A Complexity Approach to Tree Algebras: the Bounded Case

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joint work with Thomas Colcombet

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Algebras are used to characterize classes of languages

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Finite words
Monoids, semigroups

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$(A_X)_{X \text{ finite}}$

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Objective: characterize classes that can be naturally defined using infinitely sorted algebras

Infinitely sorted tree algebras: FT_Σ-algebras

Let Σ be a ranked alphabet. The free FT_Σ-algebra has as carrier $(T_X)_{X \text{ finite}}$ where the X's are finite sets of variables.

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Objects

$$\begin{array}{ccc} & a & \\ / & & \backslash \\ b & & c \end{array} \in T_\emptyset \qquad \begin{array}{ccc} & a & \\ / & & \backslash \\ x & & x \end{array} \in T_{\{x\}}$$

$$\begin{array}{ccc} & a & \\ / & & \backslash \\ x & & y \end{array} \in T_{\{x,y\}}$$

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Objects

```
  a                a
 / \ ∈ T_∅        / \ ∈ T_{x}
b   c            x   x

        a
       / \ ∈ T_{x,y}
      x   y
```

Substitution

```
  a          a          a
 / \   ·x   / \   =    / \
x   y      b   c      a   y
                     / \
                    b   c
```

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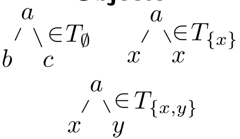

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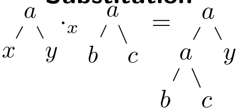
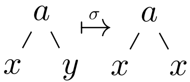


Renaming

$$\sigma(x) = \sigma(y) = x$$



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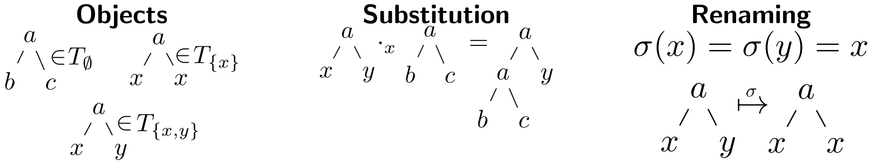

Definition (Finite Tree algebras)

A finite FT_Σ-algebra $\mathcal{A}$ consists of an infinite series of finite carrier sets A_X indexed by finite sets of variables X, together with operations:

Constants. $a(x_0, \ldots, x_{n-1})^{\mathcal{A}} \in A_{\{x_0,\ldots,x_{n-1}\}}$ for all $a \in \Sigma_n$ and variables x_i,

Substitution. $\cdot_x^{\mathcal{A}} \colon A_X \times A_Y \to A_{X \setminus \{x\} \cup Y}$ for all finite X, Y and $x \in X$,

Renaming. $\mathrm{rename}^{\mathcal{A}}[\sigma] \colon A_X \to A_Y$ for all surjective maps $\sigma \colon X \to Y$.

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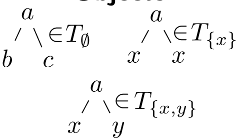

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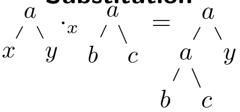

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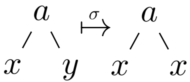

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Identities? $\quad a(x, y) \cdot_y b \quad a(x, z) \cdot_z b$

We also define morphisms, congruences...

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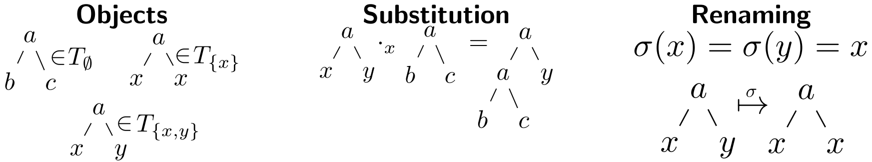

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Given a finite FT_Σ-algebra $\mathcal{A}$, there is a unique morphism from the free algebra to $\mathcal{A}$. It is called the evaluation morphism of $\mathcal{A}$.

Languages and the size of the algebra

Definition (Language recognized by an algebra)

A language L of finite trees over Σ is recognized by a finite FT_Σ-algebra $\mathcal{A}$ if there is a set $P \subseteq A_\emptyset$ such that $L = \alpha^{-1}(P)$ in which α is the evaluation morphism of $\mathcal{A}$.

