Basic operational preorders for algebraic effects in general, and for combined probability and nondeterminism in particular

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Context

Semantics

Three approaches to semantics

Operational describe evaluation steps

Denotational compositional mathematical model

Axiomatics axiomatise behaviour

Contextual preorder

1. Tied to operational semantics
2. $P_1 \sqsubseteq_{ctxt} P_2$ iff in any context C, the behaviour of $C[P_1]$ approximates the behaviour of $C[P_2]$.

[Johann et al., 2010a]

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Objective ? Give a *generic* operational semantics for a large class of languages

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How ?
1. Parametrize with a signature of effect operations Σ
2. Reduce a program to an *effect tree*
3. Define a $\preccurlyeq$ preorder on $\text{Trees}_{\text{Nat}}$ (!)

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3. Define a $\preccurlyeq$ preorder on $\mathsf{Trees}_{\mathsf{Nat}}$ (!)

Result ? Generic operational definition of contextual preorder

Contextual preorders

Morris-style

Input: A peorder $\preccurlyeq$ for type Nat

Output:

$$P_1 \sqsubseteq_{\mathsf{ctxt}} P_2 \iff \forall C[-]\,\text{context}, |C[P_1]| \preccurlyeq |C[P_2]| \quad (1)$$

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GOM

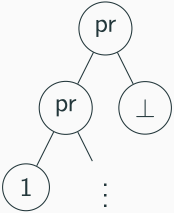
Input: A peorder $\preccurlyeq$ for type Nat

Output: A logical relation (!) on programs that characterises contextual preorder (Morris-Style)

Effect trees

Example of trees

Let $\Sigma = \{\mathsf{pr}\}$ be a signature containing one binary effect construction.



Properties

$\mathsf{Trees}_{\mathsf{Nat}}$ is a DCPO and a continuous Σ-algebra

Preorders $\preccurlyeq$

What are the conditions on $\preccurlyeq$ in GOM ?

Admissible If $t_i \preccurlyeq t'_i$ and $(t_i)_i$, $(t'_i)_i$ are an ascending chains then

$$\bigsqcup_i t_i \preccurlyeq \bigsqcup_i t'_i \tag{2}$$

Compatible with least upper bounds

Compositional If $t \preccurlyeq t'$ and $\rho \preccurlyeq \rho'$ (pointwise) then $t\rho \preccurlyeq t'\rho'$

Compositional reasoning is possible

Contributions

General Identify three different ways to produce well-behaved preorders

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Specific Examine how they apply to a specific signature

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Coincidence Prove that the three ways of defining $\preccurlyeq_{pr/nd}$ lead to the same contextual preorder

Well-behaved preorders

Methods for defining preorders

Following three common approaches to semantics

- From some operational construction $\preccurlyeq_{op}$
- From a denotation $[\![\cdot]\!]$ $\preccurlyeq_{den}$
- From axiomatic definitions $\preccurlyeq_{ax}$

Combined scheduler

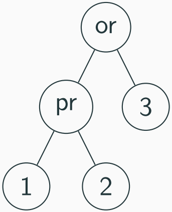
Randomised Algorithms with Scheduler

Σ coin "pr", demon "or"

$\preccurlyeq$ capture the behaviour ... and satisfies the requirements

Example of program

$(\underline{1} \text{ pr } \underline{2}) \text{ or } \underline{3}$



Operationally defined preorders

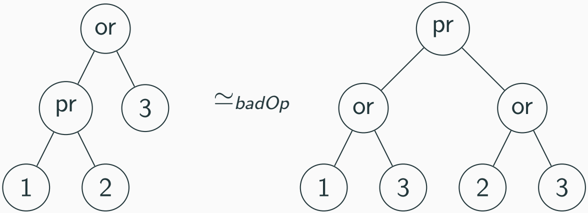
Compare Markov Decision Processes pointwise, where a point is a goal *set* $X \subseteq \mathrm{Nat}$:

$$t \preccurlyeq_{badOp} t' \iff \forall X \subseteq \mathrm{Nat}, \quad \inf_{\pi} \mathbb{E}^{\pi}(t \in X) \leq \inf_{\pi} \mathbb{E}^{\pi}(t' \in X)$$

The natural operations ... ... Counter example

The issue

1. The following trees are equated



2. If compositionality holds for $\preccurlyeq_{badOp}$ then

$$x \,\mathrm{or}(y \,\mathrm{pr}\, z) = (x \,\mathrm{or}\, y)\,\mathrm{pr}(x \,\mathrm{or}\, z) \tag{4}$$

3. Which is does not hold for $\simeq_{badOp}$ (easy substitution)
4. And should never hold [Mislove et al., 2004]

The solution

Compare Markov Decision Processes pointwise, where a point is a payoff function $h : \mathsf{Nat} \to \overline{\mathbb{R}_+}$:

$$t \preccurlyeq_{op} t' \iff \forall h : \mathsf{Nat} \to \overline{\mathbb{R}_+}, \quad \inf_{\pi} \mathbb{E}^{\pi}(h(t)) \leq \inf_{\pi} \mathbb{E}^{\pi}(h(t'))$$

Proposition

The preorder $\preccurlyeq_{op}$ is admissible and compositional

Remark

The proof requires some topological arguments...

