# Compositional Techniques for Preservation Theorems over Classes of Finite Structures

Leveraging Tools from Topology and Finite Model Theory


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November 6th 2023

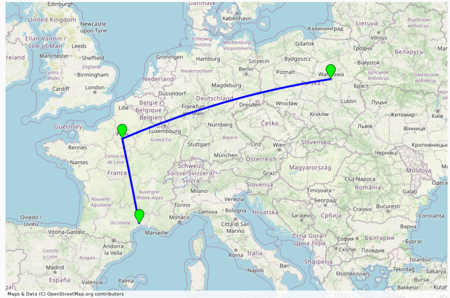
BOREAL Seminar, LIRMM, Montpellier, France

# Introduction

## Short Bio

**Where do I come from**

1. High school in Montpellier.
2. Masters at the MPRI in Paris.
3. Ph.D in Finite Model Theory and Topology.
4. With the hope that it could be applied to databases.
5. Currently in Warsaw to study MSO transductions.

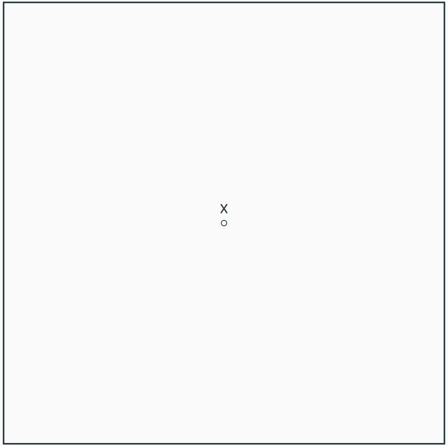


**Model Theory and Databases (see Codd [Cod70])**

| Databases | Model Theory |
|---|---|
| Infinite database | (Relational) structure |
| Database | Finite structure |
| Query | First order sentence |

**Simplifications:** no functional symbols, queries without constants.

# Topology? Finite Model Theory?




**Topological spaces**

A set X together with a collection $\mathcal{O}(X)$ of "open" subsets, closed under finite intersections and arbitrary unions.

**Examples:**

- Metric spaces $(X, d)$ — open ball topology
- Ordered spaces $(X, \leq)$ — Alexandroff topology

**Warning:** not all spaces are metric, not all spaces are separable, not all compact subsets are closed, etc...

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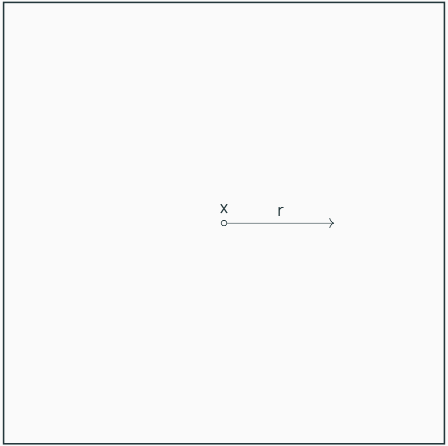


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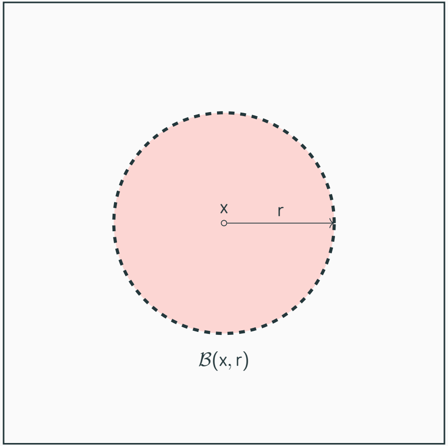


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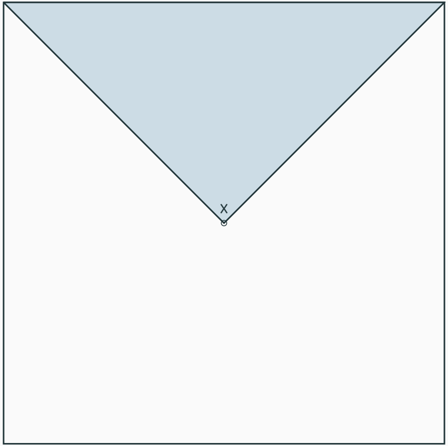


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  $\rightarrow$ this talk!

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# Introduction

## Query answering on classes of databases

### Representation of undirected graphs

- $V$ is the domain
- $E$ is a (symmetric) table with two columns

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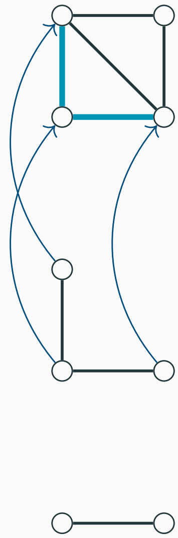
1. Adding new elements — extensions ($\subseteq_i$)
2. (and) Adding new rows — substructures ($\subseteq$)
3. (and) Merging elements — homomorphic images ($\preceq_h$)



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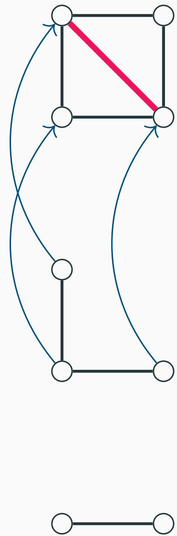

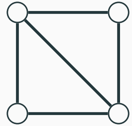

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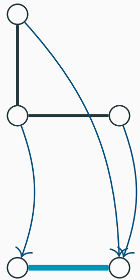

### Alternative definition

Morphisms that preserve conjunctive queries (resp. with $\neq$, resp. with $\neg$)...

