras of doubly-exponential complexity that recognizes the same language.

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This algebra has exponential complexity and is syntactic for L

Main result

Characterization theorem

A language of finite trees is recognized by an FT_Σ-algebra of bounded complexity if and only if it is a Boolean combination of languages of the following kinds:

a. The language of finite trees with unary prefix in a given regular language of words $L \subseteq \Sigma_1^*$.

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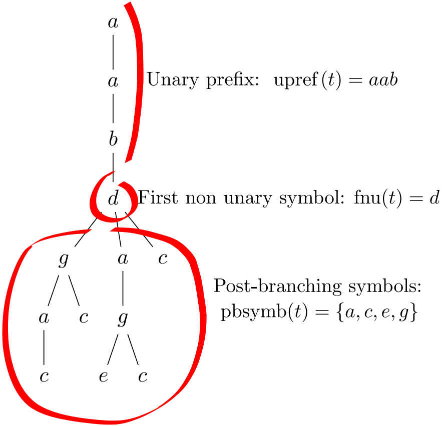
d. A regular language K of bounded branching.


a
a
b
d
g
a
c
a
c
g
c
e
c
Unary prefix: upref $(t) = aab$
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pbsymb$(t) = \{a, c, e, g\}$

Bounded branching: $\exists k$ all trees in K have at most k branches

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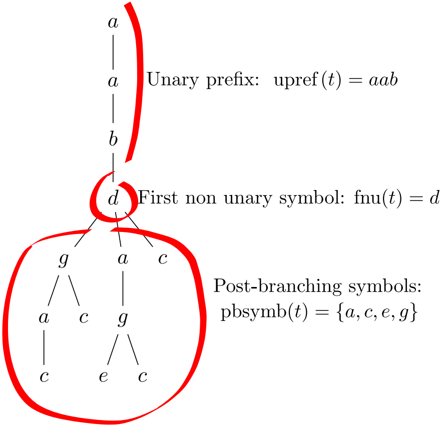
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c
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c
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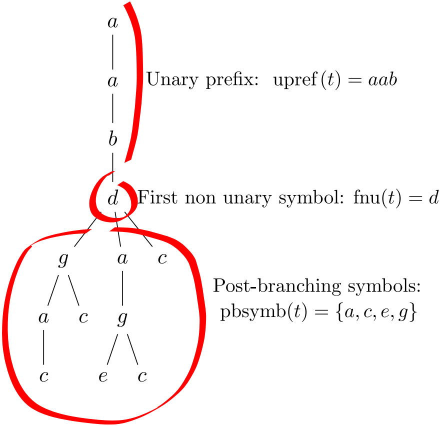
Easy direction: any Boolean combination of a.-d. is recognized by an FT_Σ-algebra of bounded complexity.

Easy direction of the characterization theorem

Lemma

The languages $\mathrm{UPref}(L)$, $\mathrm{FNU}(b)$ and $\mathrm{PBSymb}(B)$ are recognized by algebras of bounded complexity for all $b \in \Sigma_{\neq 1}$, $B \subseteq \Sigma$ and $L \subseteq \Sigma_1^*$ that is regular.



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Let $\varphi\colon \Sigma_1^* \to M$ recognize L.

$$A_X = M \times 2^{\Sigma_1} \times \Sigma_{\neq 1} \times 2^{\Sigma}$$

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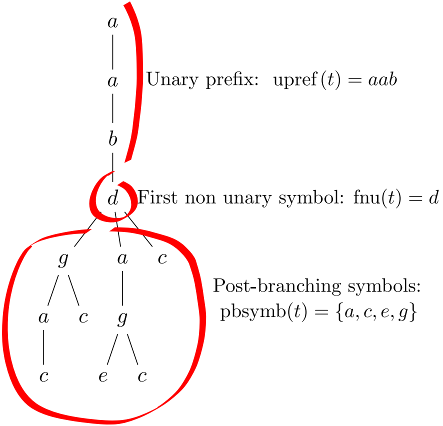

