Basic Operational Preorders for Algebraic Effects

With a pinch of non-determinism and probabilities

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A deceptively long introduction

Effects

PCF+Effects

PCF Plotkin (1977) using call-by-value with effects, similar to Plotkin and Power (2001).

Concrete implementations **with handlers**

- Eff `https://www.eff-lang.org/` Bauer and Pretnar (2012); Plotkin and Pretnar (2013)
- Haskell implementations (Fused Effects, Polysemy, Eff, ...)

What's in my bag?

Local state, global state, exceptions, non-determinism, random numbers, logging, input/output, continuations.

Many of the effects listed are in fact algebraic (modeled by a Lawvere theory) and therefore share nice properties.

Program equivalence and how to deal with it

- Dal Lago et al. (2017)
- Johann et al. (2010)

A deceptively long introduction

What does it look like concretely?

Simple programming language: Types

Non polymorphic term types

$$\tau ::= \textsf{Nat} \mid \tau \to \tau$$

Polymorphic effect types

Example	Type
Lookup	$\sigma : \alpha^{\mathsf{Nat}} \to \alpha$
Binary choice	$\sigma : \alpha^{\mathsf{n}} \to \alpha$
Update	$\sigma : \mathsf{Nat} \times \alpha^{\mathsf{n}} \to \alpha$
Generic case	$\sigma : \mathsf{Nat} \times \alpha^{\mathsf{Nat}} \to \alpha$

What about polymorphism?

Adding polymorphism for terms

- Done in Johann et al. (2010), not the main technicality
- Very useful in practice Wadler (1989); Pitts (2000): "there are not much functions of type $\forall \alpha . \alpha \to \alpha$".

Terms, $\sigma \in \Sigma$

$$\begin{aligned} \mathsf{M} := \mathsf{x} \mid \lambda \mathsf{x} : \tau.\mathsf{M} \mid \mathsf{M}\mathsf{M} \mid \mathsf{fix}\, \mathsf{M} \mid \mathsf{Z} \mid \mathsf{S}\, \mathsf{M} \\ \mid \mathsf{case}\, \mathsf{M}\, \mathsf{of}\, \mathsf{Z} \Rightarrow \mathsf{M}; \mathsf{S}(\mathsf{x}) \Rightarrow \mathsf{M} \\ \mid \sigma(\mathsf{M}, \ldots, \mathsf{M}) \\ \mid \sigma(\mathsf{M}; \mathsf{M}, \ldots, \mathsf{M}) \end{aligned}$$

Values

$$\mathsf{V} := \lambda \mathsf{x} : \tau.\mathsf{M} \mid \mathsf{Z} \mid \mathsf{S}\, \mathsf{V}$$

A deceptively long introduction

Reasoning about effects

Contextual equivalence

$M \equiv_{ctx} M'$

$$\forall C[-], \forall n, C[M] : \mathrm{Nat}, C[M'] : \mathrm{Nat}, C[M] \Downarrow \underline{n} \iff C[M'] \Downarrow \underline{n}$$

Issues

- Can capture free variables;
- More suitable for proving non-equivalence;
- Taylor made for termination.

Two amidst many alternatives

- Bisimilarity and bisimulations Dal Lago et al. (2017);
- Logical relations Johann et al. (2010).

Why can't we easily reason?

In the presence of effects, function extentionality is not a good way to reason:

$$\lambda x.\mathrm{OR}(1, 2) \quad ? \quad \mathrm{OR}(\lambda x.1, \lambda x.2)$$

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$$\mathsf{C}[-] = (\lambda f.f0 + f0)[-]$$

The zen way of building contextual preorders

The work of Johann et al. (2010)

Ground relations

Stacks, terms, trees

Reduce a pair $\langle S, M \rangle$ to a tree $|S, M|$ of effects where leaves are values.

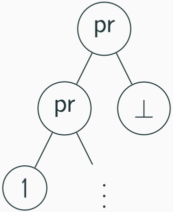
Granted $\preccurlyeq$ is a preorder over $\mathrm{Tree}_{\mathrm{Nat}}$ the free continuous Σ-algebra over Nat.

Contextual Preorder

The contextual preorder is the largest compatible (closed under context) and $\preccurlyeq$-adequate (included in $\preccurlyeq$ at ground type) relation.

Example of trees

Let $\Sigma = \{\mathsf{pr}\}$ be a signature containing one binary effect construction.



Always go concrete

Generic operational meta-theory

Input: A preorder $\preccurlyeq$ for type Nat

Output: A logical relation (!) on programs that characterises contextual preorder (Morris-Style)

Some relations are more equal than others

† Admissible If $t_i \preccurlyeq t'_i$ and $(t_i)_i$, $(t'_i)_i$ are ascending chains then

$$\bigsqcup_i t_i \preccurlyeq \bigsqcup_i t'_i$$

‡ Compatible If $t_i \preccurlyeq t'_i$ and $\sigma \in \Sigma$ then $\sigma(t_1, \ldots) \preccurlyeq \sigma(t'_1, \ldots)$.