**In general**

$\mathsf{F} \subseteq \mathsf{FO}$ **leads to a class** $\rightarrow_{\mathsf{F}}$ **of morphisms**

# Certain answers (in general)

$$\mathrm{Cert}_Q^{\rightarrow_F}(A) := \bigcap_{h \colon A \rightarrow_F B} \{\vec{a} \in A \mid h(\vec{a}) \in Q(B)\}$$

Answers, using data available in the database, that must appear, regardless of the completion.

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**How hard is computing certain answers?**

**In general:** undecidable (reduces to finite validity)

**We need to:** compute a greatest lower bound in the ordering $\rightarrow_F$ (Libkin [Lib11]).

**Candidate:** naïvely evaluate the query on the incomplete database and "hope for the best".

**Theorem (Gheerbrant, Libkin, and Sirangelo [GLS14, Corollary 4.14])**

- under OWA, naïve evaluation works for Q iff Q is preserved under homomorphisms;
- under CWA, naïve evaluation works for Q iff Q is preserved under strong onto homomorphisms;
- under WCWA, naïve evaluation works for Q iff Q is preserved under onto homomorphisms.

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**Preservation theorems are close!**

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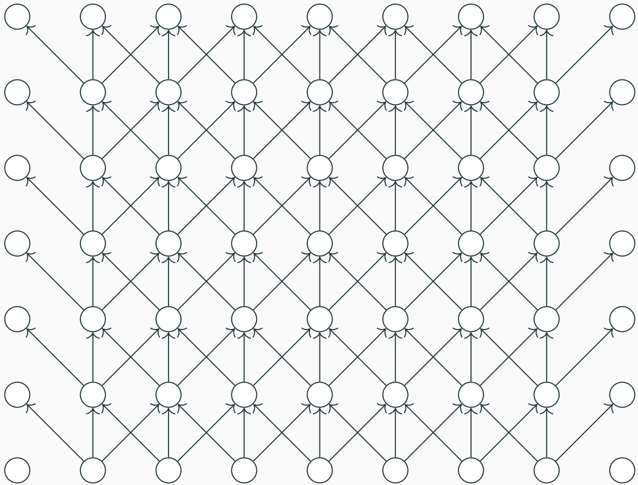

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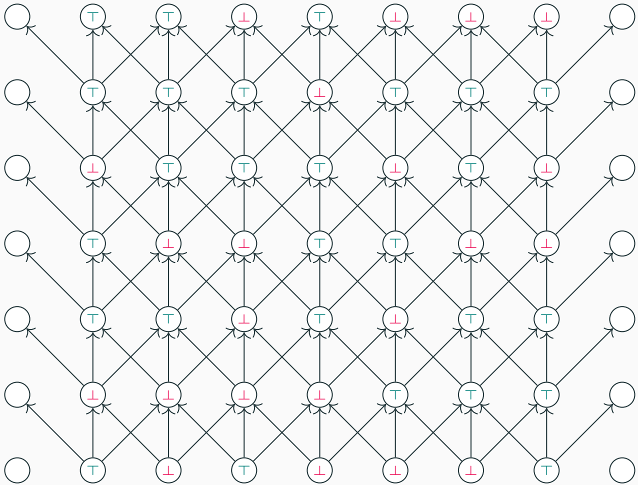

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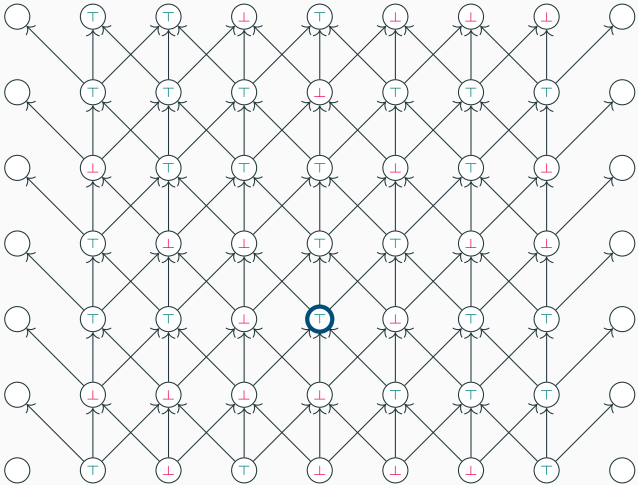

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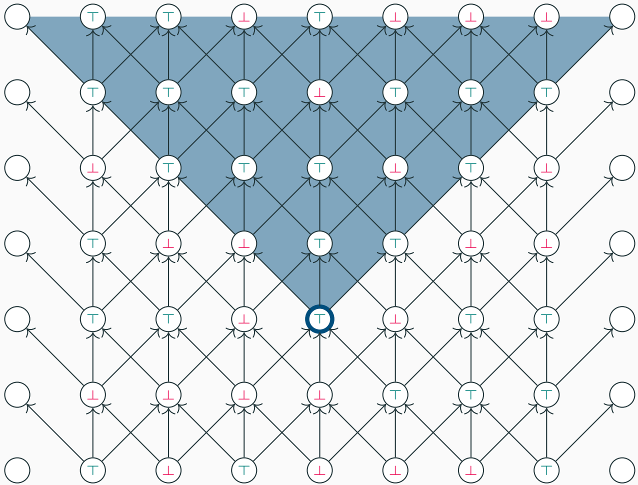