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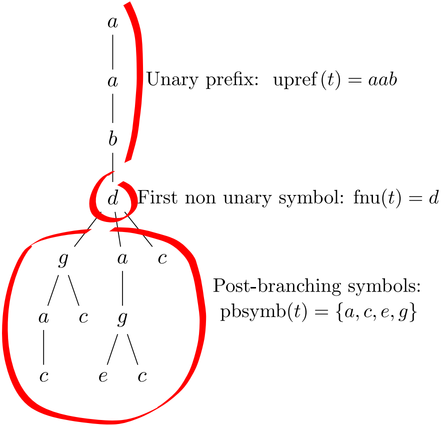

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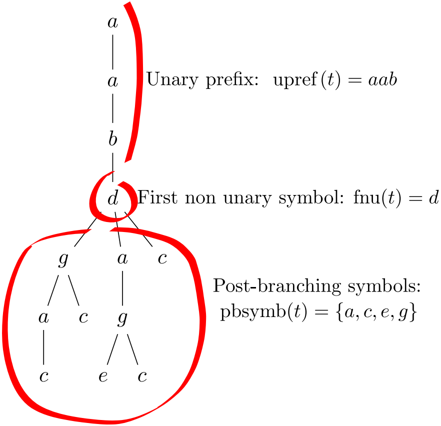

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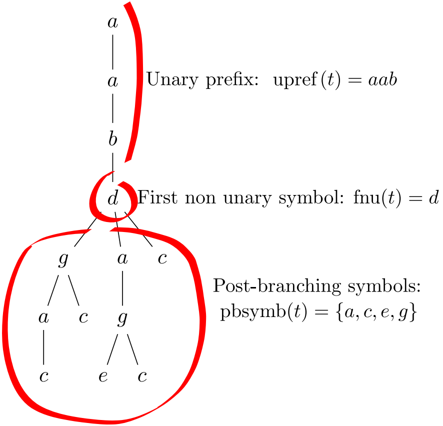

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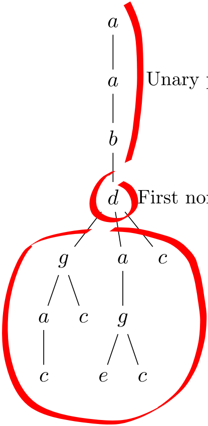

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Lemma

A regular language K of bounded branching is recognized by an algebra of bounded complexity.

Let $\mathcal{A}$ recognize K. Let k be such that trees with more than k branches never belong to K.

$$\begin{array}{cccccc}
\mathcal{A} & A_\emptyset & A_{\{x_0\}} & \dots & A_{\{x_0,\dots,x_k\}} & \dots \\
\mathcal{A}' & A_\emptyset \times \{1, \dots, k-1, \bot\} & A_{\{x_0\}} \times \{1, \dots, k-1, \bot\} & \dots & \{\bot\} & \dots
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Lemma

A Boolean combination of FT_Σ-algebras of bounded complexity has bounded complexity.

Structure of the proof of the hard direction

Characterization theorem

A language of finite trees is recognized by an FT_Σ-algebra of bounded complexity if and only if it is a Boolean combination of languages of the following kinds:

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Invariance under permutations

Consider for all X the group morphism induced by renaming

$$\begin{aligned} \varphi_X \colon \mathbf{Sym}(X) &\to \mathbf{Sym}(A_X) \\ \sigma &\mapsto \mathrm{rename}^{\mathcal{A}}[\sigma] \end{aligned}$$

Lemma 1 (Invariance under permutations)

A finite syntactic FT_Σ-algebra is of bounded complexity if and only if for all sufficiently large finite set of variables X, $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$.

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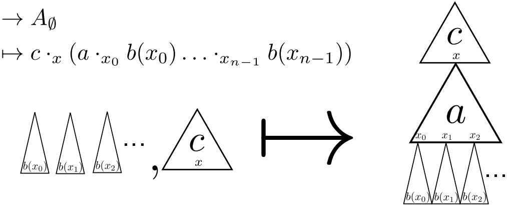
Lemma 1b: In a syntactic algebra of bounded complexity, $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ for large X.

