A Complexity Approach to Tree Algebras: the Polynomial Case

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joint work with Thomas Colcombet

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INSTITUT
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Infinitely sorted tree algebras

Let Σ be a ranked alphabet and $\mathcal{V}$ be a countably infinite set of variables. The free tree algebra has as carrier sets the $(T_X)_{X \subseteq \mathcal{V} \text{ finite}}$.

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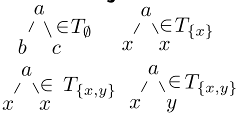

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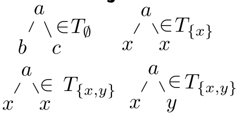

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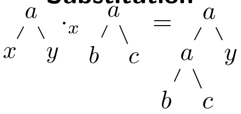

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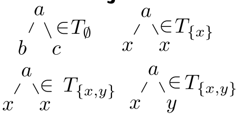

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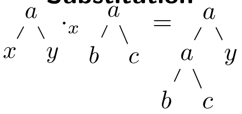

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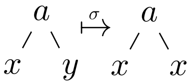

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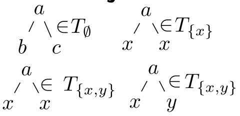

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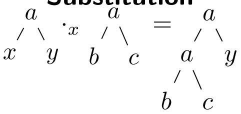

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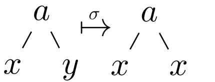

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- **Substitution.** $\cdot_x^{\mathcal{A}} \colon A_X \times A_Y \to A_{X\setminus\{x\}\cup Y}$ for all finite X, Y and variable x,
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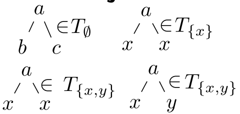

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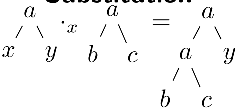

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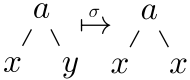

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Identities? $a(x, y) \cdot_y b$ $a(x, z) \cdot_z b$

We also define morphisms, congruences...

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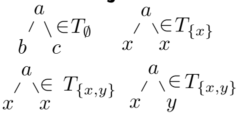

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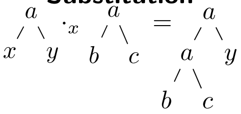

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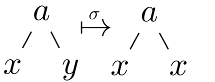

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Given a finite tree algebra $\mathcal{A}$, there is a unique morphism from the free algebra to $\mathcal{A}$. It is called the evaluation morphism of $\mathcal{A}$.

Languages and the size of the algebra

Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite algebra $\mathcal{A}$ if there is a set $P \subseteq A_\emptyset$ such that $L = \alpha^{-1}(P)$ in which α is the evaluation morphism of $\mathcal{A}$.

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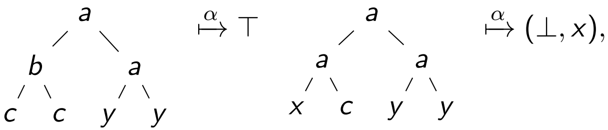

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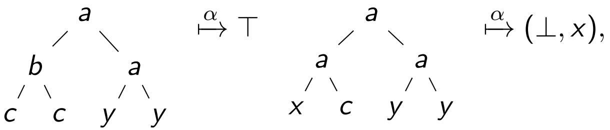

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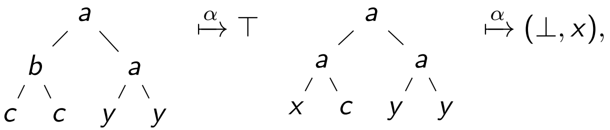

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This algebra has linear complexity.

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The objective is to identify new classes of languages and to gain a better understanding of tree algebras.