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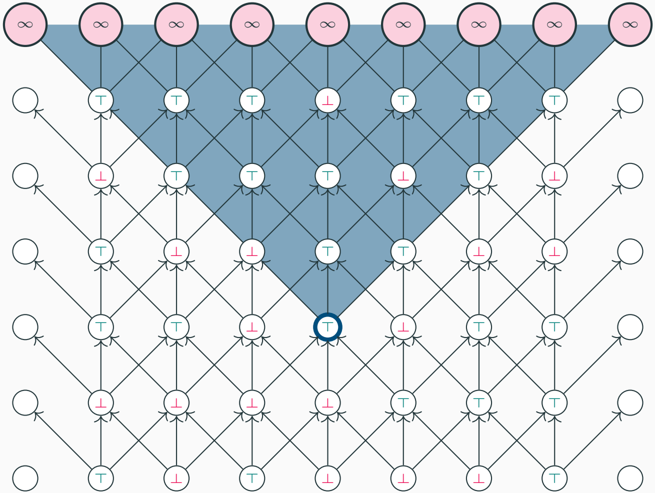


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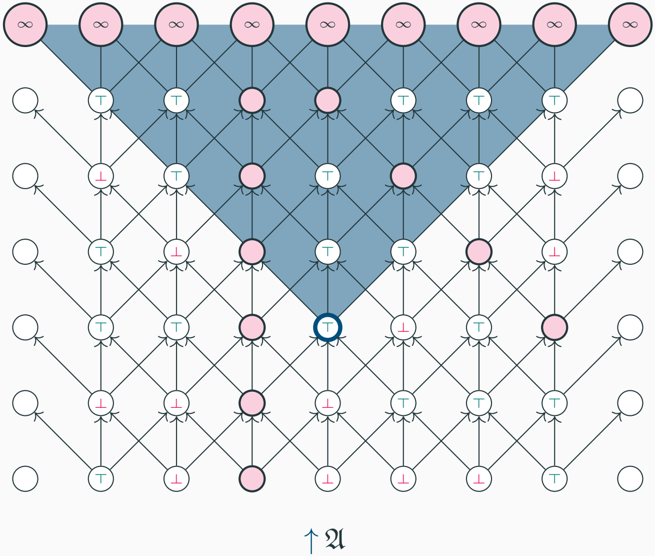

# Preservation Theorems

## What do we know about these completions?

**Theorem (Homomorphism Preservation Theorem)**

Let $Q \in \mathrm{FO}$, the following are equivalent:

1. $Q$ is equivalent to a union of conjunctive queries,
2. $Q$ is preserved under homomorphisms.

## Preservation theorems: completeness!

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1. the naïve evaluation works for all databases,
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**Easy remark**

CQs are preserved under homomorphisms (always).

# Preservation theorems: the classical results

| Preservation Under |
|---|
| homomorphisms |
| injective homomorphisms (Tarski-Lyndon) |
| strong injective homomorphisms (Łoś-Tarski) |
| surjective homomorphisms (Lyndon) |
| strong surjective homomorphism |
| $\forall$FO-embeddings (dual Chang-Łoś-Suszko) |

**These are Model Theory theorems... (using compactness of first order logic)**

Work with infinite structures, and not databases!

# Preservation theorems: the classical results

| **Preservation Under** | **Relatives to** $\mathrm{Fin}$ |
|---|---|
| homomorphisms | $\top$ [Ros08] |
| injective homomorphisms (Tarski-Lyndon) | $\perp$ [AG94, Theorem 10.2] |
| strong injective homomorphisms (Łoś-Tarski) | $\perp$ [Tai59; Gur84; DS21] |
| surjective homomorphisms (Lyndon) | $\perp$ [AG87; Sto95] |
| strong surjective homomorphism | $\perp$ [Cap+20] |
| $\forall$FO-embeddings (dual Chang-Łoś-Suszko) | $\perp$ [San+12] |

**These are Model Theory theorems... (using compactness of first order logic)**

Work with infinite structures, and not databases!

**Warning**

Preservation theorems can relativise to smaller classes of finite structures! (ex: $\emptyset$).

For the rest of the talk

we restrict our attention to classes of finite structures

and to boolean queries / first order sentences

# Preservation Theorems

## Three Specific Examples Among Classes of Finite Undirected Graphs

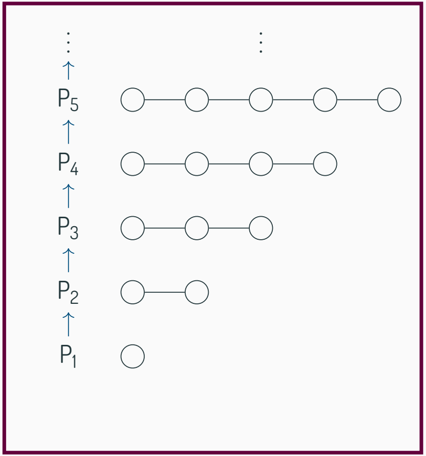
Universe: Undirected Graphs

Query: $[\![\varphi]\!]$

Order: extensions

Restriction: Finite Paths (Paths)

The Łoś-Tarski Theorem relativises to Paths!



