A Complexity Approach to Tree Algebras: The Polynomial Case

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joint work with Thomas Colcombet

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Monoids, semigroups

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We study tree algebras under the angle of asymptotic complexity

Infinitely sorted tree algebras

Let Σ be a ranked alphabet and $\mathcal{V}$ be a countably infinite set of variables. The free tree algebra has as carrier sets the $(T_X)_{X \subseteq \mathcal{V} \text{ finite}}$.

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Def. A finite tree algebra $\mathcal{A}$ consists of an infinite series of finite carrier sets A_X indexed by finite sets of variables X, together with operations:

- **Constants.** $a(x_0, \ldots, x_{n-1})^{\mathcal{A}} \in A_{\{x_0,\ldots,x_{n-1}\}}$ for all $a \in \Sigma_n$ and variables x_i,
- **Substitution.** $\cdot_x^{\mathcal{A}} \colon A_X \times A_Y \to A_{X \setminus \{x\} \cup Y}$ for all finite X, Y and variable x,
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We also define morphisms, congruences...

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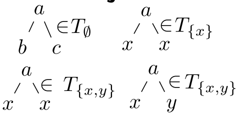

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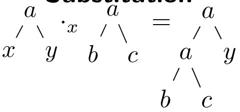

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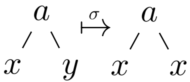

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Given a finite tree algebra $\mathcal{A}$, there is a unique morphism from the free algebra to $\mathcal{A}$. It is called the evaluation morphism of $\mathcal{A}$.

Languages and the size of the algebra

Def. A language L of finite trees over Σ is recognized by a finite algebra $\mathcal{A}$ if there is a set $P \subseteq A_\emptyset$ such that $L = \alpha^{-1}(P)$ in which α is the evaluation morphism of $\mathcal{A}$.

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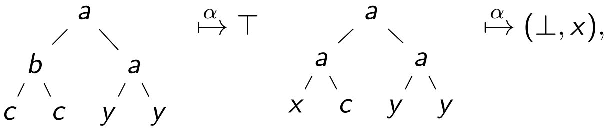

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This algebra has linear complexity.

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Bounded complexity	[Colcombet, J, 2021]
Polynomial complexity	This talk
Exponential complexity	-
Doubly-exponential complexity	All regular languages

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Orbits: c, d, $a(x, y)$, $a(x, x)$, $a(x, c)$, $a(c, x)$, $a(x, *)$, $a(*, x)$, $a(c, c)$, $a(*, *)$

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This algebra has quadratic complexity and bounded orbit complexity.

Orbit complexity

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Another bounded hierarchy of classes.

What complexity means

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Main theorem

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Equivalence between a. and b. is not obvious.

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a. Being recognized by a finite tree algebra of polynomial complexity.

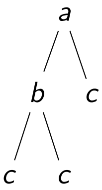
b. Being recognized by a finite tree algebra of bounded orbit complexity.

c. Being described by a coding automaton.

Equivalence between a. and b. is not obvious.

Coding automaton (high level description)

How to build the following tree ?



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x $[x]$

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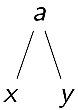

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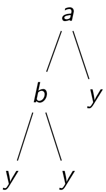

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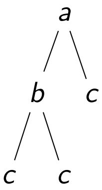

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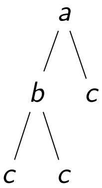

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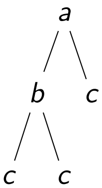

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Coding automata are register automata that map

coding $c \longmapsto$ configuration $\in Q \times \mathcal{V}^R$

They are given by

- finite set of states Q
- finite set of registers R
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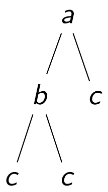

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A coding automaton describes a tree language L if either

- it accepts all codings for a tree t (then it accepts t)
- it rejects all codings for a tree t (then it rejects t)

Coding automaton (example)

$L =$ trees in which the leftmost branch ends with $*(c, d)$, where $*$ is any letter. $\Sigma = \{a : 2, b : 2, c : 0, d : 0\}$

$$Q = \{q_0, q_1, q_{12}, q_{1d}, q_{c2}, \top, \bot\}$$
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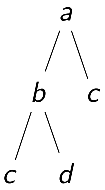

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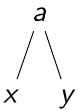

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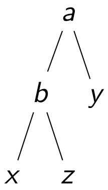

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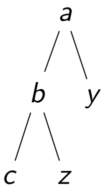

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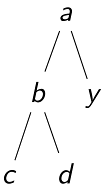

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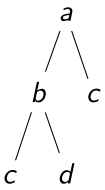

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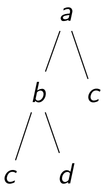

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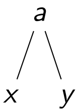

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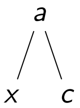

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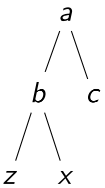

q_{12}

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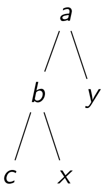

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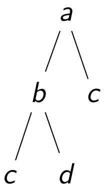

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Thm. It is decidable whether a regular tree language is recognizable by a tree algebra of polynomial complexity.

