Locally Grown Preservation Theorems

Aliaume Lopez
Ph.D Student of Sylvain Schmitz and Jean Goubault-Larrecq

Thursday, February 2nd, 2023

LaBRI, Bordeaux

université PARIS-SACLAY
école normale supérieure paris–saclay

LMF Laboratoire Méthodes Formelles

IRIF INSTITUT DE RECHERCHE EN INFORMATIQUE FONDAMENTALE

Université de Paris

As in the locality of FO

Locally Grown Preservation Theorems

As in Łós-Tarski

Aliaume Lopez
Ph.D Student of Sylvain Schmitz and Jean Goubault-Larrecq

Thursday, February 2nd, 2023

LaBRI, Bordeaux

What to expect from this talk

♦ **New things**

New "positive" Gaifman Normal Form

New local to global relativisation of Łós-Tarski

New classes where Łós-Tarski relativises

Preservation Theorems 101

Logic, models, classes and fragments

Logic, models, satisfiability

Setting: σ a finite relational signature

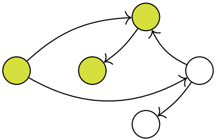
Logic first order logic, *a.k.a.* $\mathsf{FO}[\sigma]$.

Models relational structures, *a.k.a.* $\mathsf{Struct}(\sigma)$.

Setting: σ a finite relational signature

Logic first order logic, a.k.a. $\mathrm{FO}[\sigma]$.

Models relational structures, a.k.a. $\mathrm{Struct}(\sigma)$.



$\not\models \forall x.\exists y.x \to y \wedge P(y)$

Figure 1: How to check that a models satisfies a sentence over $\sigma \triangleq \{\to, P\}$.

Setting: σ a finite relational signature

Logic first order logic, a.k.a. $\mathrm{FO}[\sigma]$.

Models relational structures, a.k.a. $\mathrm{Struct}(\sigma)$.

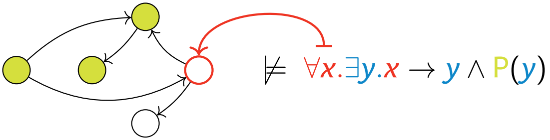


Figure 1: How to check that a models satisfies a sentence over $\sigma \triangleq \{\to, \mathrm{P}\}$.

Setting: σ a finite relational signature

Logic first order logic, a.k.a. $\mathrm{FO}[\sigma]$.

Models relational structures, a.k.a. $\mathrm{Struct}(\sigma)$.

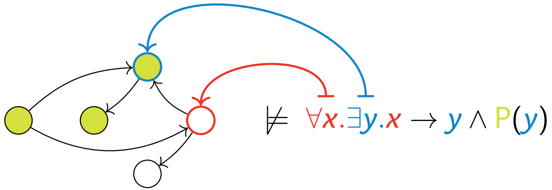


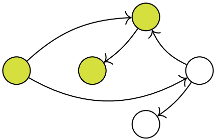
Figure 1: How to check that a models satisfies a sentence over $\sigma \triangleq \{\to, \mathrm{P}\}$.

Logic, models, satisfiability

Setting: σ a finite relational signature

Logic first order logic, a.k.a. $\mathrm{FO}[\sigma]$.

Models relational structures, a.k.a. $\mathrm{Struct}(\sigma)$.



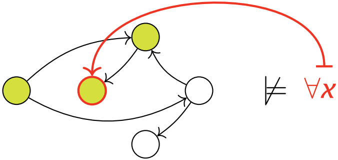
$$\not\models \forall x.\exists y.x \to y \wedge \mathrm{P}(y)$$

Figure 1: How to check that a models satisfies a sentence over $\sigma \triangleq \{\to, \mathrm{P}\}$.

Setting: σ *a finite relational signature*

Logic first order logic, a.k.a. $\mathrm{FO}[\sigma]$.

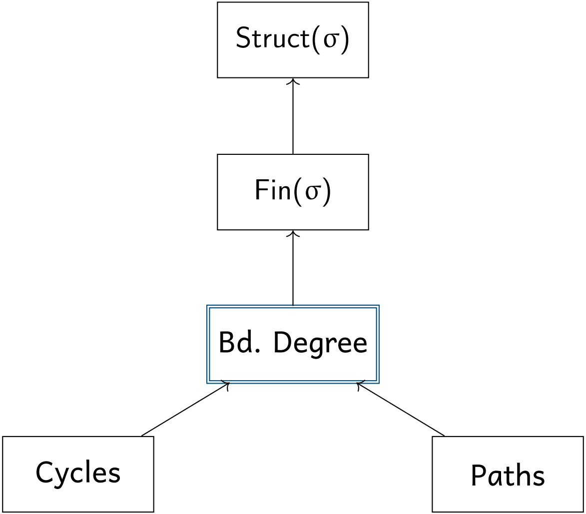
Models relational structures, a.k.a. $\mathrm{Struct}(\sigma)$.



$\not\models \forall x. \exists y. x \to y \wedge P(y)$

Figure 1: How to check that a models satisfies a sentence over $\sigma \triangleq \{\to, P\}$.

Classes of structures



Some fragments of FO

Name of the fragment	Example
FO	$\forall x.\exists y.\neg(E(x,y)) \vee x \neq y$
EFO	$\exists x.\exists y.\neg(E(x,y)) \vee x \neq x$
$\mathrm{UCQ}^{\neq}$	$\exists x.\exists y.E(x,y) \vee x \neq y$
UCQ	$\exists x.\exists y.E(x,y) \vee x = x$
CQ	$\exists x.\exists y.E(x,y) \wedge E(x,x)$

Let $\mathsf{F} \subseteq \mathsf{FO}$.

♦ **Ordering models using sentences**

$M \leq_{\mathsf{F}} M'$ when for all $\phi \in \mathsf{F}$, $M \models \phi$ implies $M' \models \phi$.

Let $\mathsf{F} \subseteq \mathsf{FO}$.

♦ **Ordering models using sentences**

$M \leq_{\mathsf{F}} M'$ when for all $\phi \in \mathsf{F}$, $M \models \phi$ implies $M' \models \phi$.

♣ **With** $\mathsf{F} = \mathsf{FO}$

Let M, M' be two finite structures. Then $M \leq_{\mathsf{FO}} M' \Leftrightarrow M \simeq M'$.

♦ Ordering models using formulas

$M \to_{\mathrm{F}} M'$ when there exists a map $h\colon M \to M'$ such that for every $\phi \in F$, $M, \vec{x} \models \phi$ implies $M', h(\vec{x}) \models \phi$.

♦ **Ordering models using formulas**

$M \to_{\mathsf{F}} M'$ when there exists a map $h\colon M \to M'$ such that for every $\phi \in F$, $M, \vec{x} \models \phi$ implies $M', h(\vec{x}) \models \phi$.

♣ **With** $\mathsf{F} = \mathsf{EFO}$

Let $G \triangleq \mathbb{Z} \times \mathbb{Z}$ be a infinite grid. Then:

- $\biguplus_{\lambda \in \mathbb{R}} G \leq_{\mathsf{EFO}} G$,
- $\biguplus_{\lambda \in \mathbb{R}} G \not\to_{\mathsf{EFO}} G$.

Fragments and their orderings.