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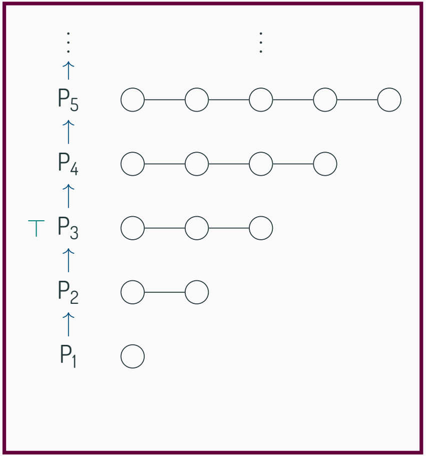

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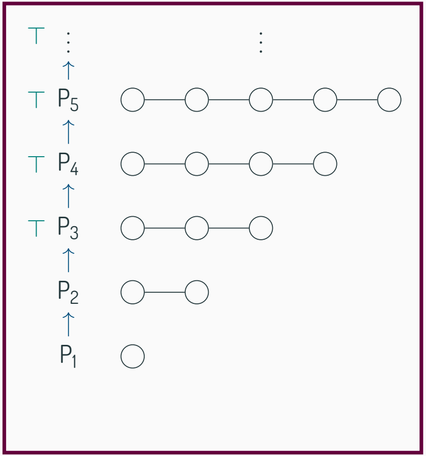

# Finite Paths (folklore)



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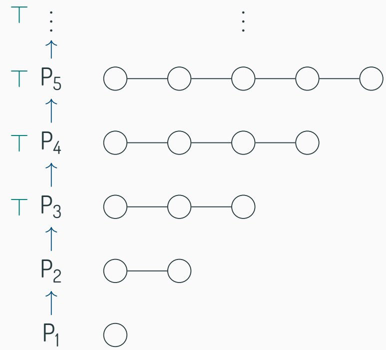


$$[\![\exists^{\geq 3} x. \top]\!] \cap \text{Paths} = [\![\varphi]\!] \cap \text{Paths}$$

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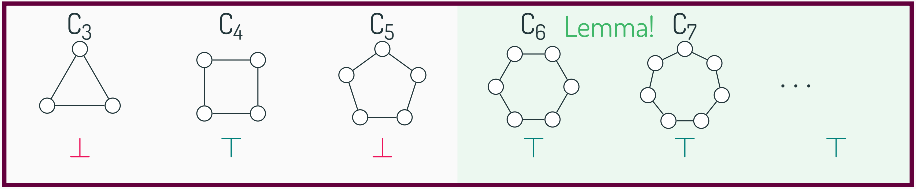
Query: $[\![\varphi]\!]$
Restriction: Finite Cycles (Cycles)



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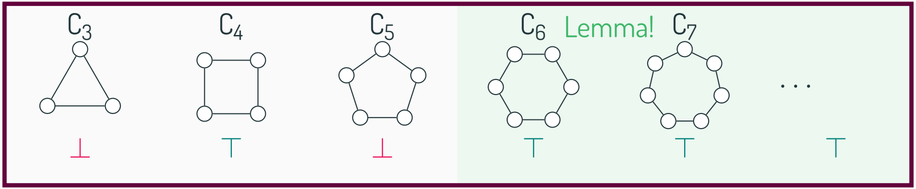

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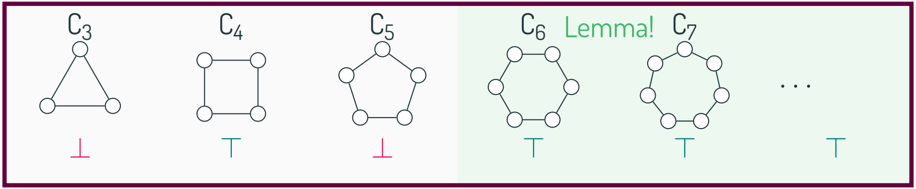


**Lemma (folklore)**

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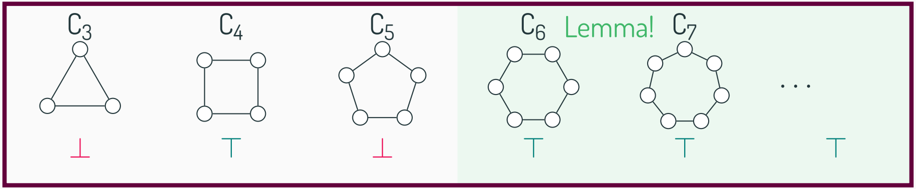


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$$[\![\varphi]\!] \cap \mathrm{Cycles} = [\![\exists^{=4}x.\top \vee \exists^{\geq 6}x.\top]\!] \cap \mathrm{Cycles}$$

**Theorem ([ADG08, Theorem 4.3])**

The Łoś-Tarski Theorem relativises to every class $\mathcal{C}$ of finite structures such that:

1. There exists a bound d on the maximal degree in the structures
2. The class is hereditary (neither Paths, nor Cycles)
3. The class is closed under disjoint unions (neither Paths, nor Cycles)

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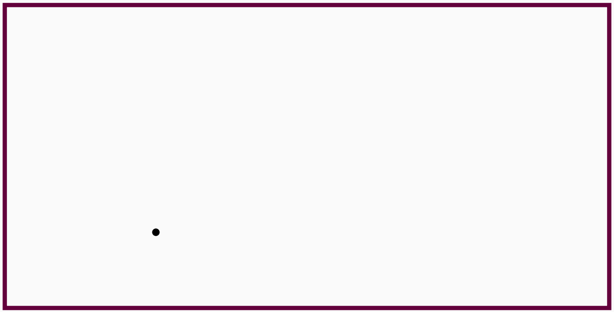