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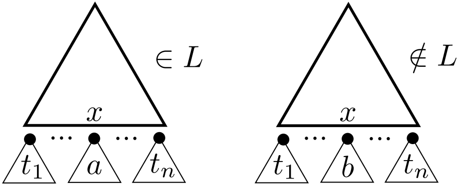
Fix L. A tree $t \in T_{\{\bullet\}}$ is L-sensitive to a leaf x if there exist trees $a, b, t_1, \ldots, t_n$ such that



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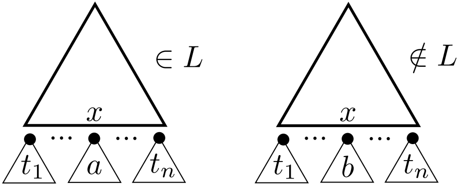

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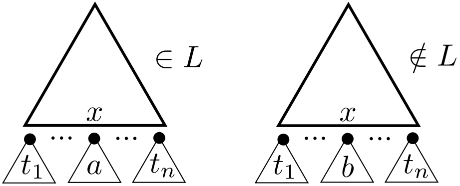

Lem. A regular language of trees L is described by a coding automaton if and only if there is a bound on the number of L-sensitive leaves in trees.

The existence of such a bound can be encoded into cost-MSO. Thus, it is decidable.

Summary

Notions: tree algebra, complexity, orbit complexity, coding, coding automaton

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For a regular language of finite trees, the following properties are equivalent:

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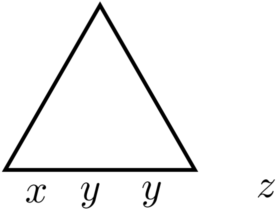
Different types of tree algebras

Unrestrained tree algebras



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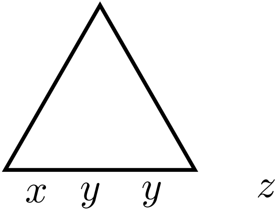

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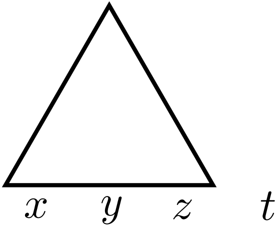

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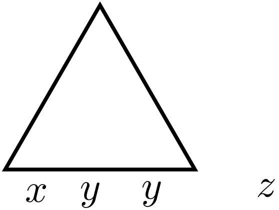

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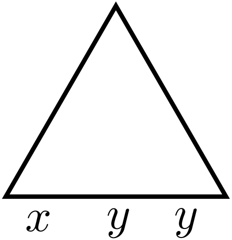

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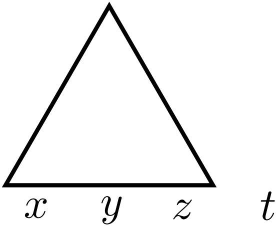

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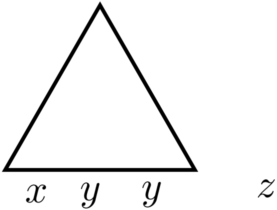

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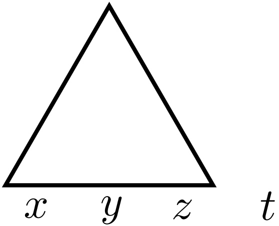

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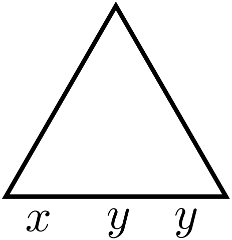

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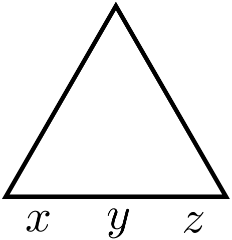

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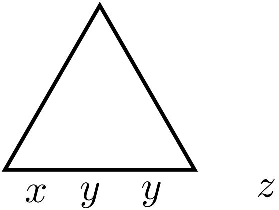

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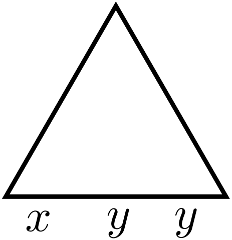

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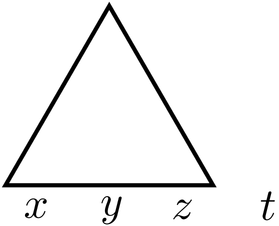

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