Fragment F	Specialisation	Symbol $\rightarrow_{\mathsf{F}}$
CQ	homomorphism	$\rightarrow$
UCQ	homomorphism	$\rightarrow$
$\mathsf{UCQ}^{\neq}$	substructure	$\subseteq$
EFO	extension	$\subseteq_i$
FOLoc	local elementary embedding	$\Rrightarrow$
FO	elementary extension	$\preceq$

Fragments and their orderings.

	Fragment F	**Specialisation**	**Symbol** $\to_{\mathsf{F}}$
	CQ	homomorphism	$\to$
	UCQ	homomorphism	$\to$
	$\mathsf{UCQ}^{\neq}$	substructure	$\subseteq$
This Talk!	EFO	extension	$\subseteq_i$
	FOLoc	local elementary embedding	$\Rrightarrow$
	FO	elementary extension	$\preceq$

Preservation Theorems 101

Example of Preservation Theorems

Upwards closed sets

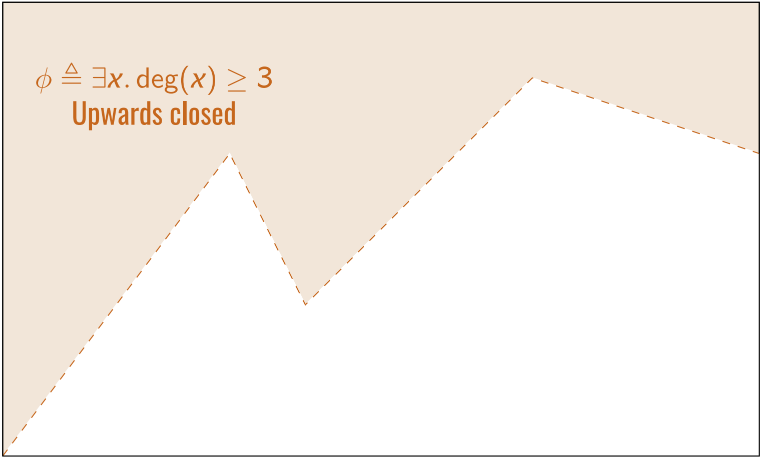


Figure 2: Finite graphs encoded using $\Sigma \triangleq \{E\}$

Upwards closed sets

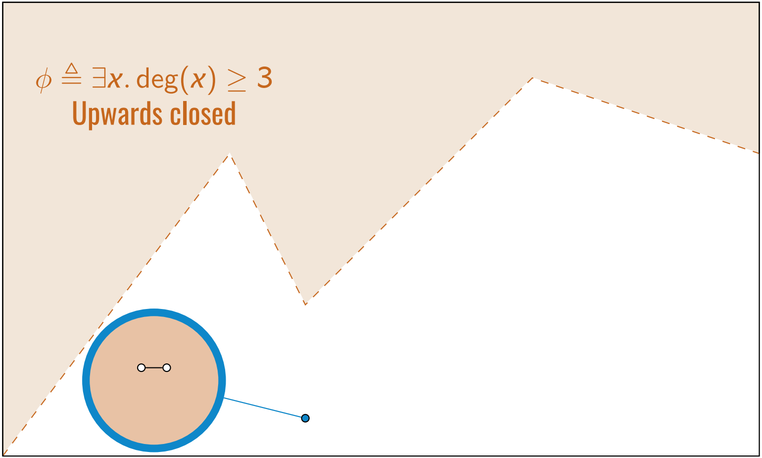


Figure 2: Finite graphs encoded using $\Sigma \triangleq \{E\}$

Upwards closed sets

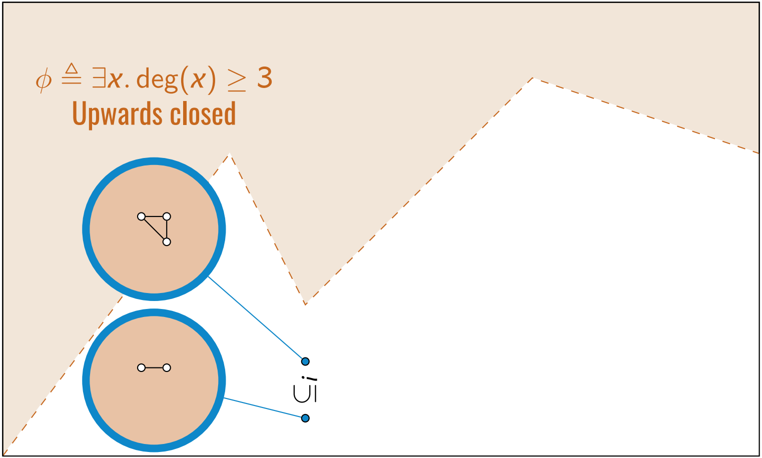


Figure 2: Finite graphs encoded using $\Sigma \triangleq \{E\}$

Upwards closed sets

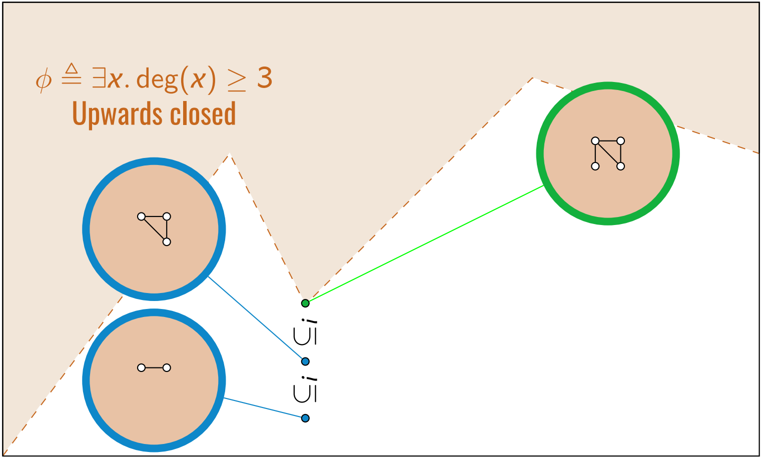


Figure 2: Finite graphs encoded using $\Sigma \triangleq \{E\}$

Upwards closed sets

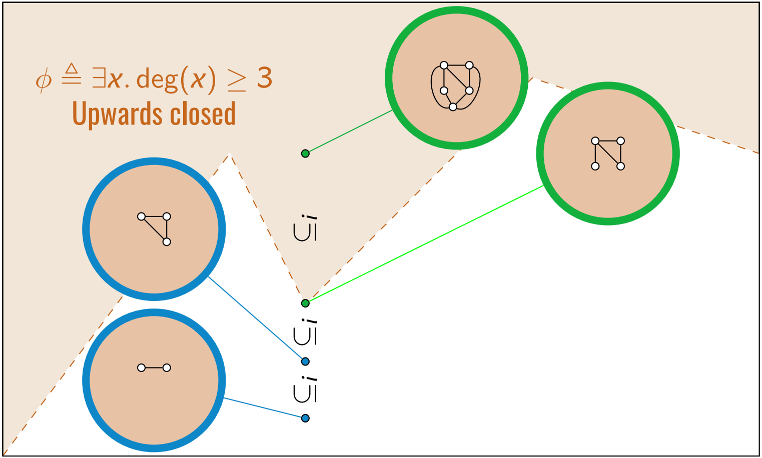


Figure 2: Finite graphs encoded using $\Sigma \triangleq \{E\}$

Preservation Under Extensions: Łoś (1955); Tarski (1954)