Another example

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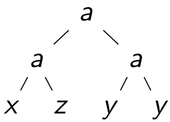

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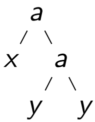

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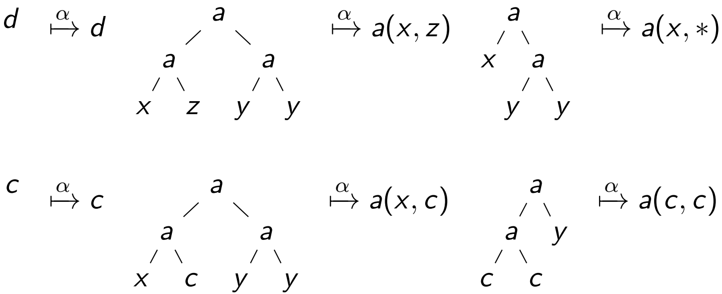

Orbits: c, d, $a(x,y)$, $a(x,x)$, $a(x,c)$, $a(c,x)$, $a(x,*)$, $a(*,x)$, $a(c,c)$, $a(*,*)$

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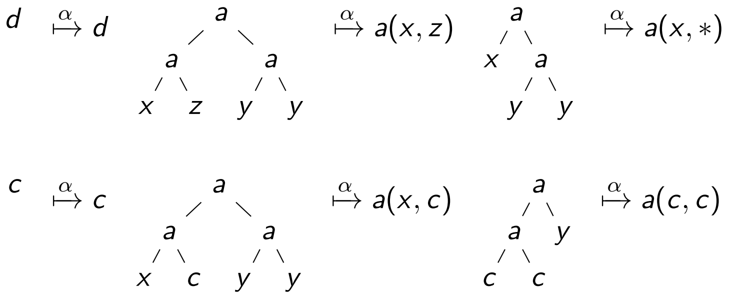

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This algebra has quadratic complexity and bounded orbit complexity.

Orbit complexity

Let $|A_X/\mathbf{Sym}(X)|$ be the number of orbits of A_X under the action of $\mathbf{Sym}(X)$ induced by renamings.

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Another hierarchy of classes:

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Polynomial orbit complexity	-
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a. Being recognized by a finite tree algebra of polynomial complexity.
b. Being recognized by a finite tree algebra of bounded orbit complexity.
c. Being described by a coding automaton.

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- X (resp. f) is called equivariant if it is supported by the empty set.

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- X is called nominal if its elements are finitely supported.
- $f : X \to Y$ is supported by $S \subseteq \mathcal{V}$ if $f(\sigma(x)) = \sigma(f(x))$, for all $x \in X$, $\sigma \in \mathbf{Sym}(\mathcal{V} \setminus S)$.
- X (resp. f) is called equivariant if it is supported by the empty set.

A deterministic orbit-finite nominal automaton is given by

- an orbit-finite nominal set A (the alphabet),
- an orbit-finite nominal set Q (the states),
- equivariant subsets $\{q_I\}$ and F of Q (the initial state and the final states),
- and an equivariant transition function $\delta : Q \times A \to Q$.

Nominal automata

Let $\mathbf{Sym}(\mathcal{V})$ act upon sets X and Y.

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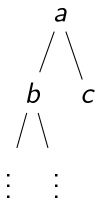
Example: a deterministic register automaton can be seen as a deterministic orbit-finite nominal automaton.

Coding of trees

How to build the following tree ?



Coding of trees

How to build the following tree ?

x $[x]$

Coding of trees

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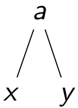

$[x]$

$[\cdot_x a(x, y)]$

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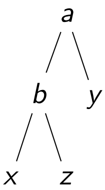

$[x]$

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$[\cdot_x b(x, z)]$

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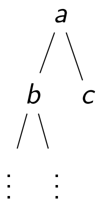

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$\dots$

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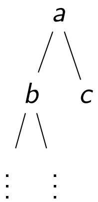