Syntactic FT_Σ-algebras

Let $\mathcal{A}$ be some FT_Σ-algebra and let $X = \{x_0, ..., x_{n-1}\}$ be a finite set of variables. Define for all $a \in A_X$

$$\begin{aligned}\langle a\rangle \colon (A_\emptyset)^X \times A_{\{x\}} &\rightarrow A_\emptyset \\ (b, c) &\mapsto c \cdot_x (a \cdot_{x_0} b(x_0) \ldots \cdot_{x_{n-1}} b(x_{n-1}))\end{aligned}$$



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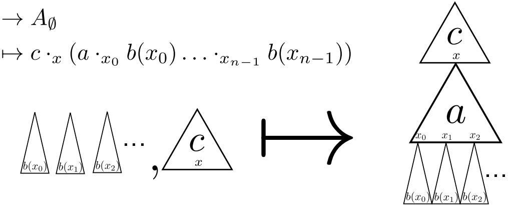

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If $\mathcal{A}$ is a *syntactic algebra* then $a = b$ iff $\langle a \rangle = \langle b \rangle$, for all a, b in $\mathcal{A}$.

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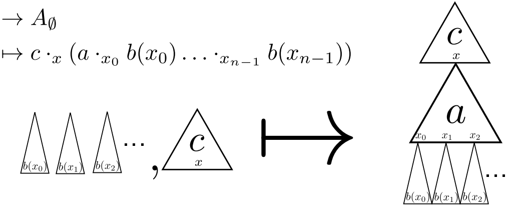

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Corollary: A *syntactic algebra* is of complexity at most doubly-exponential:

$$|A_X| \leq |A_\emptyset|^{|A_{\{x\}}||A_\emptyset|^{|X|}}$$

Lemmas 1a and 1b

Lemma 1a

In a syntactic algebra $\mathcal{A}$, there is an integer M such that for all X of cardinal at least M, either $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ or $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$.

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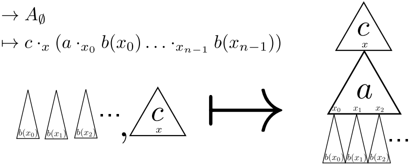

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$M = \max(5, |A_\emptyset| + 1)$

Suppose for the sake of contradiction that $|X| \geq M$ and $\mathrm{Ker}(\varphi_X) = \mathbf{Alt}(X)$

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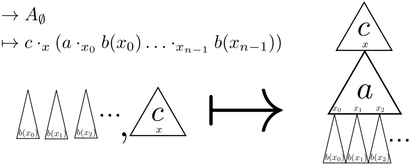

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Prove $\mathrm{rename}^{\mathcal{A}}[t] = \mathrm{id}_{A_X}$ by showing $\langle \mathrm{rename}^{\mathcal{A}}[t](a)\rangle = \langle a \rangle$ for all $a \in A_X$.

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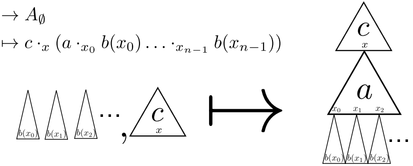

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Fix $a \in A_X$

$$c \in A_{\{x\}}, b \in (A_\emptyset)^X$$

$$x \neq y \text{ with } b(x) = b(y)$$

$$\begin{aligned}\langle \mathrm{rename}^{\mathcal{A}}[t](a) \rangle (b, c) &= \langle \tau(a) \rangle (b, c) \\ &= \langle \mathrm{rename}^{\mathcal{A}}[(x\ y)](a) \rangle (b, c) \\ &= \langle a \rangle (b, c)\end{aligned}$$

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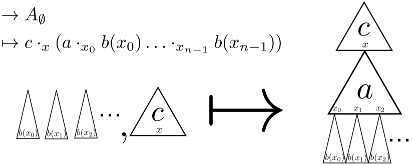

Lemmas 1a and 1b

Lemma 1a

In a syntactic algebra $\mathcal{A}$, there is an integer M such that for all X of cardinal at least M, either $\mathrm{Ker}(\varphi_X) = \textbf{Sym}(X)$ or $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$.