♦ "Query Optimisation" over a class $\mathcal{C}$

Input Some FO sentence φ

Promise $M \models \varphi \wedge M \subseteq_i M' \Rightarrow M' \models \varphi$

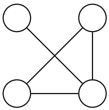
Output A simplified query (existential) over $\mathcal{C}$

♦ **"Query Optimisation" over a class $\mathcal{C}$**

Input there exists no vertex cover of size 1 in G

Promise $M \models \varphi \wedge M \subseteq_i M' \Rightarrow M' \models \varphi$

Output A simplified query (existential) over $\mathcal{C}$



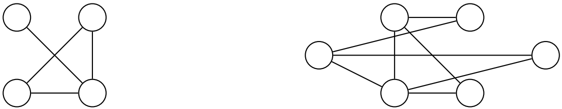
$G \models \varphi$

♦ **“Query Optimisation” over a class $\mathcal{C}$**

Input $\neg(\exists x.\forall yz.E(y,z) \implies z = y \lor z = x)$

Promise Covers restrict to induced subgraphs

Output A simplified query (existential) over $\mathcal{C}$



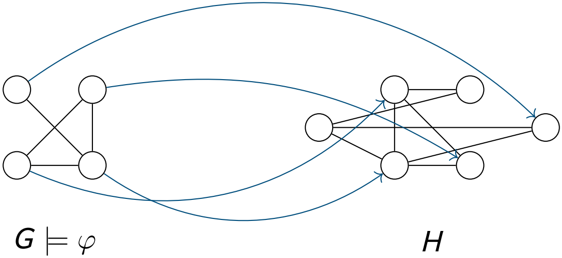
$G \models \varphi$ $\qquad$ H

- **"Query Optimisation" over a class $\mathcal{C}$**

Input $\neg(\exists x.\forall yz.E(y,z) \implies z = y \lor z = x)$

Promise Covers restrict to induced subgraphs

Output A simplified query (existential) over $\mathcal{C}$

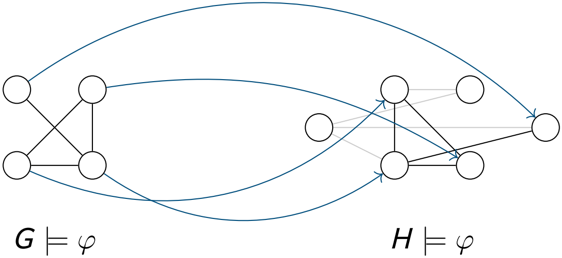


♦ **"Query Optimisation" over a class $\mathcal{C}$**

Input $\neg(\exists x. \forall yz. E(y,z) \implies z = y \lor z = x)$

Promise Covers restrict to induced subgraphs

Output A simplified query (existential) over $\mathcal{C}$

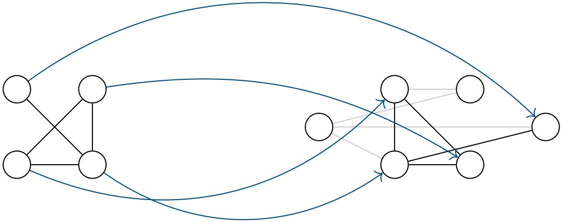


♦ **"Query Optimisation" over a class $\mathcal{C}$**

Input $\neg(\exists x.\forall yz.E(y,z) \implies z = y \lor z = x)$

Promise Covers restrict to induced subgraphs

Output Finitely many graphs M_i to check



$G \models \varphi$ $H \models \varphi$

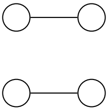
Preservation Under Extensions: Łoś (1955); Tarski (1954)

♦ **"Query Optimisation" over a class $\mathcal{C}$**

Input $\neg(\exists x.\forall yz.E(y,z) \implies z = y \vee z = x)$

Promise Covers restrict to induced subgraphs

Output Finitely many graphs M_i to check



M (for $\subseteq$)

Preservation Under Extensions: Łoś (1955); Tarski (1954)

Theorem (Łoś (1955); Tarski (1954))

This algorithm exists when $\mathcal{C} = \mathrm{Struct}(\sigma)$.

Proof.

- an equivalent existential sentence exists (heavy use of compactness)
- one can enumerate proofs $\vdash \psi \leftrightarrow \varphi$ with ψ existential. $\square$

Behind the scenes...

In general for $\mathcal{C} = \mathrm{Struct}(\sigma)$

When F is a reasonable subset of FO, a sentence ϕ preserved by $\to_{\mathsf{F}}$ can be rewritten as a sentence in $\exists$F.

Behind the scenes...

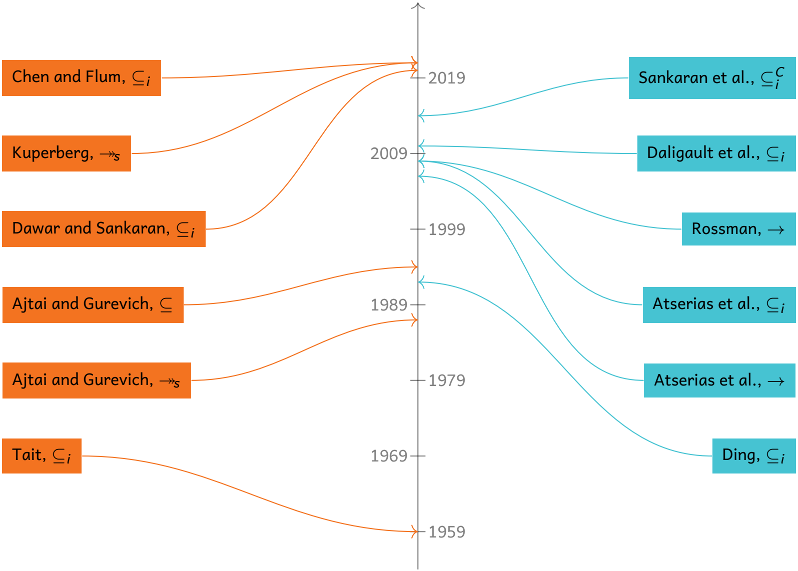
In general for $\mathcal{C} = \mathrm{Struct}(\sigma)$

When F is a reasonable subset of FO, a sentence ϕ preserved by $\to_{\mathsf{F}}$ can be rewritten as a sentence in $\exists$F.

♦ A non exhaustive list

Name	Specialisation	Fragment
Łós-Tarski	$\subseteq_i$	EFO
Tarski-Lyndon	$\subseteq$	$\mathrm{UCQ}^{\neq}$
H.P.T.	$\to$	UCQ

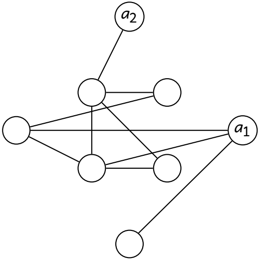
In the finite ($\mathcal{C} \subseteq \mathsf{Fin}(\sigma)$), the picture is not so clear



Gaifman Locality

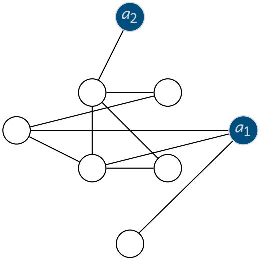
Neighbourhoods

Locality in a graph



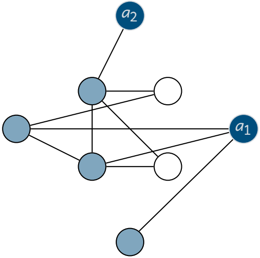
A structure A.