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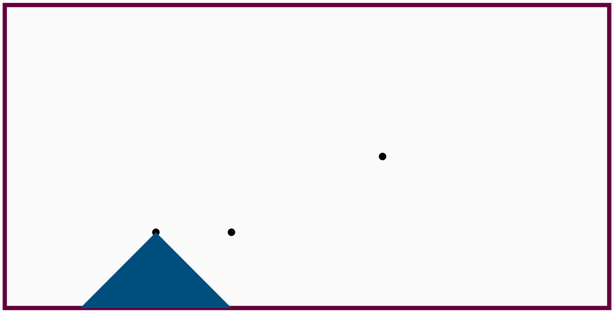

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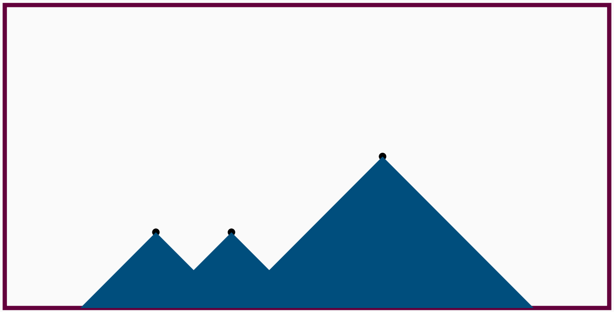

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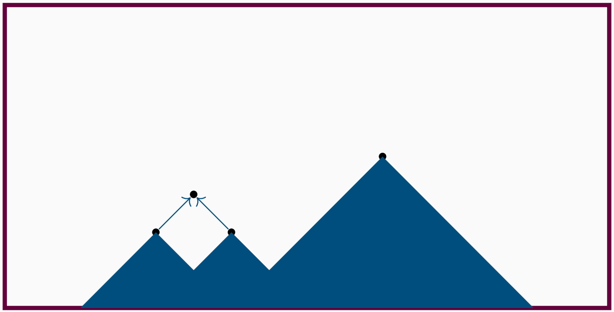

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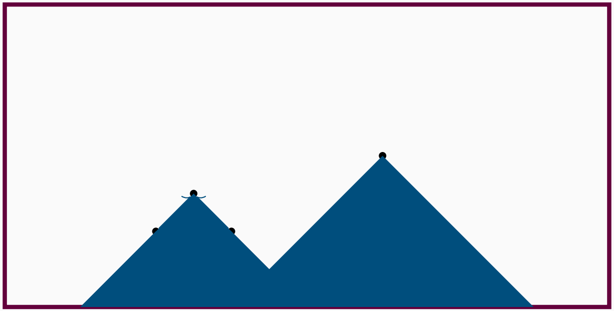

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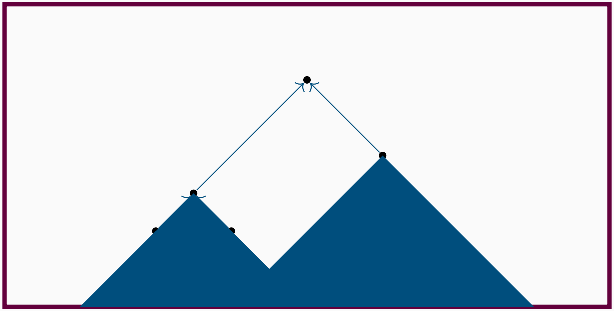

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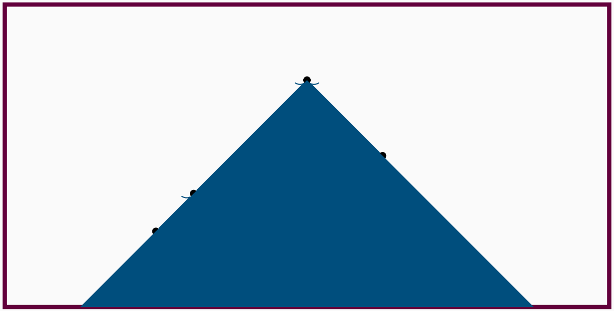

**Three Non Overlapping Internal Approaches**

1. Upwards closed subsets are "simple" (Paths) – $\uparrow E$ where $E$ is finite
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**An external approach?**

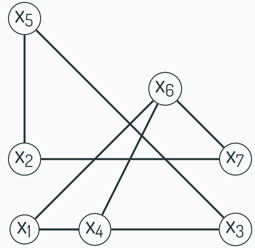
Is it possible to avoid starting from scratch every time?

- Cycles $\cup$ Paths? None of the above apply!

**Lemma (Chandra and Merlin [CM77])**

For every finite structure $\mathfrak{A}$ of size n, there exists a query $Q_{\mathfrak{A}}$ with n free variables, such that forall $\mathfrak{B}$ and $h : \mathfrak{A} \to \mathfrak{B}$, the following are equivalent

1. h is a homomorphism
2. $B, h(\mathfrak{A}) \models Q_{\mathfrak{A}}$.



