A Complexity Approach to Tree Algebras

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joint work with Thomas Colcombet

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June 28, 2021

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Finite words

Monoids, semigroups

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Objective: characterize classes that can be naturally defined using infinitely sorted algebras

Infinitely sorted tree algebras: FT_Σ-algebras

Let Σ be a ranked alphabet. The free FT_Σ-algebra has as carrier $(T_X)_{X \text{ finite}}$ where the X's are finite sets of variables.

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$$\sigma(x) = \sigma(y) = x$$

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Definition (Finite Tree algebras)

A finite FT_Σ-algebra $\mathcal{A}$ consists of an infinite series of finite carrier sets A_X indexed by finite sets of variables X, together with operations:

Constants. $a(x_0, \ldots, x_{n-1})^{\mathcal{A}} \in A_{\{x_0,\ldots,x_{n-1}\}}$ for all $a \in \Sigma_n$ and variables x_i,

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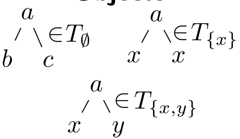

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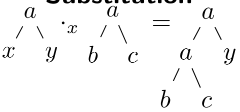

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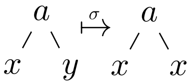

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Identities? $a(x,y) \cdot_y b$ $\quad a(x,z) \cdot_z b$

We also define morphisms, congruences...

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Given a finite FT_Σ-algebra $\mathcal{A}$, there is a unique morphism from the free algebra to $\mathcal{A}$. It is called the evaluation morphism of $\mathcal{A}$.

Languages and the size of the algebra

Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite FT_Σ-algebra $\mathcal{A}$ if there is a set $P \subseteq A_\emptyset$ such that $L = \alpha^{-1}(P)$ in which α is the evaluation morphism of $\mathcal{A}$.

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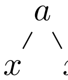
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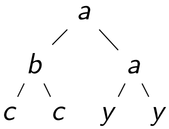
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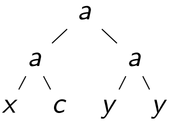
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$$A_X = 2^\Sigma \uplus (2^\Sigma \times X)$$

$|A_X| = 2^{|\Sigma|} + 2^{|\Sigma|}|X|$ is linear in $|X|$.

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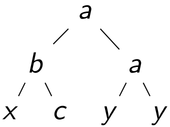

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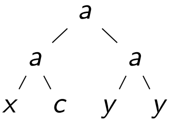

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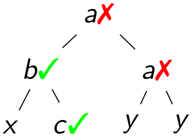

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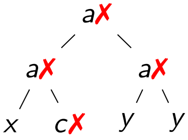

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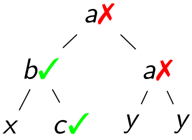

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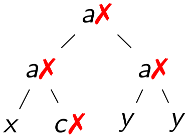

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Languages recognized by top-down deterministic automata

All languages recognized by top-down deterministic automata are recognized by FT_Σ-algebras of exponential complexity.

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Doubly exponential complexity

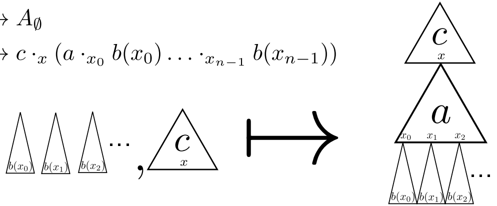
Regular languages

A top-down nondeterministic automaton can be transformed into a FT_Σ-algebras of doubly-exponential complexity that recognizes the same language.
Conversely, any language recognized by a finite FT_Σ-algebra is regular.

Syntactic algebras

Let $\mathcal{A}$ be some FT_Σ-algebra and let $X = \{x_0, ..., x_{n-1}\}$ be a finite set of variables. Define for all $a \in A_X$

$$\begin{aligned}\langle a\rangle \colon (A_\emptyset)^X \times A_{\{x\}} &\to A_\emptyset \\ (b, c) &\mapsto c \cdot_x (a \cdot_{x_0} b(x_0) \ldots \cdot_{x_{n-1}} b(x_{n-1}))\end{aligned}$$



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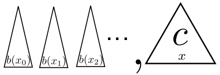

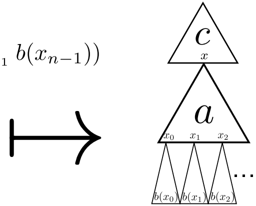

Lemma

If $\mathcal{A}$ is a *syntactic algebra* then $a = b$ iff $\langle a \rangle = \langle b \rangle$, for all a, b in $\mathcal{A}$.