Locality in a graph



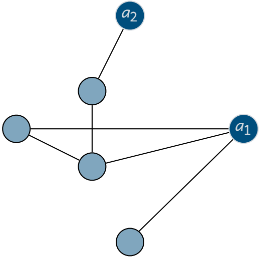
A structure A, with 2 selected nodes.

Locality in a graph



A structure A, with 2 selected nodes, and a 1-local neighborhood.

Locality in a graph



$$\mathcal{N}_A(a_1 a_2, 1) \subseteq_i A.$$

$$
\begin{aligned}
\mathcal{N}_A(\vec{a}, r) &\triangleq \{b \in A \colon \exists a \in \vec{a}.\, \mathsf{dist}_A(a, b) \leq r\} \\
&= \bigcup_{a \in \vec{a}} \mathcal{N}_A(a, r)
\end{aligned}
$$

$$\mathcal{N}_A(\vec{a}, r) \triangleq \{b \in A \colon \exists a \in \vec{a}.\, \mathrm{dist}_A(a, b) \leq r\}$$
$$= \bigcup_{a \in \vec{a}} \mathcal{N}_A(a, r)$$

♣ **What about higher arities?**

Representing relations with arity greater than 2.

The Gaifman Graph

$$\rightarrow(x,y) \triangleq \bigvee_{(R,n)\in\sigma} \exists z_1,\ldots,z_n.R(z_1,\ldots,z_n) \wedge \bigvee_{1\le i,j\le n}^{n} x = z_i \wedge y = z_j$$

Representing relations with arity greater than 2.

The Gaifman Graph

$$\rightarrow (x, y) \triangleq \bigvee_{(R,n)\in\sigma} \exists z_1, \ldots, z_n . R(z_1, \ldots, z_n) \wedge \bigvee_{1\leq i,j\leq n}^{n} x = z_i \wedge y = z_j$$

2	2	1
2	1	0
2	0	2
1	2	0
1	1	2
1	0	1
0	2	2
0	1	1
0	0	0

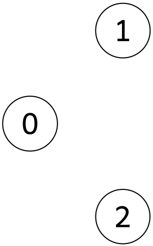


Figure 3: From a table to a graph.

The Gaifman Graph

$$\rightarrow (x, y) \triangleq \bigvee_{(R,n)\in\sigma} \exists z_1, \ldots, z_n . R(z_1, \ldots, z_n) \wedge \bigvee_{1 \leq i,j \leq n}^{n} x = z_i \wedge y = z_j$$

R			
	2	2	1
	2	1	0
	2	0	2
	1	2	0
	1	1	2
	1	0	1
	0	2	2
	0	1	1
	0	0	0

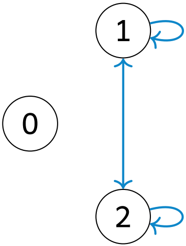


Figure 3: From a table to a graph.

The Gaifman Graph

$$\rightarrow (x, y) \triangleq \bigvee_{(R,n)\in\sigma} \exists z_1, \ldots, z_n . R(z_1, \ldots, z_n) \wedge \bigvee_{1\leq i,j\leq n}^{n} x = z_i \wedge y = z_j$$

R	2	2	1
	2	1	0
	2	0	2
R	1	2	0
	1	1	2
	1	0	1
	0	2	2
	0	1	1
	0	0	0

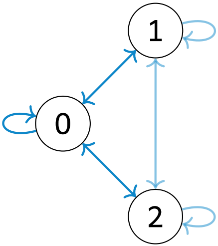


Figure 3: From a table to a graph.

Representing relations with arity greater than 2.

The Gaifman Graph

$$\rightarrow(x,y) \triangleq \bigvee_{(R,n)\in\sigma} \exists z_1,\ldots,z_n.R(z_1,\ldots,z_n) \wedge \bigvee_{1\leq i,j\leq n}^{n} x = z_i \wedge y = z_j$$

R	2	2	1
	2	1	0
	2	0	2
R	1	2	0
	1	1	2
	1	0	1
	0	2	2
	0	1	1
	0	0	0

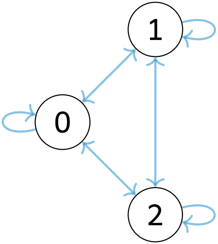


Figure 3: From a table to a graph.

Gaifman Locality

Local Formulas

Two equivalent definitions of FOLoc

Semantically local

$A, \vec{a} \models \phi(\vec{x})$ if and only if $\mathcal{N}_A(\vec{a}, r), \vec{a} \models \phi(\vec{x})$.

Two equivalent definitions of FOLoc

Semantically local

$A, \vec{a} \models \phi(\vec{x})$ if and only if $\mathcal{N}_A(\vec{a}, r), \vec{a} \models \phi(\vec{x})$.

Syntactically local

$A, \vec{a} \models \phi(\vec{x})$ if and only if $A, \vec{a} \models (\!|\phi|\!)^r_{\vec{x}}$.

Two equivalent definitions of FOLoc

Semantically local

$A, \vec{a} \models \phi(\vec{x})$ if and only if $\mathcal{N}_A(\vec{a}, r), \vec{a} \models \phi(\vec{x})$.

Syntactically local

$A, \vec{a} \models \phi(\vec{x})$ if and only if $A, \vec{a} \models (\!|\phi|\!)^r_{\vec{x}}$.

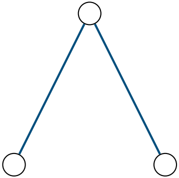
♦ **A non-trivial sentence cannot be local**

Fragments and their orderings (BIS)

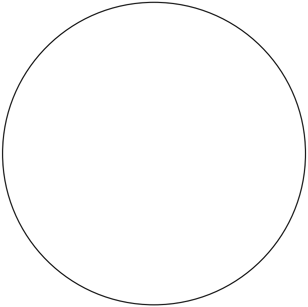
Fragment F	**Specialisation**	**Symbol** $\to_F$
CQ	homomorphism	$\to$
UCQ	homomorphism	$\to$
$UCQ^{\neq}$	substructure	$\subseteq$
EFO	extension	$\subseteq_i$
FOLoc	local elementary embedding	$\Rrightarrow$
FO	elementary extension	$\preceq$

What is a local elementary embedding?

Let $A, B \in \mathrm{Fin}(\sigma)$ such that $A \Rrightarrow B$.