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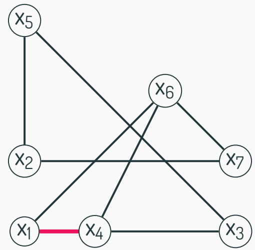


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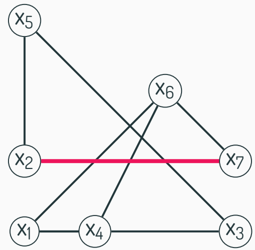


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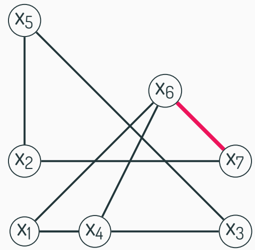


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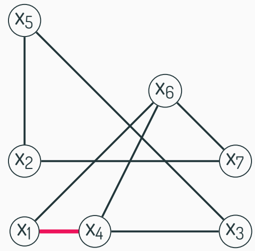


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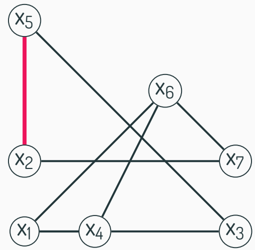


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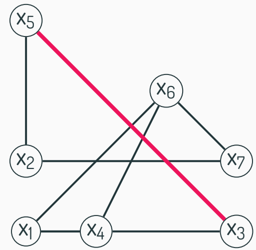


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# Diagram Queries and Minimal Models

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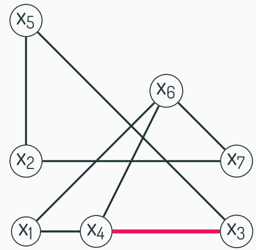


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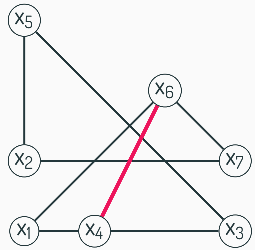


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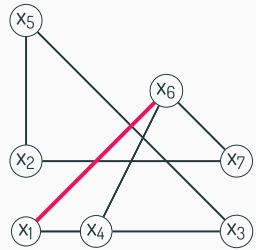


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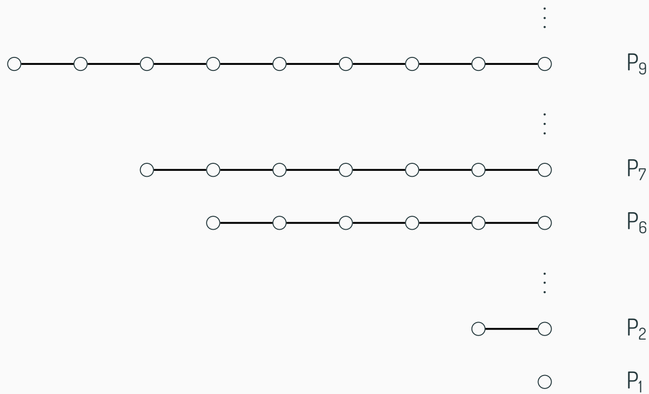
We work over $\mathcal{C}$ = Paths $\cup$ Cycles, ordered by $\subseteq_i$, and search for a query $\varphi$ preserved under extensions but not equivalent to an existential sentence.



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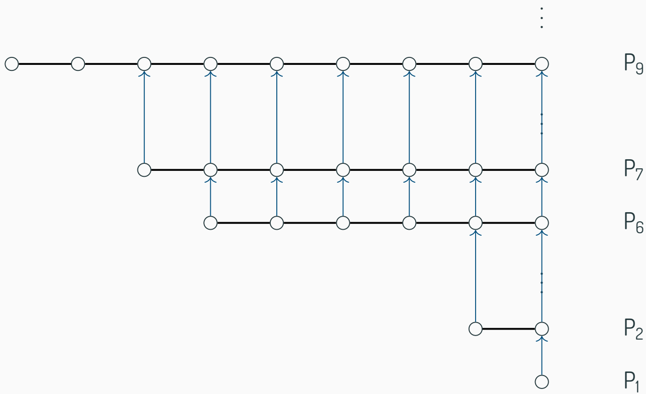

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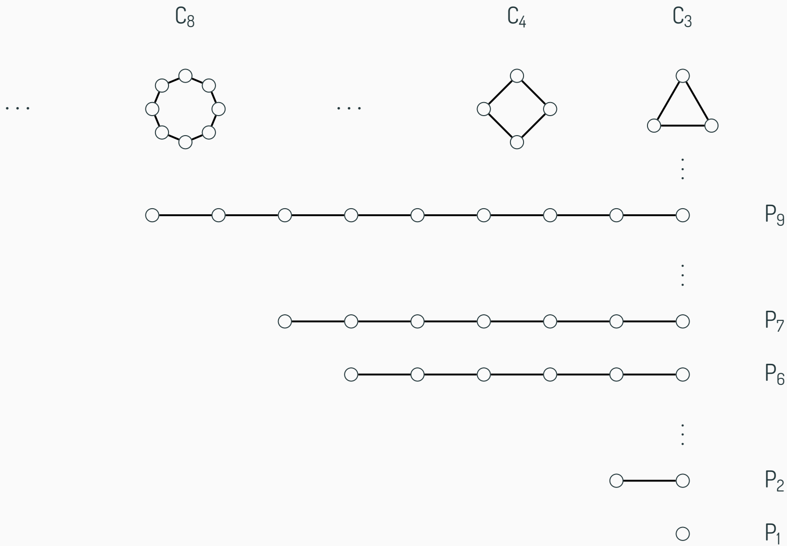