Characterizing bounded complexity

What are the languages recognized by FT_{Σ}-algebras of bounded complexity?

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Consider for all X the group morphism induced by renaming

$$\begin{aligned}\varphi_X \colon \mathbf{Sym}(X) &\to \mathbf{Sym}(A_X)\\ \sigma &\mapsto \mathrm{rename}^{\mathcal{A}}[\sigma]\end{aligned}$$

Kernel of φ_X

In a syntactic algebra $\mathcal{A}$, there is an integer M such that for all X of cardinal at least M, either $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ or $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$.

Invariance under permutations

A finite syntactic FT_Σ-algebra is of bounded complexity if and only if for all sufficiently large finite set of variables X, $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$.

Proofs

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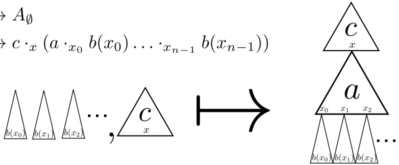
$$
\begin{aligned}
\langle a \rangle \colon (A_\emptyset)^X \times A_{\{x\}} &\to A_\emptyset \\
(b, c) &\mapsto c \cdot_x (a \cdot_{x_0} b(x_0) \ldots \cdot_{x_{n-1}} b(x_{n-1}))
\end{aligned}
$$



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$M = \max(5, |A_\emptyset| + 1)$

Suppose for the sake of contradiction that $|X| \geq M$ and $\mathrm{Ker}(\varphi_X) = \textbf{Alt}(X)$

$\mathrm{Im}(\varphi_X) = \{\mathrm{id}_{A_X}, \tau\}$

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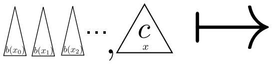

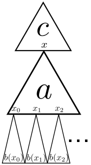

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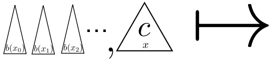

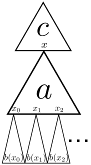

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Fix $a \in A_X$

$$c \in A_{\{x\}}, b \in (A_\emptyset)^X$$

$$x \neq y \text{ with } b(x) = b(y)$$

$$\begin{aligned}\langle \mathrm{rename}^{\mathcal{A}}[t](a)\rangle(b, c) &= \langle \tau(a)\rangle(b, c)\\ &= \langle \mathrm{rename}^{\mathcal{A}}[(x\ y)](a)\rangle(b, c)\\ &= \langle a\rangle(b, c)\end{aligned}$$

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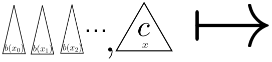

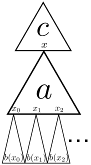

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Invariance under permutations (easy direction)

In a syntactic algebra of bounded complexity, $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ whenever X is large enough.

Suppose $|A_X| \leq k$ for all X and $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$

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id and **Sym** do not alternate

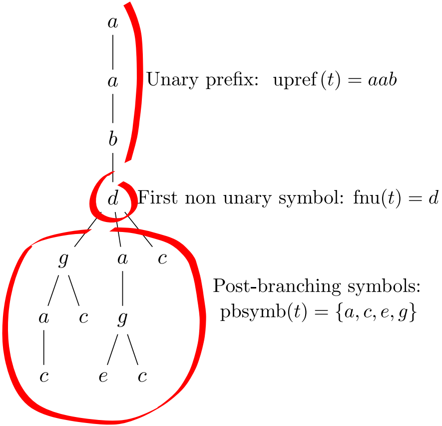
In a syntactic algebra, either $\mathrm{Ker}(\varphi_X) = \{\mathrm{id}_X\}$ for large X, or $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$ for large X.

Characterisation of bounded complexity

Characterization theorem

A language of finite trees is recognized by an FT_Σ-algebra of bounded complexity if and only if it is a Boolean combination of languages of the following kinds:

a. The language of finite trees with unary prefix in a given regular language of words $L \subseteq \Sigma_1^*$.
b. The language of finite trees with first non unary symbol b for a fixed non unary symbol b.
c. The language of finite trees with post-branching symbols B, for $B \subseteq \Sigma$.
d. A regular language K of bounded branching.



Bounded branching: $\exists k$ all trees in K have at most k branches

Conclusion

Complexity map: $c_{\mathcal{A}}(|X|) = |A_X|$

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$$c^{\circ}_{\mathcal{A}}(|X|) = |A_X/\mathbf{Sym}(X)|$$

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A similar characterization of languages of infinite regular trees as Boolean combinations of a.-d. and other languages

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