The structure A



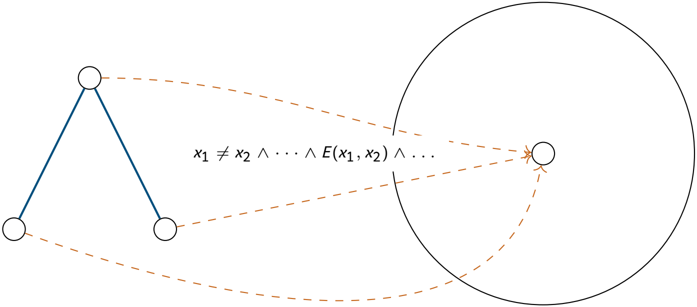
The structure B

♣ In the finite...

$A \Rrightarrow B$ if and only if there exists B' such that $A \uplus B' = B$.

What is a local elementary embedding?

Let $A, B \in \mathrm{Fin}(\sigma)$ such that $A \Rrightarrow B$.



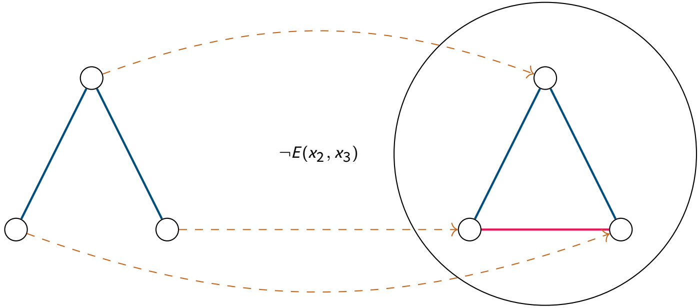
The structure A The structure B

♣ **In the finite...**

$A \Rrightarrow B$ if and only if there exists B' such that $A \uplus B' = B$.

What is a local elementary embedding?

Let $A, B \in \mathrm{Fin}(\sigma)$ such that $A \Rrightarrow B$.



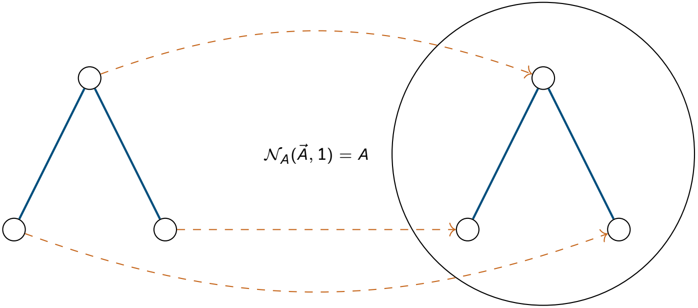
The structure A The structure B

♣ In the finite...

$A \Rrightarrow B$ if and only if there exists B' such that $A \uplus B' = B$.

What is a local elementary embedding?

Let $A, B \in \mathrm{Fin}(\sigma)$ such that $A \Rrightarrow B$.



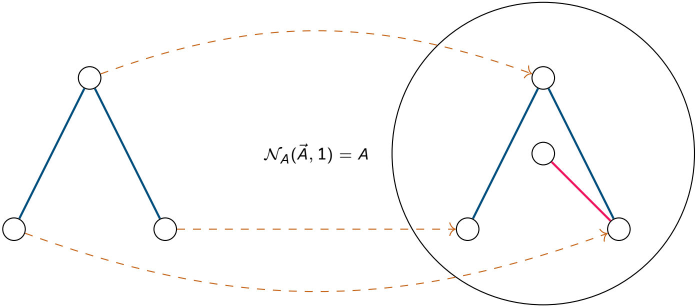
The structure A The structure B

♣ In the finite...

$A \Rrightarrow B$ if and only if there exists B' such that $A \uplus B' = B$.

What is a local elementary embedding?

Let $A, B \in \mathrm{Fin}(\sigma)$ such that $A \Rrightarrow B$.



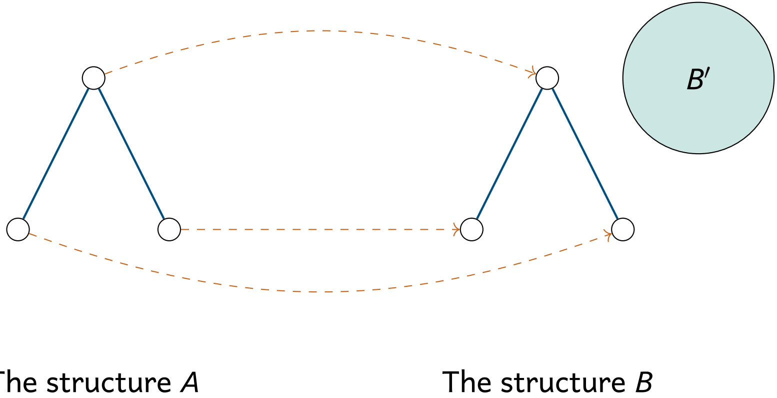
The structure A The structure B

♣ In the finite...

$A \Rrightarrow B$ if and only if there exists B' such that $A \uplus B' = B$.

What is a local elementary embedding?

Let $A, B \in \mathrm{Fin}(\sigma)$ such that $A \Rrightarrow B$.



♣ In the finite...

$A \Rrightarrow B$ if and only if there exists B' such that $A \uplus B' = B$.

A trivial theorem

Theorem ((L., 2022, Over $\mathsf{Struct}(\sigma)$))

First order sentences preserved under local elementary embeddings ($\Rrightarrow$) are existential local sentences ($\exists\mathsf{FOLoc}$).

A trivial theorem

Theorem ((L., 2022, Over $\mathrm{Struct}(\sigma)$))

First order sentences preserved under local elementary embeddings ($\Rrightarrow$) are existential local sentences ($\exists\mathsf{FOLoc}$).

♣ existential 0-local = existential

A locality theorem

Theorem (Gaifman (1982))

*Every first order sentence (*FO*) is equivalent to a boolean combination of basic local sentences.*

♦ **Basic Local Sentence**

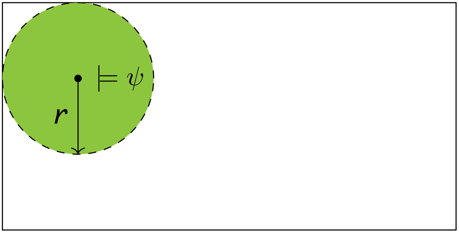
$$\exists_r^{\geq n} x.\psi(x)$$

Theorem (Gaifman (1982))

Every first order sentence (FO) is equivalent to a boolean combination of basic local sentences.

♦ Basic Local Sentence

$$\exists_r^{\geq n} x.\psi(x)$$

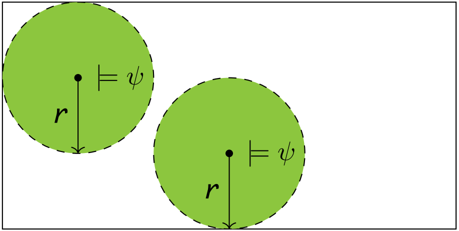


Theorem (Gaifman (1982))

Every first order sentence (FO) is equivalent to a boolean combination of basic local sentences.