Lemma 1b

In a syntactic algebra of bounded complexity, $\mathrm{Ker}(\varphi_X) = \textbf{Sym}(X)$ whenever X is large enough.

Suppose $|A_X| \leq k$ for all X and $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$

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Suppose $|A_X| \leq k$ for all X and $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$

$$|X|! = |\mathrm{Im}(\varphi_X)| \leq |\mathbf{Sym}(A_X)| = |A_X|! \leq k!$$

Invariance under permutations

Consider for all X the group morphism induced by renaming

$$\begin{aligned}\varphi_X \colon \mathbf{Sym}(X) &\to \mathbf{Sym}(A_X)\\ \sigma &\mapsto \mathrm{rename}^{\mathcal{A}}[\sigma]\end{aligned}$$

Lemma 1 (Invariance under permutations)

A finite syntactic FT_Σ-algebra is of bounded complexity if and only if for all sufficiently large finite set of variables X, $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$.

Lemma 1a: In a syntactic algebra, $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$ or $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ for large X.

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Lemma 1c: A syntactic algebra in which $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ for every sufficiently large X is of bounded complexity.

Lemma 1c

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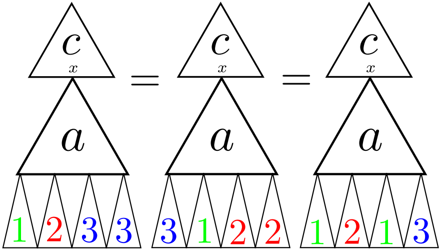
Suppose that $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ whenever $|X| \in \{n, n-1\}$. Then for all $a \in A_X$ with $|X| = n$, and all $b, b' \in (A_\emptyset)^X$, $c \in A_{\{x\}}$

$$\langle a \rangle(b, c) = \langle a \rangle(b', c)$$

whenever $\mathrm{Im}(b) = \mathrm{Im}(b')$.

$$\langle a \rangle \colon (A_\emptyset)^X \times A_{\{x\}} \to A_\emptyset$$



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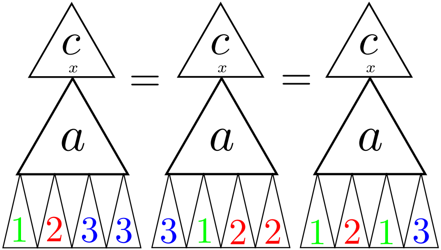

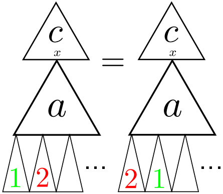

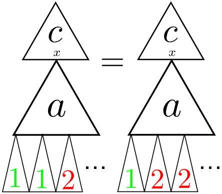

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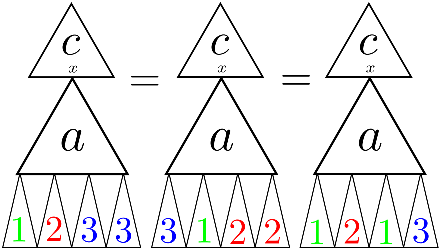

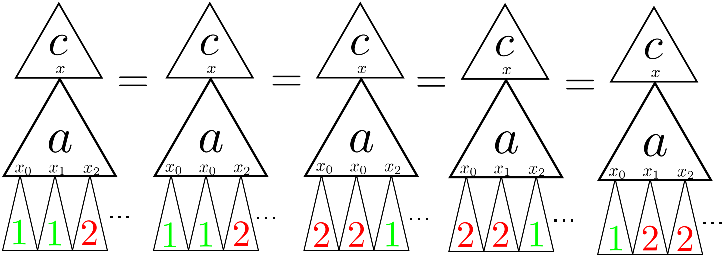

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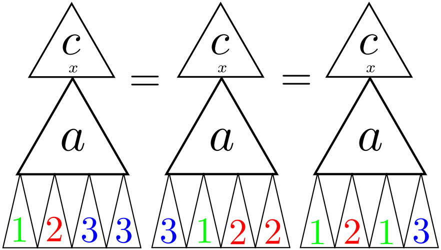

Lemma 1c

A finite syntactic algebra such that $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ for all sufficiently large set of variables X has bounded complexity.