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$$C_{\mathcal{V}} = \{[x] \mid x \in \mathcal{V}\}$$

$$C_{\mathcal{V},\Sigma} = \{[\cdot_x a(x_0, \ldots, x_{n-1})] \mid a \in \Sigma_n, x, x_0, \ldots, x_{n-1} \in \mathcal{V}\}$$

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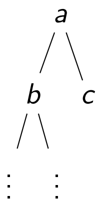

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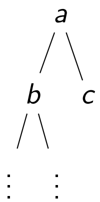

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Coding languages describing tree languages

A language L of codings describes a language $K \subseteq T_\emptyset$ of trees if, for every coding c such that $T(c) \in T_\emptyset$, $c \in L$ if and only if $T(c) \in K$.

Dealing with missing variables

Let $c = [x][\cdot_x a(x, y)][\cdot_z c]$. What is $T(c)$?

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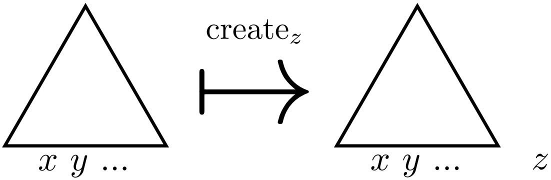

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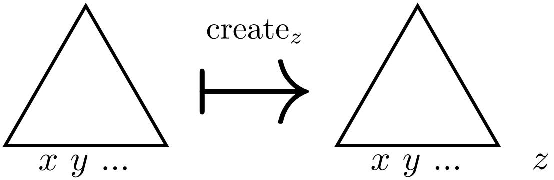

$$T(c) = \mathrm{create}_z(a(x,y)) \cdot_z c = a(x,y)$$

Coding automata

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We assume that there is no transition toward the initial state q_0.

Language described by a coding automaton

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q_0	x	$\bot$	$a\{x, y\}$	$a\{x\}$	$a\{x\}$	$a\{x\}$	$\bot$	$\top$	$\top$	$\top$

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A state is an abstraction of a tree, that possibly *forgot* some variables.

Minimizing coding automata

Myhill-Nerode relation of a tree language L. Let $c, c' \in C_{\mathcal{V}} C^*_{\mathcal{V},\Sigma}$ be tree codings. $c \equiv_L c'$ if

$$T(cv) \in L \Leftrightarrow T(c'v) \in L \text{ for all } v \in C^*_{\mathcal{V},\Sigma} \text{ such that}$$
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The minimal automaton Min_L of L is defined as follows:

- the set of states is $Q = \{q_0\} \uplus C_{\mathcal{V}} C^*_{\mathcal{V},\Sigma} / \equiv_L$,
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Minimal automaton

For L a tree language described by a coding automaton, Min_L is a coding automaton which describes L.

From coding automata to tree algebras 1/2

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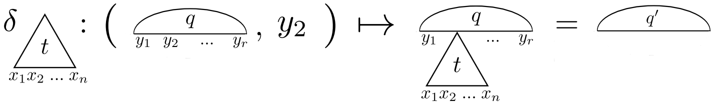

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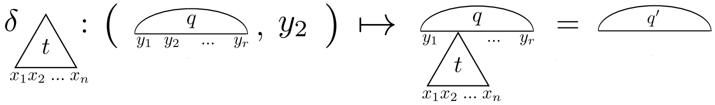

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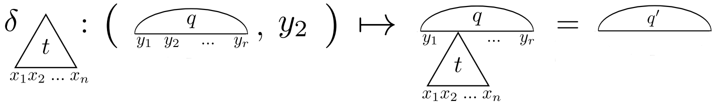

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δ_t is well defined

The definition of δ_t does not depend on a particular choice of coding.
Let $\mathrm{Trans}(\mathrm{Min}_L)$ be the set of all functions δ_t.

From coding automata to tree algebras 2/2

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$\mathcal{A}$ **has polynomial complexity.** A_X has boundedly many orbits. On any orbit, there are at most $\frac{|X|!}{(|X|-k)!}$ elements under the action of $\mathbf{Sym}(X)$.