♦ Basic Local Sentence

$$\exists_r^{\geq n} x.\psi(x)$$



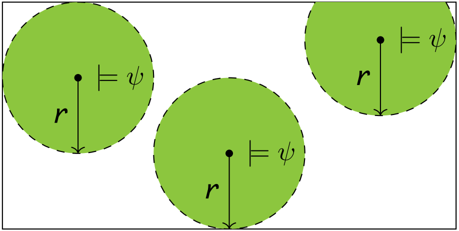
A locality theorem

Theorem (Gaifman (1982))

*Every first order sentence (*FO*) is equivalent to a boolean combination of basic local sentences.*

♦ Basic Local Sentence

$$\exists_r^{\geq n} x.\psi(x)$$



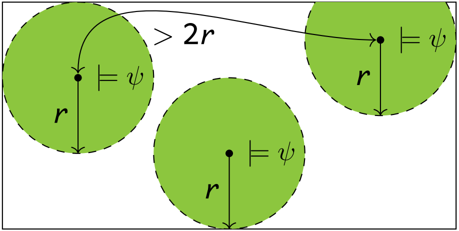
A locality theorem

Theorem (Gaifman (1982))

Every first order sentence (FO) is equivalent to a boolean combination of basic local sentences.

♦ Basic Local Sentence

$$\exists_r^{\geq n} x.\psi(x)$$



Equivalent fragments

Fragment	**Shape**
Basic local	$\exists_r^{\geq n} x. (\!|\psi|\!)_r^x$
Existential local	$\exists \vec{x}. (\!|\psi|\!)_r^{\vec{x}}$

Theorem (L. (2022))

The following fragments are equivalent over any $\mathcal{C} \subseteq \mathrm{Struct}(\sigma)$*:*

- *Positive boolean combinations of basic local sentences*
- *Existential local sentences*

Equivalent fragments

Fragment	Shape
Basic local	$\exists_r^{\geq n} x. (\!|\psi|\!)_r^x$
Existential local	$\exists \vec{x}. (\!|\psi|\!)_r^{\vec{x}}$

Theorem (L. (2022))

The following fragments are equivalent over any $\mathcal{C} \subseteq \mathrm{Struct}(\sigma)$*:*

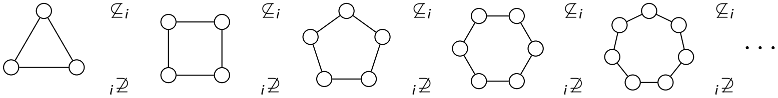
- *Positive boolean combinations of basic local sentences*
- *Existential local sentences*

♣ Preservation under local embeddings

Sentences preserved under local elementary embeddings ($\Rrightarrow$) are equivalent to a *positive* Boolean combination of basic local sentences.

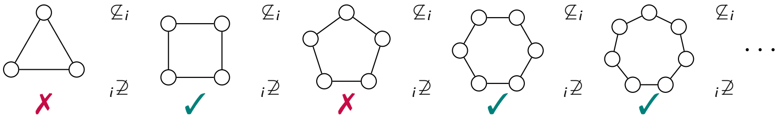
♦ Ultimately

All cycles locally look the same



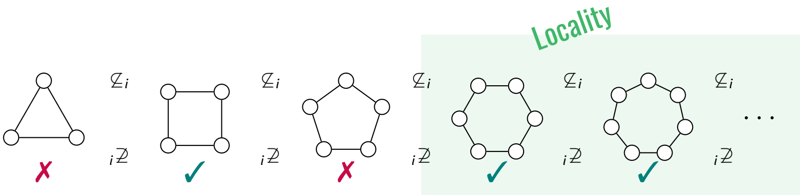
♦ Ultimately

All cycles locally look the same



♦ Ultimately

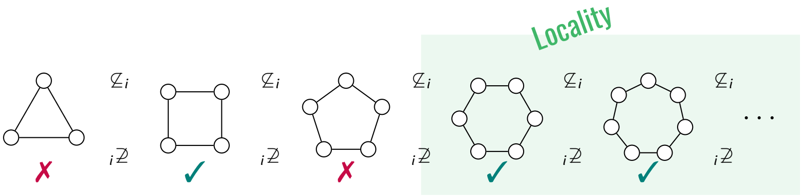
All cycles locally look the same



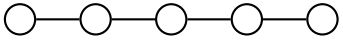
Application to preservation under extensions

♦ Ultimately

All cycles locally look the same



Exactly one 2-neighbourhood



A mostly wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a class $\mathcal{C}$**

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under local elementary embeddings,
3. Hence $\phi \in \exists$FOLoc,
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential.

A mostly wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a class $\mathcal{C}$**

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under local elementary embeddings, ✓
3. Hence $\phi \in \exists$FOLoc,
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential.

A mostly wrong proof scheme

♣ To prove that Łós-Tarski relativises to a class $\mathcal{C}$

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under local elementary embeddings, ✓
3. Hence $\phi \in \exists$FOLoc, ✗
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential.

A mostly wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a class $\mathcal{C}$**

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under local elementary embeddings, ✓
3. Hence $\phi \in \exists$FOLoc, ✗
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential. ✗

Relativisation to the finite

Failure in the finite

A connectivity issue

Let ϕ_{CC} be such that $A \models \phi_{CC}$ if and only if A as more than one connected component.

The following is preserved under $\uplus$

$\phi_B \triangleq \forall x. \neg B(x) \vee \phi_{CC}$

A connectivity issue

Let ϕ_{CC} be such that $A \models \phi_{CC}$ if and only if A as more than one connected component.

The following is preserved under $\uplus$

$\phi_B \triangleq \forall x. \neg B(x) \vee \phi_{CC}$

♦ Cannot be rewritten in $\exists$FOLoc

Distinguish two connected components with one black node, or one connected component with one black node.

A connectivity issue

Let ϕ_{CC} be such that $A \models \phi_{CC}$ if and only if A as more than one connected component.

The following is preserved under $\uplus$

$\phi_B \triangleq \forall x. \neg B(x) \vee \phi_{CC}$

♦ Cannot be rewritten in $\exists$FOLoc

Distinguish two connected components with one black node, or one connected component with one black node.

♣ Wait, is ϕ_{CC} definable ??

It generalises

Theorem ((L., 2022, in the finite))

There exists $\phi \in \mathsf{FO}$ *preserved under disjoint unions over* $\mathsf{Fin}(\sigma)$ *but not equivalent to an existential local sentence over* $\mathsf{Fin}(\sigma)$.

It generalises

Theorem ((L., 2022, in the finite))

There exists $\phi \in \mathsf{FO}$ *preserved under disjoint unions over* $\mathsf{Fin}(\sigma)$ *but not equivalent to an existential local sentence over* $\mathsf{Fin}(\sigma)$*.*

Theorem (L. (2022))

The following are undecidable:

- *Is* φ *preserved under disjoint unions over* $\mathsf{Fin}(\sigma)$*?*
- *Is* φ *equivalent to an existential local sentence over* $\mathsf{Fin}(\sigma)$*?*

It generalises

Theorem ((L., 2022, in the finite))

There exists $\phi \in$ FO preserved under disjoint unions over $\mathrm{Fin}(\sigma)$ but not equivalent to an existential local sentence over $\mathrm{Fin}(\sigma)$.

Theorem (L. (2022))

The following are undecidable:

- *Is φ preserved under disjoint unions over $\mathrm{Fin}(\sigma)$?*
- *Is φ equivalent to an existential local sentence over $\mathrm{Fin}(\sigma)$?*

Theorem (L. (2022))

There exists no algorithm that given $\phi \in$ FO and the promise that an equivalent local sentence exists over $\mathrm{Fin}(\sigma)$, comptes such a sentence.