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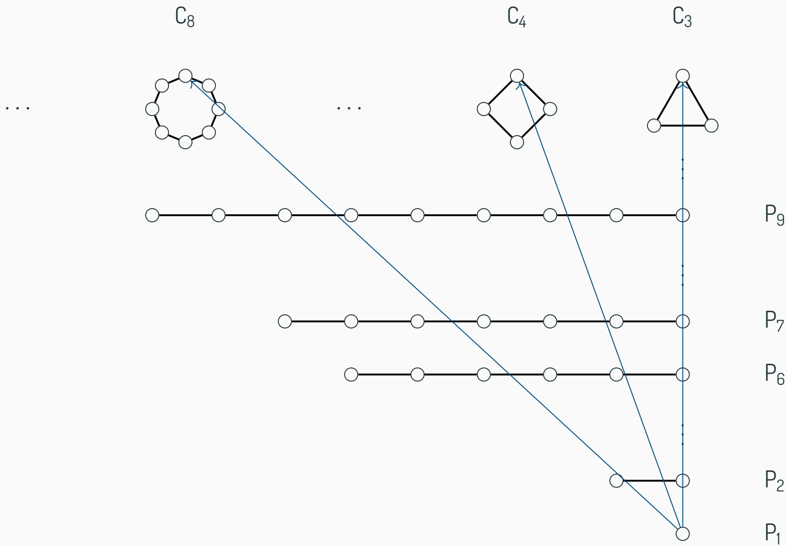

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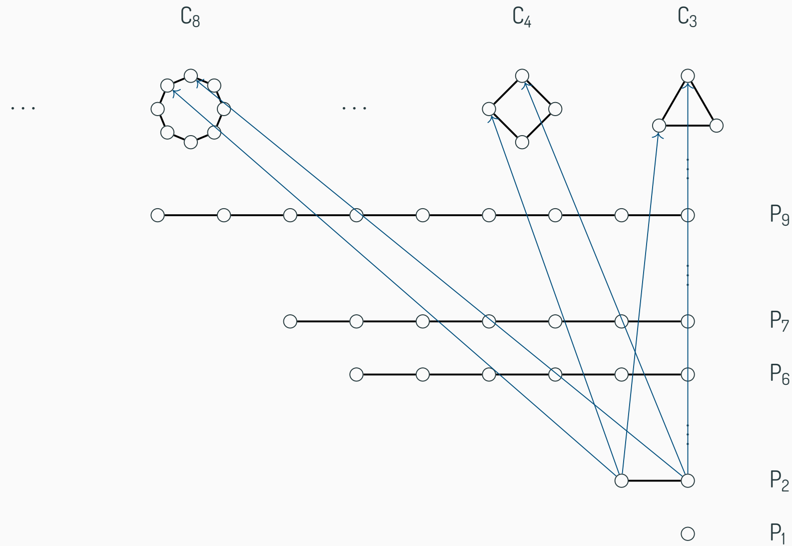

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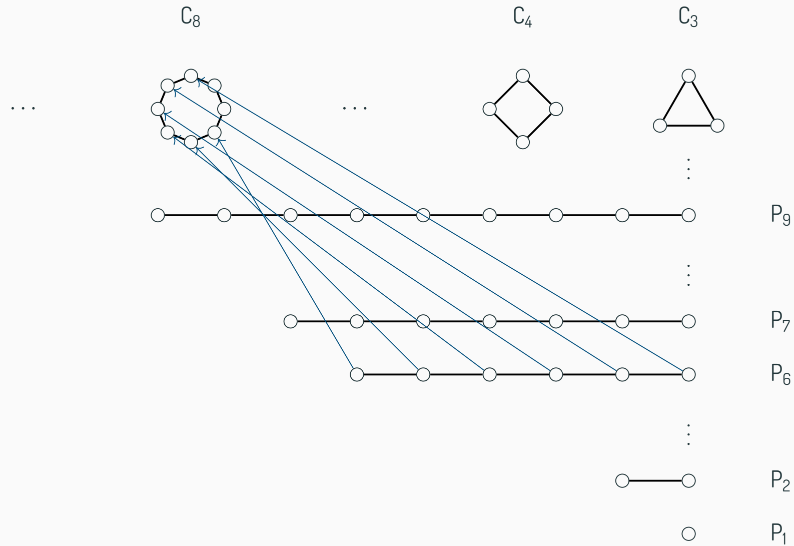

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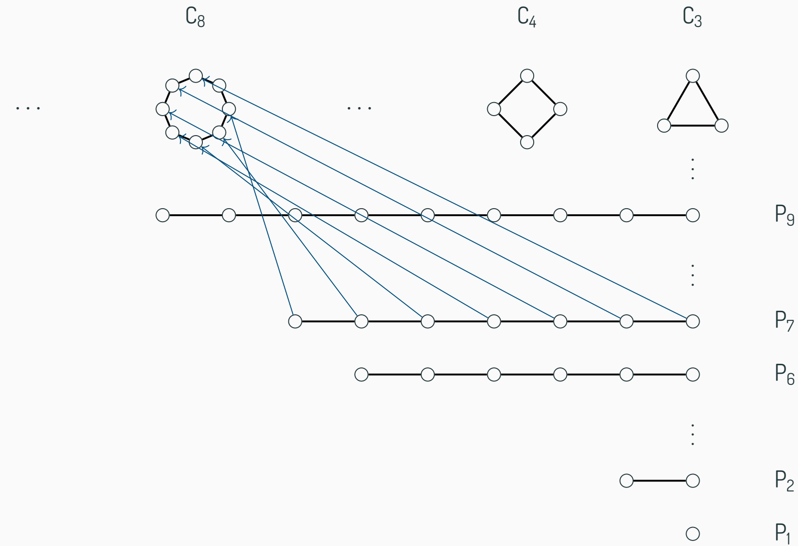