From tree algebras to coding automata

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Every language of trees L recognized by a tree algebra of polynomial complexity or of bounded orbit complexity is described by a coding automaton.

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2. Prove that tree algebras of polynomial complexity or bounded orbit complexity have supports of bounded size (say K).
3. Thus, only the elements in sorts A_X where $|X| \leq K$ matter. Let

$$Q = \bigcup_{|X| \leq K} A_X \ .$$

This is used to define a coding automaton that describes L.

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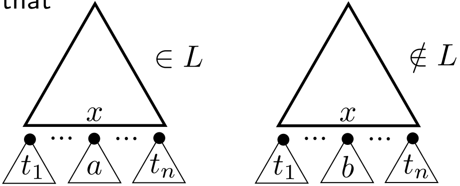
Fix L. A tree $t \in T_{\{\bullet\}}$ is L-sensitive to a leaf x if there exist trees $a, b, t_1, \ldots, t_n$ such that



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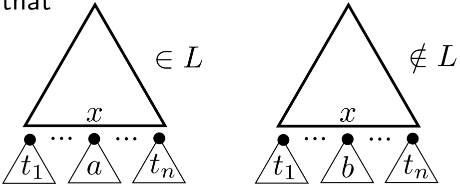

Lemma

A regular language of trees L is described by a coding automaton if and only if there is a bound on the number of L-sensitive leaves in trees.

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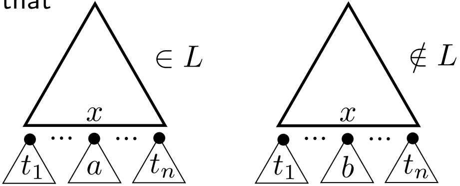

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The existence of such a bound can be encoded into cost-MSO. Thus, it is decidable.

From a coding automaton to a bound

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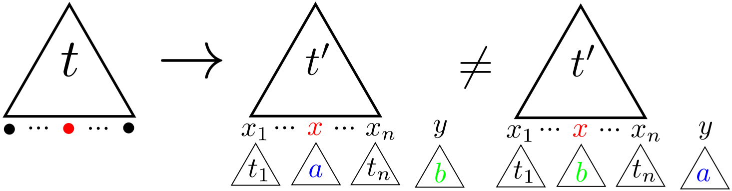
If L is described by a coding automaton, then there is a bound on the number of L-sensitive leaves in trees.

Suppose $t \in T_{\{\bullet\}}$ is L-sensitive to some leaf (and let $a, b, t_1, \ldots, t_n$ be the associated trees).



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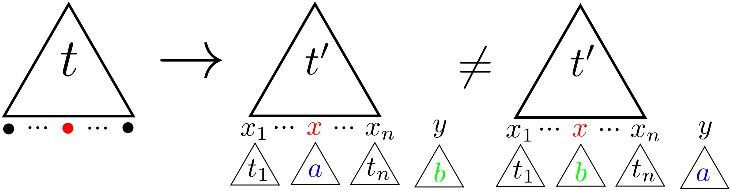

Let c be a coding such that $T(c) = t'$.

$$t'[\cdot_x a][\cdot_y b] \neq ((x\ y)t')[\cdot_x a][\cdot_y b]$$

Then x is in the support of $\delta(q_0, c)$ in Min_L.

From a bound to a coding automaton 1/2

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Let $\mathcal{A} = (Q, q_0, F, \delta)$ be the minimal tree automaton for L, and suppose K is a bound on the number of L-sensitive leaves.

