ACT I: Where queries are optimised
Query Optimisation

**Input** Some FO sentence \( \varphi \)

**Promise** Upwards closure for induced substructures \((\subseteq_i)\) – a.k.a. extensions

**Output** A simplified query (existential)
Query Optimisation

Input  $\varphi = \text{there exists no vertex cover of size 1 in } G$

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Output  A simplified query (existential)

$G \models \varphi$
Query Optimisation

**Input** $\varphi = \text{there exists no vertex cover of size 1 in } G$

**Promise** When $G \subseteq_i H$, a vertex cover of $H$ induces a vertex cover of $G$

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$G \models \varphi$

$H$
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**Promise** When \( G \subseteq_i H \), a vertex cover of \( H \) induces a vertex cover of \( G \)

**Output** Finitely many graphs \( M_i \) to check

\[ G \models \varphi \]

\[ H \models \varphi \]
Query Optimisation

**Input** \( \varphi = \text{there exists no vertex cover of size 1 in } G \)

**Promise** When \( G \subseteq H \), a vertex cover of \( H \) induces a vertex cover of \( G \)

**Output** Finitely many graphs \( M_i \) to check
Theorem (Łoś (1955); Tarski (1954))

This algorithm exists.

Proof.

• an equivalent existential sentence exists (heavy use of compactness)

• one can enumerate proofs \( \vdash \psi \leftrightarrow \varphi \) with \( \psi \) existential. \( \square \)
Łoś (1955); Tarski (1954) in a diagram?

\[ \text{FO} \supset \text{EFO} \]

\[ \sublattice \]

\[ \text{UpSet}(C) \]

\[ \text{vertex cover of size 1} \]

\[ \mathcal{P}(C) \]

\[ \sublattice \]

Beware: In computer science \( C \) is a class of finite structures!
Łoś (1955); Tarski (1954) in a diagram?

\[ \{ A \in C \mid A \models \varphi \} \]

\[ \mathcal{P}(C) \]

\[ \text{vertex cover of size 1} \]

\[ \text{sublattice} \]

\[ \text{sublattice} \]

\[ \{ A \in C \mid A \models \varphi \} \]

\[ \text{FO} \]

\[ \text{EFO} \]

\[ \text{UpSet}(C) \]
Łoś (1955); Tarski (1954) in a diagram?

\[\neg \exists x \forall y, z. E(y, z) \implies x = y \lor x = z\]

\[\{A \in C \mid A \models \varphi\}\]

\[\neg \text{ vertex cover of size 1}\]

\[G_i H^G_2 S = H_2 S\]

\[\text{sublattice}\]

\[\text{sublattice}\]

\[\text{FO} \rightarrow \mathcal{P}(C)\]

\[\text{EFO} \rightarrow \text{UpSet}(C)\]

\[f A_2 C j A_j = \phi\]
Łoś (1955); Tarski (1954) in a diagram

\[ \neg \exists x. \forall y, z. E(y, z) \implies x = y \lor x = z \]

\[ \{ A \in C \mid A \models \varphi \} \]

\[ \neg \text{vertex cover of size 1} \]

\[ \text{FO} \quad \xrightarrow{\text{sublattice}} \quad \text{EFO} \]

\[ \text{FO} \quad \xrightarrow{\text{sublattice}} \quad \mathcal{P}(C) \]

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Beware: In computer science \( C \) is a class of finite structures!
Łoś (1955); Tarski (1954) in a diagram?

\[
\neg \exists x \forall y, z.E(y, z) \implies x = y \lor x = z
\]

\[
\{ A \in C \mid A \models \varphi \}
\]

\[
\neg \text{ vertex cover of size 1}
\]

\[
G \subseteq H \wedge G \in S \implies H \in S \]

\[
\text{EFO} \xrightarrow{\text{sublattice}} \text{FO}
\]

\[
\text{FO} \xrightarrow{\text{vertex cover of size 1}} \mathcal{P}(C)
\]

\[
\mathcal{P}(C) \xrightarrow{\text{sublattice}} \text{UpSet}(C)
\]

\[
\text{UpSet}(C) \xrightarrow{\text{vertex cover of size 1}} \{ A \in C \mid A \models \varphi \}
\]

\[
\text{FO} \xrightarrow{\text{sublattice}} \text{EFO}
\]

Beware: In computer science \( C \) is a class of finite structures!
Łoś (1955); Tarski (1954) in a diagram?

Beware: In computer science $\mathcal{C}$ is a class of finite structures!
ACT II: Where all is lost in a fire
Query Optimisation \( \star \) over \( \mathcal{C} \) \( \star \)

**Input** Some FO sentence \( \varphi \)

**Promise** Upwards closure \( \star \) over \( \mathcal{C} \) \( \star \)

**Output** An existential sentence \( \psi \) equivalent to \( \varphi \) \( \star \) over \( \mathcal{C} \) \( \star \)

Over finite structures

- Tait (1959): no such \( \psi \)!
- Chen and Flum (2021): no algorithm even if \( \psi \) exists.
ACT III: Where the problem is solved
Finding the right nails for our hammer

In computer science $\mathcal{C}$ is a class of finite structures!
Finding the right nails for our hammer

Easy case: $C$ is a *finite* class of *finite* structures.
Finding the right nails for our hammer

\[
\text{FO} \longrightarrow \mathcal{P}(C) \quad \{A \in C \mid A \models \varphi\}
\]

\[
\text{EFO} \longrightarrow \text{UpSet}(C) \quad \{A \in C \mid A \models \varphi\}
\]

Combinatorics: \( C \) is \textit{well-quasi-ordered} (WQO).
Non Trivial: \( C \) is hereditary (downwards closed), wide, and closed under \( \uplus \) (Atserias et al., 2008).
Finding the right nails for our hammer

Property implication over hereditary classes of finite structures closed under $\biguplus$.
Property implication over **hereditary** classes of **finite structures** closed under $\sqcup$.
Finding the right nails for our hammer

Property implication over hereditary classes of finite structures closed under $\cup$.