◇ Substitutive Given $\rho \colon \mathsf{Tree}_{\mathsf{Nat}} \to \mathsf{Tree}_{\mathsf{Nat}}$, if $t \preccurlyeq t'$ then $t\rho \preccurlyeq t'\rho$

□ Compositional Given $\rho, \rho' \colon \mathsf{Tree}_{\mathsf{Nat}} \to \mathsf{Tree}_{\mathsf{Nat}}$, if $t \preccurlyeq t'$ and $\rho \preccurlyeq \rho'$ then $t\rho \preccurlyeq t'\rho'$

$$\dagger \wedge \ddagger \wedge \diamond \iff \dagger \wedge \Box$$

Examples, counter examples

Effect	Admissible	Compatible	Substitutive	Compositional
Hoare	✓	✓	✓	✓
Smyth	✓	✓	✓	✓
Countable Smyth	✗	?	?	?
Valuations	✓	✓	✓	✓
Exceptions	✓	✓	✓	✓
Mixed Pr/Nd	✓	✓	✗	✗

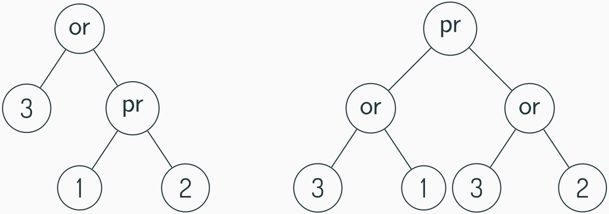
Idea

Denotational semantics provides admissibility and compatibility for free...

Probability and non-determinism

Bad operational preorder (not substitutive)

$$\forall H \subseteq \mathrm{Nat}, \sup_s \mathbb{P}(t/s \in H) \leq \sup_s \mathbb{P}(t'/s \in H)$$



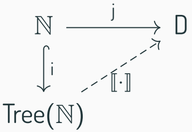
Hint: substitute using $1 \mapsto \frac{7}{8}\underline{0} + \frac{1}{8}\underline{1}$, $2 \mapsto 1$, $3 \mapsto \frac{3}{4}\underline{0} + \frac{1}{4}\underline{1}$ and compute for $H = \{1\}$ ...

The zen way of building contextual preorders

Nice, but what does this mean?

It really looks like denotational semantics...

Given a continuous Σ-algebra D and a morphism $[\![\cdot]\!] : \mathbb{N}_\perp \to D$ one can build the preorder $\preccurlyeq_{\mathrm{den}}$.



$$t \preccurlyeq_{\mathrm{den}} t' \iff [\![t]\!] \leq_D [\![t']\!] \quad (1)$$

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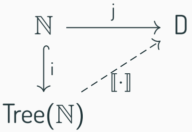


$$t \preccurlyeq_{den} t' \iff [\![t]\!] \leq_D [\![t']\!] \qquad (1)$$

Properties of $\preccurlyeq_{den}$

1. Automatically admissible (continuity)
2. Automatically compatible (Σ-algebra)
3. Not always compositional! ($+, \times$ interpreted naturally)

Denotationally defined preorders

Factorisation

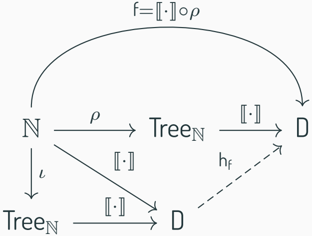
The map $[\![\cdot]\!] : \mathbb{N} \to D$ is said to have the factorisation property if, for every function $f : \mathbb{N} \to D$, there exists a continuous homomorphism $h_f : D \to D$ such that $f = h_f \circ [\![\cdot]\!]$.

$$\mathbb{N} \xrightarrow{[\![\cdot]\!]} D \overset{h_f}{\dashrightarrow} D, \qquad f : \mathbb{N} \to D$$

Consequence

We then have $[\![t\sigma]\!] = h_\sigma([\![t]\!])$ which is continuous in t with a fixed σ.

Proof



In practice...Monads

If (T, η, μ) is a monad over continuous Σ-algebras, the map $\eta \colon \mathsf{Nat} \to \mathsf{TNat}$ satisfies the factorisation property with $\mathsf{h_f} \triangleq \mathsf{f}^\dagger$.

Solution(s) for $\preccurlyeq^{\text{den}}$

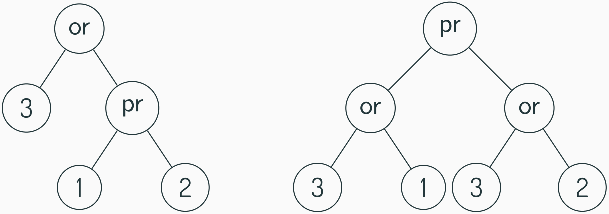
Kegelspitze Keimel and Plotkin (2017)

Scott closed convex subsets of valuations over X ordered by inclusion.

Previsions Goubault-Larrecq (2016)

$\mathcal{L}(X)$ is the set of lower semicontinuous maps from X to $\overline{\mathbb{R}}$. We use $\mathcal{L}(\mathcal{L}(X))$ to represent previsions.