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```
   a
  / \   ↦α  {a} ∈ A_{x}
 x   x
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$$A \cdot_x B = A \cup B$$

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```
        a                                 a
      /   \                             /   \
     b     a     --α-->  {a, b, c}     a     a     --α-->  ({a}, x),
    / \   / \                         / \   / \
   c   c y   y                       x   c y   y
```

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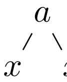

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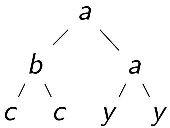
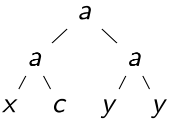


$\overset{\alpha}{\mapsto} \{a, b, c\}$



$\overset{\alpha}{\mapsto} (\{a\}, x)$,

$A_X = 2^\Sigma \uplus (2^\Sigma \times X)$

$|A_X| = 2^{|\Sigma|} + 2^{|\Sigma|}|X|$ is linear in $|X|$.

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Definition (Complexity)

Given a finite FT_{Σ}-algebra $\mathcal{A}$ with carrier

$$(A_X)_{X \text{ finite}}, \text{ all } A_X \text{ finite}$$

its complexity map is $c_{\mathcal{A}}(|X|) = |A_X|$. $(|X| = |Y| \text{ implies } |A_X| = |A_Y|)$

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Linear complexity

Example: The language of all trees with at least a b on every branch

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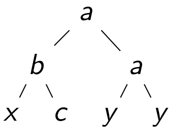
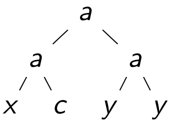


$\overset{\alpha}{\mapsto} (\{y\}, \top)$



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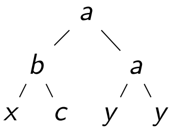

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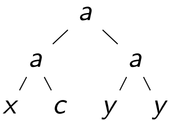

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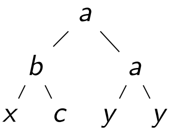

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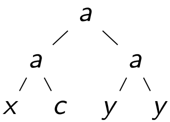

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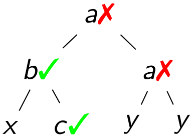

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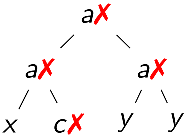

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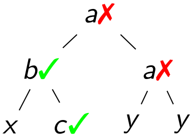

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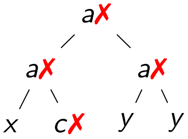

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Languages recognized by top-down deterministic automata

All languages recognized by top-down deterministic automata are recognized by FT_Σ-algebras of exponential complexity.

What complexity means

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Doubly exponential complexity

Regular languages

A top-down nondeterministic automaton can be transformed into a FT_Σ-algebras of doubly-exponential complexity that recognizes the same language.
Conversely, any language recognized by a finite FT_Σ-algebra is regular.

Characterizing bounded complexity

Is complexity meaningful?

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What are the languages recognized by FT_Σ-algebras of bounded complexity?

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Consider for all X the group morphism induced by renaming

$$\begin{aligned}\varphi_X \colon \mathbf{Sym}(X) &\to \mathbf{Sym}(A_X)\\ \sigma &\mapsto \mathrm{rename}^{\mathcal{A}}[\sigma]\end{aligned}$$

Invariance under permutations

A finite syntactic FT_Σ-algebra is of bounded complexity if and only if for all sufficiently large finite set of variables X, $\mathrm{Ker}(\varphi_X) = \mathbf{Sym}(X)$.

Main result

Characterization theorem

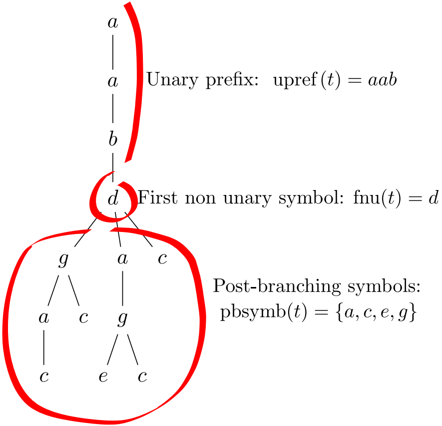
A language of finite trees is recognized by an FT_Σ-algebra of bounded complexity if and only if it is a Boolean combination of languages of the following kinds:

a. The language of finite trees with unary prefix in a given regular language of words $L \subseteq \Sigma_1^*$.

b. The language of finite trees with first non unary symbol b for a fixed non unary symbol b.

c. The language of finite trees with post-branching symbols B, for $B \subseteq \Sigma$.

d. A regular language K of bounded branching.



Bounded branching: $\exists k$ all trees in K have at most k branches

Conclusion

Complexity map: $c_{\mathcal{A}}(|X|) = |A_X|$

Bounded complexity ✓

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a. The language of finite trees with unary prefix in a given regular language of words $L \subseteq \Sigma_1^*$.

b. The language of finite trees with first non unary symbol b for a fixed non unary symbol b.

c. The language of finite trees with post-branching symbols B, for $B \subseteq \Sigma$.

d. A regular language K of bounded branching.

Conclusion

Complexity map: $c_{\mathcal{A}}(|X|) = |A_X|$

Bounded complexity ✓

Polynomial complexity ?

Exponential complexity ?

Orbit complexity: renaming yields an action of $\mathbf{Sym}(X)$ over A_X.

$$c^{\circ}_{\mathcal{A}}(|X|) = |A_X/\mathbf{Sym}(X)|$$

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A similar characterization of languages of infinite regular trees as Boolean combinations of a.-d. and other languages

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