A Complexity Approach to Tree Algebras: the Polynomial Case

Arthur Jaquard

joint work with Thomas Colcombet

Université de Paris, CNRS, IRIF, F-75006, Paris, France

September 17, Highlights 2021

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Infinitely sorted tree algebras

The free tree algebra has as carrier $(T_X)_{X \text{ finite}}$ where the X's are finite sets of variables.

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Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite tree algebra $\mathcal{A}$ if it is the inverse image of a subset of $A_\emptyset$ under a morphism from the free tree algebra to $\mathcal{A}$.

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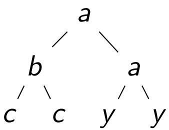

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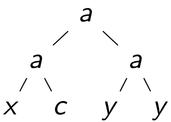

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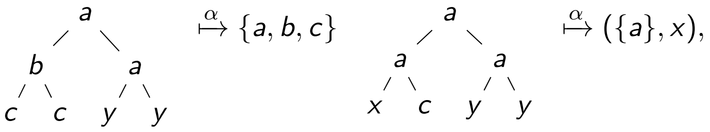

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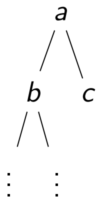
The corresponding automaton model

How to build the following tree ?



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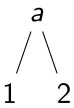

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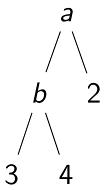

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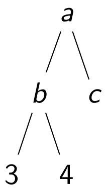

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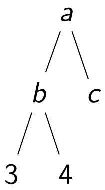

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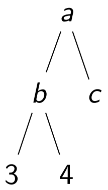

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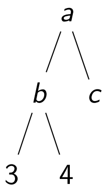

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Finite deterministic automaton $\mathcal{A}$ with k registers that store integers

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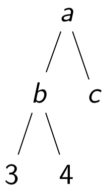

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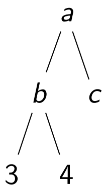

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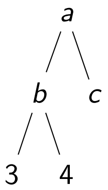

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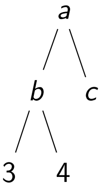

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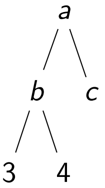

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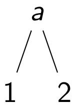

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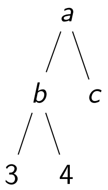

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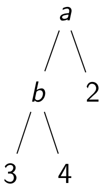

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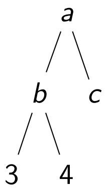

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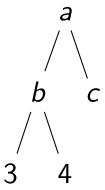

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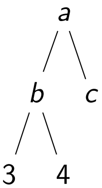

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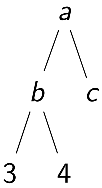

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Same expressive power as algebras of polynomial complexity.

Theorem on permutation groups

Theorem 5.2A. *Let $A := Alt(\Omega)$ where $n := |\Omega| \geq 5$,, and let r be an integer with $1 \leq r \leq n/2$. Suppose that $G \leq A$ has index $|A : G| < \binom{n}{r}$. Then one of the following holds:*

(i) *for some $\Delta \subseteq \Omega$ with $|\Delta| < r$ we have $A_{(\Delta)} \leq G \leq A_{\{\Delta\}}$;*

(ii) *$n = 2m$ is even, G is imprimitive with two blocks of size m, and $|A : G| = \frac{1}{2}\binom{n}{m}$; or*

(iii) *one of six exceptional cases hold where:*

  (a) *G is imprimitive on Ω and $(n, r, |A : G|) = (6,3,15)$;*

  (b) *G is primitive on Ω and $(n, r, |A : G|, G) = (5,2,6,5{:}2)$, $(6,2,6,PSL_2(5))$, $(7,2,15,PSL_3(2))$, $(8,2,15,AGL_3(2))$, or $(9,4,120,P\Gamma L_2(8))$.*

[DIXON, MORTIMER, Permutation groups. Springer Science & Business Media, 1996]

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[DIXON, MORTIMER, Permutation groups. Springer Science & Business Media, 1996]

Corollary: existence of small pseudo-support

Fix $k \in \mathbb{N}$, and let **Sym**(n) act transitively on A with $|A| \leq n^k$. If n is large enough then, for every $a \in A$, there is some $\Delta \subseteq n$ with $|\Delta| \leq k$ such that

$$\mathbf{Alt}(n \setminus \Delta) \subseteq \mathbf{Stabilizer}(a)$$