Denotationally defined preorders

Denotationally defined preorders

The idea

Input
1. Continuous Σ-algebra D
2. $[\![\cdot]\!] \colon \mathbb{N}_\perp \to D$ continuous Σ-algebra homomorphism

Output The preorder $\preccurlyeq_{den}$

$$\begin{array}{ccc} \mathbb{N} & \xrightarrow{\ j\ } & D \\ {\scriptstyle i}\big\downarrow & \nearrow_{[\![\cdot]\!]} & \\ \mathrm{Trees}(\mathbb{N}) & & \end{array}$$

$$t \preccurlyeq_{den} t' \iff [\![t]\!] \leq_D [\![t']\!] \quad (5)$$

Denotationally defined preorders

Properties of $\preccurlyeq_{den}$

1. Automatically *admissible* (continuity)
2. Automatically *compatible* (Σ-algebra)
3. Not always *compositional* !

Denotationally defined preorders

Factorisation

The map $j\colon \mathbb{N} \to D$ is said to have the *factorisation property* if, for every function $f\colon \mathbb{N} \to D$, there exists a continuous homomorphism $h_f : D \to D$ such that $f = h_f \circ j$.

$$\mathbb{N} \xrightarrow{\;j\;} D \overset{h_f}{\dashrightarrow} D \qquad (f\colon \mathbb{N} \to D)$$

Idea

We then have $[\![t\sigma]\!] = h_\sigma([\![t]\!])$ which is continuous in t with a fixed σ.

Well behaved denotational preorder

Proposition

If $j : \mathbb{N} \to D$ has the factorisation property then the relation $\preccurlyeq_D$ is substitutive, hence it is an admissible compositional precongruence.

In practice [Proposition 16]

It is usually not necessary to prove the factorisation property directly. Instead it holds as a consequence of the continuous algebra D and map $j : \mathrm{Nat} \to D$ being derived from a suitable monad.

Applying to the running example

Using Kegelpsitze [Keimel and Plotkin, 2017]

$\mathcal{V}_{\leq 1}X$ ωCPO of (discrete) subprobability distributions over X.

$\mathcal{S}\mathcal{V}_{\leq 1}X$ ωCPO of nonempty Scott-compact convex upper-closed subsets of $\mathcal{V}_{\leq 1}X$ ordered by reverse inclusion $\supseteq$.

$$\mathrm{or}(A, B) = \mathrm{Conv}(A \cup B) \tag{6}$$

$$\mathrm{pr}(A, B) = \left\{ \frac{1}{2}a + \frac{1}{2}b \mid a \in A, b \in B \right\} \tag{7}$$

Axiomatically defined preorders

Generic definition

Theories

Equation $e \leq e'$ with $e, e' \in \text{Trees}(\text{Vars})$

Clause (Infinitary) Horn-Clause of equations

Theory Set of Horn-Clauses

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Theory Set of Horn-Clauses

Axiomatically defined preorder

Definition There exists a smallest admissible preorder $\preccurlyeq_{ax}$ that models T

Property $\preccurlyeq_{ax}$ is compositional

Axioms for Pr and Nd

Bot: $\bot \leq x$

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Prob: $x \,\mathrm{pr}\, x = x$, $x \,\mathrm{pr}\, y = y \,\mathrm{pr}\, x$,
$(x \,\mathrm{pr}\, y) \,\mathrm{pr}\, (z \,\mathrm{pr}\, w) = (x \,\mathrm{pr}\, z) \,\mathrm{pr}\, (y \,\mathrm{pr}\, w)$

Appr: $x \,\mathrm{pr}\, y \leq y \implies x \leq y$ (!)

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Nondet: $x \operatorname{or} x = x,\ x \operatorname{or} y = y \operatorname{or} x,\ x \operatorname{or} (y \operatorname{or} z) = (x \operatorname{or} y) \operatorname{or} z$

Dem: $x \operatorname{or} y \geq x$

Axioms for Pr and Nd

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Dem: $x \,\mathsf{or}\, y \geq x$

Dist: $x \,\mathsf{pr}\, (y \,\mathsf{or}\, z) = (x \,\mathsf{pr}\, y) \,\mathsf{or}\, (x \,\mathsf{pr}\, z)$ (!)

The coincidence theorem

Coincidence

For probability and non-determinism

$$\preccurlyeq_{op} = \preccurlyeq_{den} = \preccurlyeq_{ax}$$

Proof sketch

1. Equality on trees without or nodes
2. Equality for trees with *finite* number of or nodes (!)
3. General equality using finite approximations and admissibility

Summary and limitations

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What has been done

- Denotational and Axiomatic definitions of preorders
- Applied to a specific signature $\Sigma = \{\mathsf{pr}, \mathsf{or}\}$

Limitations

- Some effects are not algebraic
- The preorder for *countable* non-determinism is not admissible

Thank You!

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