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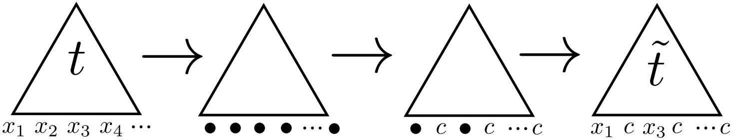

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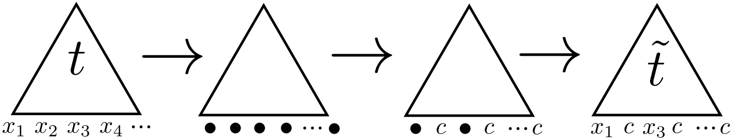

The reduct of a tree has at most K different variables.

From a bound to a coding automaton 2/2

Let s and t be Σ, X-trees. Suppose $p_{\tilde{t}} = p_{\tilde{s}}$, then

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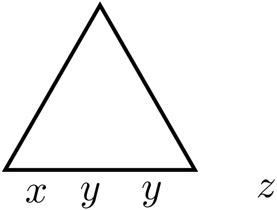
Different types of tree algebras

Unrestrained tree algebras



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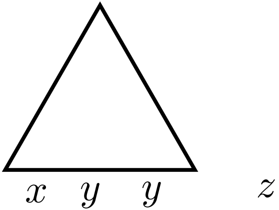

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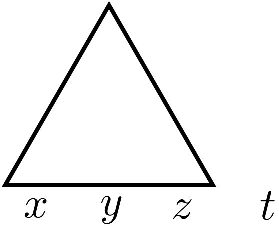

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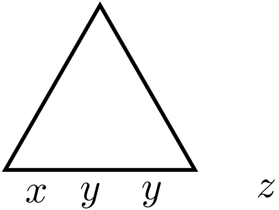

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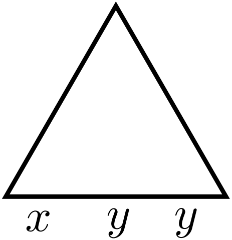

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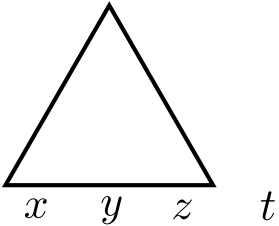

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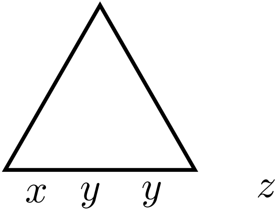

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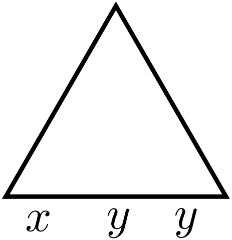

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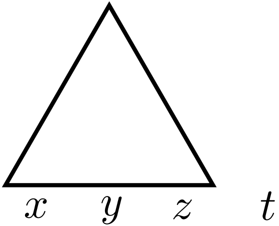

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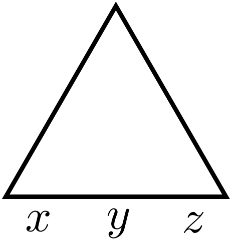

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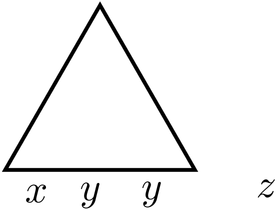

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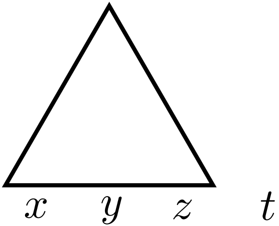

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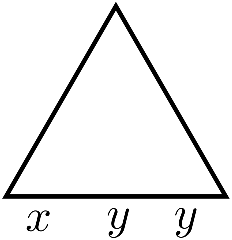

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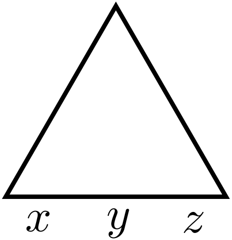

Conclusion

Equivalence theorem

For a regular language of finite trees, the following properties are equivalent:

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Future work. Tree algebras of exponential complexity.