For all a, $\langle a \rangle$ must be chosen in a set of at most $|A_\emptyset|^{|A_{\{x\}}|2^{|A_\emptyset|}}$ functions.
Lemma: for all a, b, $a = b$ if and only if $\langle a \rangle = \langle b \rangle$.

Structure of the proof of the hard direction

Characterization theorem

A language of finite trees is recognized by an FT_Σ-algebra of bounded complexity if and only if it is a Boolean combination of languages of the following kinds:

a. The language of finite trees with unary prefix in a given regular language of words $L \subseteq \Sigma_1^*$.

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1. In syntactic algebras of bounded complexity, the elements of A_X are invariant under permutations for large X.

   The converse is also true.

2. For all finite trees s and t with sufficiently many branches, if $\mathrm{upref}(s) = \mathrm{upref}(t)$, $\mathrm{fnu}(s) = \mathrm{fnu}(t)$ and $\mathrm{pbsymb}(s) = \mathrm{pbsymb}(t)$ then $\mathcal{A}$ does not distinguish between s and t.

3. A language recognized by an algebra of bounded complexity is a Boolean combination of a.-d.

Lemma 2 (Trees with many branches) 1/2

Fix a syntactic FT_Σ-algebra $\mathcal{A}$ of bounded complexity, with evaluation morphism α. Write $s \simeq_{\mathcal{A}} t$ if $\alpha(s) = \alpha(t)$.

Permutation lemma

If a tree $t(x, y)$ has sufficiently many branches then, for all trees t_1, t_2,

$$t(t_1, t_2) \simeq_{\mathcal{A}} t(t_2, t_1)$$

Duplication lemma

If a tree $t(x, y, z)$ has sufficiently many branches then, for all trees t_1, t_2,

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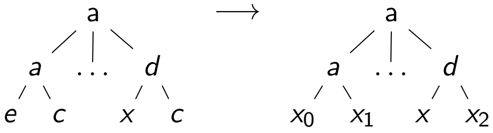

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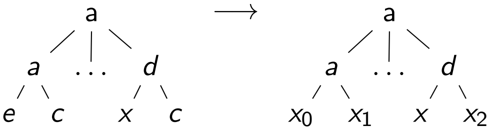

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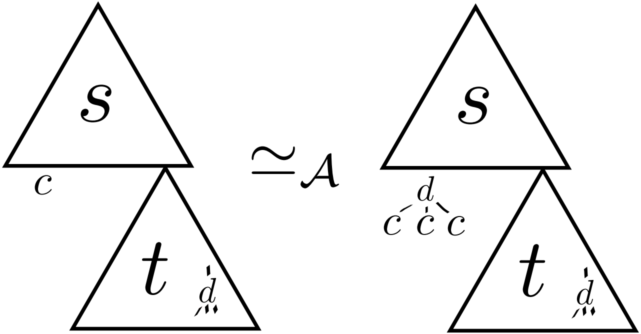

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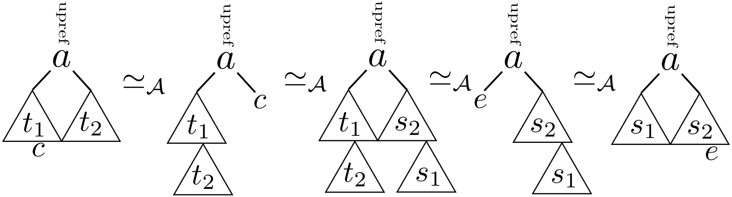

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Ongoing: polynomial complexity, bounded orbit complexity...

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