Concretely



The zen way of building contextual preorders

Axiomatics

Let Vars be a set of countably many distinct variables

$$\left(\bigwedge_{i\in I} e_i \leq e_i'\right) \implies e \leq e' \quad ,$$

An effect theory T is a set of Horn clauses.

Order associated to a theory

There exists a smallest admissible and compositional preorder $\preccurlyeq_T$ satisfying T.

Concretely... $\preccurlyeq^{ax}$

Bot:	$\bot \leq x$
Prob:	$x \,\mathrm{pr}\, x = x,\; x \,\mathrm{pr}\, y = y \,\mathrm{pr}\, x,\; (x \,\mathrm{pr}\, y) \,\mathrm{pr}\, (z \,\mathrm{pr}\, w) = (x \,\mathrm{pr}\, z) \,\mathrm{pr}\, (y \,\mathrm{pr}\, w)$
Appr:	$x \,\mathrm{pr}\, y \leq y \implies x \leq y$
Nondet:	$x \,\mathrm{or}\, x = x,\; x \,\mathrm{or}\, y = y \,\mathrm{or}\, x,\; x \,\mathrm{or}\, (y \,\mathrm{or}\, z) = (x \,\mathrm{or}\, y) \,\mathrm{or}\, z$
Ang:	$x \,\mathrm{or}\, y \leq x$
Dem:	$x \,\mathrm{or}\, y \geq x$
Dist:	$x \,\mathrm{pr}\, (y \,\mathrm{or}\, z) = (x \,\mathrm{pr}\, y) \,\mathrm{or}\, (x \,\mathrm{pr}\, z)$

Figure 1: Horn theory for mixed probability and non determinism

Universal approximation scheme

Let $\frac{2^0-1}{2^0}t = \bot$ and $\frac{2^{(n+1)}-1}{2^{(n+1)}}t = t \text{ pr } \frac{2^n-1}{2^n}t$, extend with $\frac{2^\infty-1}{2^\infty}t = \bigsqcup_n \frac{2^n-1}{2^n}t$.

In a reasonable interpretation of trees $\frac{2^n-1}{2^n}t \ll t$ and $\frac{2^\infty-1}{2^\infty}t = t$.

Removing implications

For any effect theory containing the Bot and Prob axioms, an admissible model satisfies the Appr axiom if and only if it satisfies the equation $\frac{2^\infty-1}{2^\infty}x = x$.

The hard part $\preccurlyeq^{den} \subseteq \preccurlyeq^{ax}$

(i) Over trees without or: well-known since Heckmann (1994)

- Finite case: normal form
- Infinite case: approximation

(ii) Over trees with finitely many or

- Use distributivity to have a finite hat of or nodes
- $t' \equiv_{ax,den} t'$ or k, when $k = \lambda_1 t_1 + \cdots + \lambda_n t_n$
- $\forall i, \exists k_i := \lambda_1 t_1' + \cdots + \lambda_n t_n', t_i \preccurlyeq^{den} k_i$
- $t \preccurlyeq^{ax} t'$ or k_1 or $\cdots$ or $k_n \equiv_{ax,den} t'$

(iii) Over arbitrary trees using $\frac{2^n - 1}{2^n} t$, admissibility and the way-below relation

What admissibility? Why the way-below relation?

$$\left[\!\left[\frac{2^n-1}{2^n}t\right]\!\right] \ll [\![t]\!] \leq \bigsqcup_m \left[\!\left[\frac{2^m-1}{2^m}t'\right]\!\right]$$

Hence there exists an m such that

$$\left[\!\left[\frac{2^n-1}{2^n}t\right]\!\right] \leq \left[\!\left[\frac{2^m-1}{2^m}t'\right]\!\right]$$

And conclude using the approximation.

The zen way of building contextual preorders

Can we forget about domain theory?

Self-contained preorder definition

Given a strategy $s\colon \{l, r\}^* \to \{l, r\}$ and a tree t evaluate $t \restriction s$.

$$t \preccurlyeq^{op} t' \Leftrightarrow \forall h\colon \mathbb{N} \to [0, \infty] \ \sup_s E_{t \restriction s}(h) \leq \sup_s E_{t' \restriction s}(h)$$

1. Compatible: easy
2. Substitutive: easy
3. Admissible: the function $G_h\colon (s, t) \mapsto \mathbb{E}_{t \restriction s}(h)$ is continuous and the set of strategies is compact.

The correspondance between the operational preorder and the denotational one is observed through the isomorphism between $\mathcal{L}(\mathcal{L}(X))$ and $\mathcal{SV}_{\leq 1}X$ noticed by Keimel and Plotkin (2017) and Goubault-Larrecq (2016)

$$\Lambda\colon A \mapsto \left(f \mapsto \inf_{\mu \in A} \int_{n \in \mathbb{N}} f(n) d\mu \right)$$

Missing

What I did not tell

Probabilities and non-determinism are still a thing

- Call-by-push-value and full abstraction for PCF with probabilities and non-determinism Goubault-Larrecq (2019)
- Weak distributive laws allow to combine “naturally” probabilities and non-determinism Goy and Petrisan (2020)
- And many more!

Thank you ☺

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