(Folklore) (Atserias et al., 2008)
Locally satisfying a property?

\[ \text{Local}(C, r, k) \triangleq \{ N_A(\vec{a}, r) \mid A \in C, \vec{a} \in A^k \} \]

A structure \( A \).
Locally satisfying a property?

Local\( (C, r, k) \triangleq \{ \mathcal{N}_A(\tilde{a}, r) \mid A \in C, \tilde{a} \in A^k \} \)

A structure \( A \), with 2 selected nodes.
Locally satisfying a property?

\[ \text{Local}(C, r, k) \triangleq \{ \mathcal{N}_A(\bar{a}, r) \mid A \in C, \bar{a} \in A^k \} \]

A structure \( A \), with 2 selected nodes, and a 1-local neighborhood.
Locally satisfying a property?

\[ \text{Local}(C, r, k) \triangleq \{ N_A(\vec{a}, r) \mid A \in C, \vec{a} \in A^k \} \]

An element of Local\((C, 1, 2)\).
Locally satisfying a property?

\[ \text{Local}(\mathcal{C}, r, k) \triangleq \{ \mathcal{N}_A(\vec{a}, r) \mid A \in \mathcal{C}, \vec{a} \in A^k \} \]
Locally satisfying a property?

\[
\text{Local}(\mathcal{C}, r, k) \equiv \{ \mathcal{N}_A(\bar{a}, r) \mid A \in \mathcal{C}, \bar{a} \in A^k \}
\]

♣ Localise Bounded Degree

\(\mathcal{C}\) is of bounded degree if and only if \(\text{Local}(\mathcal{C}, r, k)\) is finite for all \(k, r \geq 0\), i.e., \textit{locally} finite
Locally satisfying a property?

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\text{Local}(\mathcal{C}, r, k) \equiv \{ \mathcal{N}_A(\tilde{a}, r) \mid A \in \mathcal{C}, \tilde{a} \in A^k \}
\]

♣ Localise Bounded Degree

\( \mathcal{C} \) is of bounded degree if and only if \( \text{Local}(\mathcal{C}, r, k) \) is finite for all \( k, r \geq 0 \), i.e., locally finite

Corollary (of theorem \( \circlearrowright \); known from Atserias et al. (2008))

Hereditary classes of bounded degree, closed under \( \oplus \), satisfy preservation under extensions.
ACT IV: Behind the scenes
Why assume $\mathcal{C}$ to be hereditary (downwards closed)?

Let $\varphi \in \text{FO}$ be upwards closed, t.f.a.e.:

(i) $\varphi \equiv_{\mathcal{C}} \psi$ with $\psi \in \text{EFO}$.
(ii) $\varphi$ has finitely many minimal models in $\mathcal{C}$.
(iii) minimal models of $\varphi$ in $\mathcal{C}$ have bounded size.
Proof scheme of Theorem ①

Assume $\varphi$ is upwards-closed with respect to $\subseteq_i$ over $C$.

<table>
<thead>
<tr>
<th>Step</th>
<th>Minimal Models</th>
<th>Sentence</th>
</tr>
</thead>
<tbody>
<tr>
<td>∼</td>
<td>bounded radius $(r, k)$</td>
<td>$\exists x_1, \ldots, x_k. \psi(\vec{x}), \psi$ r-local</td>
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<tr>
<td>∗</td>
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<td>$\exists x_1, \ldots, x_\ell. \psi(\vec{x}), \psi$ quantifier free</td>
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Proof scheme of Theorem

Assume $\varphi$ is upwards-closed with respect to $\subseteq_i$ over $\mathcal{C}$.

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</table>

- **existential**
- **definable and upwards closed**
- **existential local and upwards closed**
ACT V: The friends made along the way
Locality and preservation under extensions

Existential Gaifman normal form

Existential Sentences

Extensions ($\subseteq_i$)

Gaifman normal form

Arbitrary Sentences

None

(Grohe and Wöhrle, 2004)

(Łoś, 1955; Tarski, 1954)

(Gaifman, 1982)

Over arbitrary structures
Locality and preservation under extensions

- Existential Gaifman normal form
  - (Grohe and Wöhrle, 2004)

- Existential Sentences
  - (Łoś, 1955; Tarski, 1954)

- Extensions ($\subseteq_i$)

- Positive Gaifman normal form
  - Thm.

- Existential local Sentences
  - Thm.

- Local elementary embeddings

- Gaifman normal form
  - (Gaifman, 1982)

- Arbitrary Sentences

Over arbitrary structures

(Grohe and Wöhrle, 2004)

(Łoś, 1955; Tarski, 1954)

(Gaifman, 1982)
Locality and preservation under extensions

Existential Gaifman normal form

Existential Sentences

Extensions (⊆i)

Positive Gaifman normal form

Existential local Sentences

Disjoint unions (⊔)

Gaifman normal form

Arbitrary Sentences

None

(Over finite structures)

(Grohe and Wöhrle, 2004)

(Tait, 1959; Chen and Flum, 2021)

(Gaifman, 1982)