Relativisation to the finite

Salvaging some relativisation

Fixing some parameters r, q, k

Parameters of a local sentence

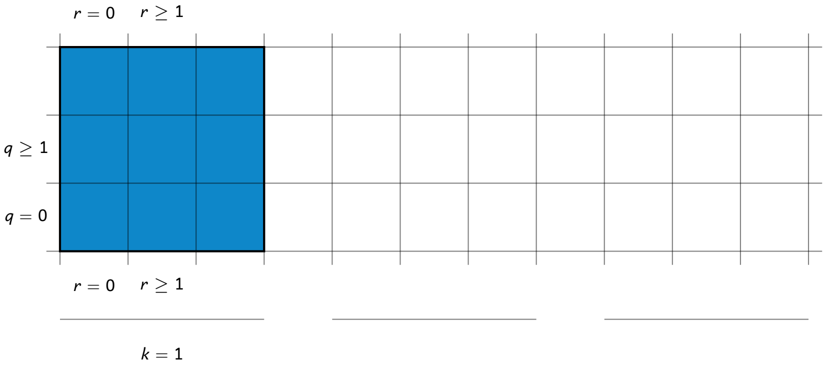
$$\exists x_1, \ldots, x_k . \llparenthesis Q_1 y_1 . Q_2 y_2 . \ldots . Q_q y_q . \theta(\vec{x}, \vec{y}) \rrparenthesis_r^{\vec{x}}$$

♣ Fixing all parameters…

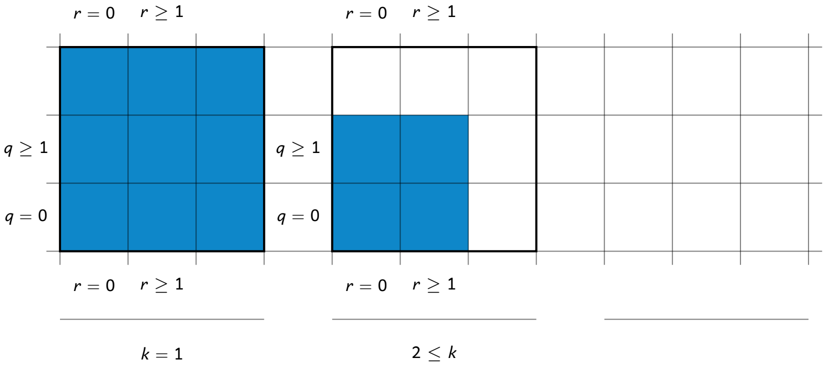
A sentence φ preserved under (r, q, k)-local elementary embeddings is equivalent to an existential local sentence.

Expanded Cube

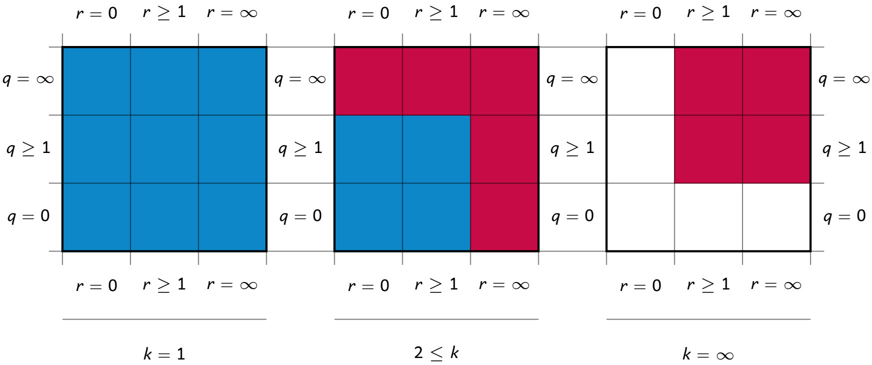
Expanded Cube



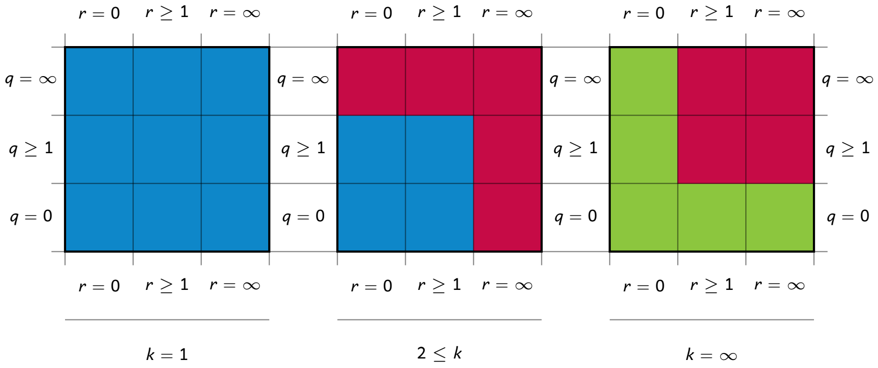
Expanded Cube



Expanded Cube



Expanded Cube



L. (2022)

Let $\mathcal{C} \subseteq \mathrm{Fin}(\sigma)$ be hereditary and closed under disjoint unions ($\uplus$).
Let $\phi \in$ FO be preserved under extensions, then ϕ is preserved under (r, q, k)-local elementary embeddings for some $0 \leq r, q, k < \infty$.

The saviour of our proof scheme

L. (2022)

Let $\mathcal{C} \subseteq \mathsf{Fin}(\sigma)$ be hereditary *and* closed under disjoint unions ($\uplus$). Let $\phi \in \mathsf{FO}$ be preserved under extensions, then ϕ is preserved under (r, q, k)-local elementary embeddings for some $0 \leq r, q, k < \infty$.

♣ **Almost Łós-Tarski!**

A partially wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a hereditary class $\mathcal{C}$ closed under disjoint unions ($\uplus$)**

1. Consider $\phi \in \mathrm{FO}$ preserved under $\subseteq_i$,
2. It is preserved under (r, q, k)-local elementary embeddings for some $0 \leq r, q, k < \infty$
3. We know that $\phi \in \exists\mathrm{FOLoc}$,
4. A sentence in $\exists\mathrm{FOLoc}$ preserved under $\subseteq_i$ must be existential.

A partially wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a hereditary class $\mathcal{C}$ closed under disjoint unions ($\uplus$)**

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under (r, q, k)-local elementary embeddings for some $0 \leq r, q, k < \infty$ ✓
3. We know that $\phi \in \exists$FOLoc,
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential.

A partially wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a hereditary class $\mathcal{C}$ closed under disjoint unions ($\uplus$)**

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under (r, q, k)-local elementary embeddings for some $0 \leq r, q, k < \infty$ ✓
3. We know that $\phi \in \exists$FOLoc, ✓
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential.

