A Complexity Approach to Tree Algebras: the Exponential Case (ongoing work)

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joint work with Thomas Colcombet

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INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

Université de Paris

Infinitely sorted tree algebras

Let Σ be a ranked alphabet and $\mathcal{V}$ be a countably infinite set of variables. The free tree algebra has as carrier sets the $(T_X)_{X \subseteq \mathcal{V} \text{ finite}}$.

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Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite tree algebra $\mathcal{A}$ if it is the inverse image of a subset of $A_\emptyset$ under a morphism from the free tree algebra to $\mathcal{A}$.

Languages and the size of the algebra

Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite algebra $\mathcal{A}$ if there is a set $P \subseteq A_{\emptyset}$ such that $L = \alpha^{-1}(P)$ in which α is the evaluation morphism of $\mathcal{A}$.

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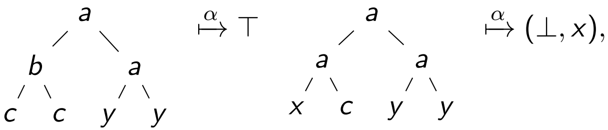

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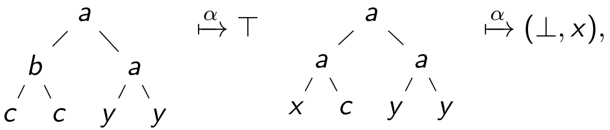

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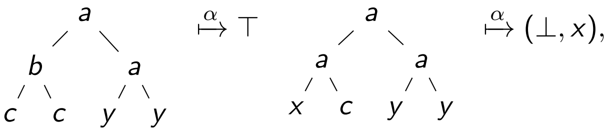

$A_X = \{\top, \perp\} \uplus (\{\top, \perp\} \times X)$ $\qquad$ $|A_X| = 2 + 2|X|$ is linear in $|X|$.

This algebra has linear complexity.

Complexity

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The objective is to identify new classes of languages and to gain a better understanding of tree algebras.

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Equivalence theorem

For a regular language of finite trees, the following are equivalent:

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For a regular language of finite trees, the following are equivalent:

a. Being recognized by a finite tree algebra of exponential complexity.
b. Being recognized by a color tree automaton.

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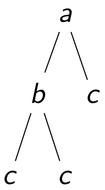
What is a color tree automaton?

Color tree automaton

How to build the following tree ?



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x

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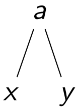

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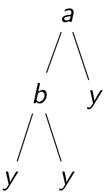

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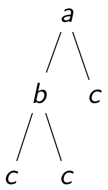

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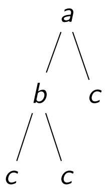

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Such a word is called a tree coding.

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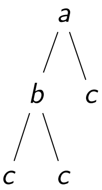

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A color tree automaton works on a tree coding. It is given by

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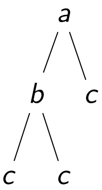

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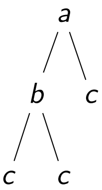

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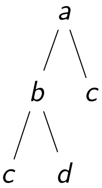
An automaton accepts a tree if it accepts all of its codings.

Example

L = trees in which there is a d below a b, $\Sigma = \{(a,2),(b,2),(c,0),(d,0)\}$

$$Q = \{\top, \bot\} \qquad\qquad K = \{\text{red}, \mathit{green}\}$$



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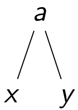

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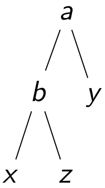

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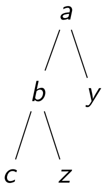

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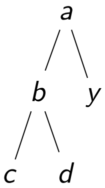

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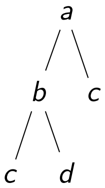

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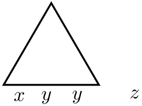
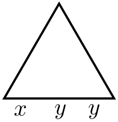
Expressive power of the different types of algebras

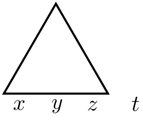
Unrestrained tree algebras $T_{\{x,y,z\}}$



Superlinear tree algebras $T_{\{x,y\}}$



Sublinear tree algebras $T_{\{x,y,z,t\}}$



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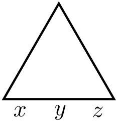

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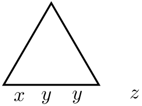

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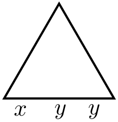

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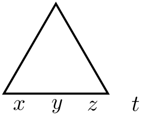

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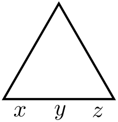

Expressive power

If we allow exponential complexity, the four variants of tree algebras have the same expressive power.

This is not the case for bounded complexity and polynomial complexity.

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Questions?