A partially wrong proof scheme

♣ **To prove that Łós-Tarski relativises to a hereditary class $\mathcal{C}$ closed under disjoint unions ($\uplus$)**

1. Consider $\phi \in$ FO preserved under $\subseteq_i$,
2. It is preserved under (r, q, k)-local elementary embeddings for some $0 \leq r, q, k < \infty$ ✓
3. We know that $\phi \in \exists$FOLoc, ✓
4. A sentence in $\exists$FOLoc preserved under $\subseteq_i$ must be existential. ✗

Localising Preservation

Sufficient and necessary condition

Locally satisfying a property?

$$\mathrm{Local}(\mathcal{C}, r, k) \triangleq \{\mathcal{N}_A(\vec{a}, r) \colon A \in \mathcal{C}, \vec{a} \in A^k\}$$

Locally satisfying a property?

$$\mathsf{Local}(\mathcal{C}, r, k) \triangleq \{\mathcal{N}_A(\vec{a}, r) \colon A \in \mathcal{C}, \vec{a} \in A^k\}$$

♣ Localise Bounded Degree

$\mathcal{C}$ is of bounded degree if and only if $\mathsf{Local}(\mathcal{C}, r, k)$ is finite for all $k, r \geq 0$, i.e., *locally* finite

Locally satisfying a property?

$$\text{Local}(\mathcal{C}, r, k) \triangleq \{\mathcal{N}_A(\vec{a}, r) \colon A \in \mathcal{C}, \vec{a} \in A^k\}$$

♣ Localise Bounded Degree

$\mathcal{C}$ is of bounded degree if and only if $\text{Local}(\mathcal{C}, r, k)$ is finite for all $k, r \geq 0$, i.e., *locally* finite

Theorem (Atserias et al. (2008))

Hereditary classes that are locally finite and closed under $\uplus$, satisfy preservation under extensions.

L. (2022)

For a hereditary classe of finite structures $\mathcal{C}$, the following are equivalent

- Łós-Tarski relativises locally (i.e. relativises to $\mathsf{Local}(\mathcal{C}, r, k)$ for all $r, k \geq 0$),
- Existential local sentences preserved under extensions are equivalent to existential sentences.

L. (2022)

For a hereditary classe of finite structures $\mathcal{C}$, the following are equivalent

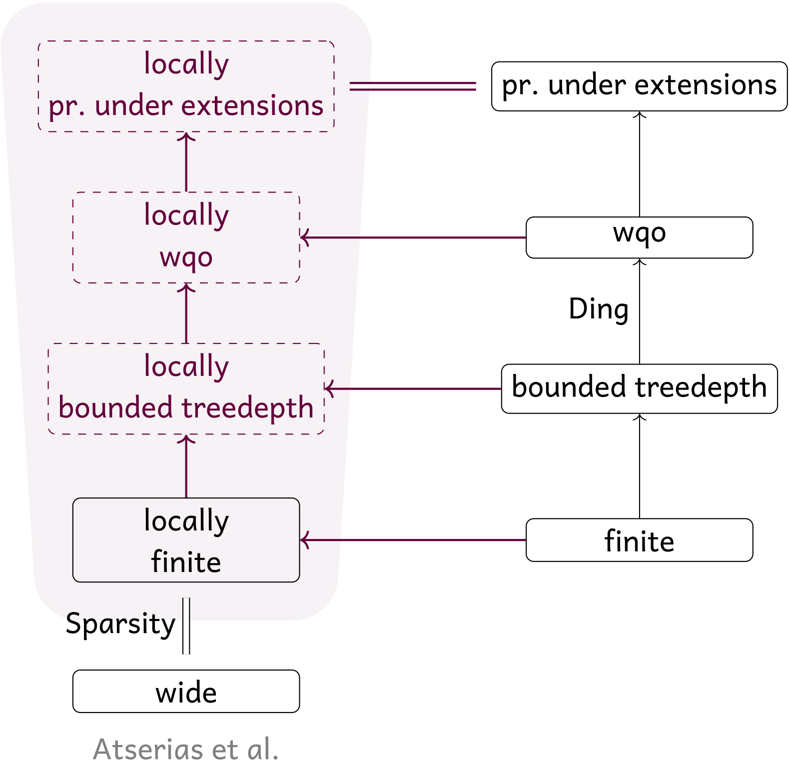
- Łós-Tarski relativises locally (i.e. relativises to $\mathsf{Local}(\mathcal{C}, r, k)$ for all $r, k \geq 0$),
- Existential local sentences preserved under extensions are equivalent to existential sentences.

♦ Main Result (assuming hereditary and closed under disjoint unions)

Łós-Tarski relativises to $\mathcal{C}$ if and only if it relativises to $\mathsf{Local}(\mathcal{C}, r, k)$ for all $r, k \geq 0$.

Localising Preservation

Applications, *a.k.a.*, “Was it worth it?”



locally
pr. under extensions
pr. under extensions
locally
wqo
wqo
Ding
locally
bounded treedepth
bounded treedepth
locally
finite
finite
Sparsity
wide
Atserias et al.

Thank you!

Ajtai, M. and Gurevich, Y. (1987). Monotone versus positive. *J. ACM*, 34(4):1004–1015.

Ajtai, M. and Gurevich, Y. (1994). Datalog vs first-order logic. *J. Comput. Syst. Sci.*, 49(3):562–588.

Atserias, A., Dawar, A., and Grohe, M. (2008). Preservation under extensions on well-behaved finite structures. *SIAM J. Comput.*, 38(4):1364–1381.

Atserias, A., Dawar, A., and Kolaitis, P. G. (2006). On preservation under homomorphisms and unions of conjunctive queries. *J. ACM*, 53(2):208–237.

Chen, Y. and Flum, J. (2021). Forbidden induced subgraphs and the łoś–tarski theorem. In *Proc. LICS'21*, pages 1–13.

Daligault, J., Rao, M., and Thomassé, S. (2010). Well-quasi-order of relabel functions. *Order*, 27(3):301–315.

Dawar, A. and Sankaran, A. (2021). Extension Preservation in the Finite and Prefix Classes of First Order Logic. In *Proc. CSL'21*, volume 183 of *LIPIcs*, pages 18:1–18:13. LZI.

Ding, G. (1992). Subgraphs and well-quasi-ordering. *J. Graph Theory*, 16:489–502.

Gaifman, H. (1982). On local and non-local properties. In *Proc. Herbrand Symposium*, volume 107 of *Studies in Logic and the Foundations of Mathematics*, pages 105–135. Elsevier.

Kuperberg, D. (2021). Positive first-order logic on words. In *Proc. LICS'21*, pages 1–13. IEEE.

L., A. (2022). When Locality Meets Preservation. In *Proceedings of the 37th Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '22, pages 1–14, New York, NY, USA. Association for Computing Machinery.

Łoś, J. (1955). On the extending of models (I). *Fund. Math.*, 42(1):38–54.

Rossman, B. (2008). Homomorphism preservation theorems. *J. ACM*, 55(3):15:1–15:53.

Sankaran, A., Adsul, B., and Chakraborty, S. (2014). A generalization of the łoś-tarski preservation theorem over classes of finite structures. In *International Symposium on Mathematical Foundations of Computer Science*, pages 474–485. Springer.

Tait, W. W. (1959). A counterexample to a conjecture of Scott and Suppes. *J. Symb. Logic*, 24(1):15–16.

Tarski, A. (1954). Contributions to the theory of models. I. *Indag. Math. (Proc.)*, 57:572–581.