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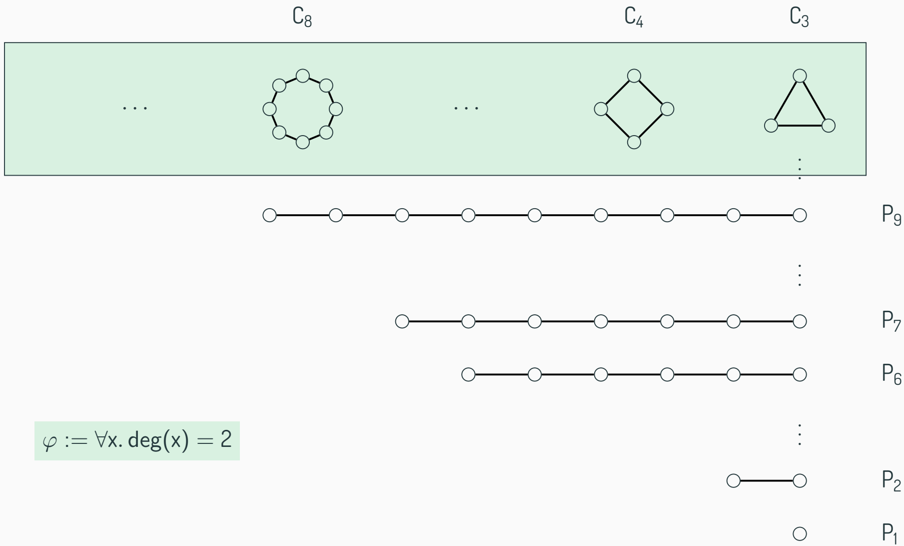
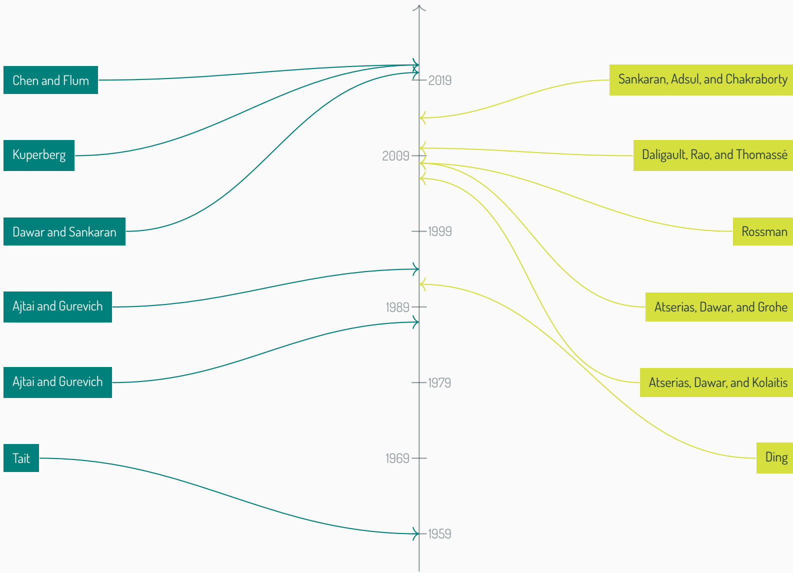



$\varphi := \forall x.\ \deg(x) = 2$

# Positive and negative results … A non exhaustive timeline



# But do they compose?

## What do you think?

We are working with the Łoś-Tarski Theorem for simplicity.
That is, ordering structures with $\subseteq_i$ and using the fragment $\exists$FO.

| **Class** | **Relativisation?** |
|---|---|
| Paths | $\top$ |
| Cycles | $\top$ |
| $\mathrm{Deg}^{\leq 2}, \uplus, \downarrow$ | $\top$ |
| Labelled(Paths, L) | $\bot$ |
| $\mathrm{Labelled}(\mathrm{Deg}^{\leq 2}, \mathrm{L}), \uplus, \downarrow$ | $\top$ |
| Cliques | $\top$ |
| Paths $\cup$ Cycles | $\bot$ |
| Paths $\times$ Cycles | $\top$ |
| Fin | $\bot$ |
| Struct | $\top$ |

**Multiple sources**

- Datasets come from different sources, with different assumptions on their completeness.
- Queries operating on incomplete databases can be seen as operating on classes of databases.

# But do they compose?

## Logically Presented Pre-Spectral Spaces (it is too late to change the name now)

**Definition (Compact subset)**

$U \in \mathcal{O}(X)$ is compact if for every sequence $(U_i)_{i \in I}$,

$$U \subseteq \bigcup_{i \in I} U_i \implies \exists n \in \mathbb{N}, U \subseteq U_{i_1} \cup \cdots \cup U_{i_n}$$

| Space | Subset | Compact? |
|---|---|---|
| $\mathbb{R}$ | $\{1\}$ | $\top$ |
| $\mathbb{R}$ | $[0, 1]$ | $\top$ |
| $\mathbb{R}$ | $]0, 1]$ | $\bot$ |
| $\mathbb{R}$ | $\mathbb{R}$ | $\bot$ |
| $(\mathrm{Paths}, \subseteq_i)$ | $\uparrow P_3$ | $\top$ |
| $(\mathrm{Cycles}, \subseteq_i)$ | Cycles | $\bot$ |

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**Ideal for preservation theorems**

Let $\varphi$ be preserved under $\to_F$. We enumerate models $\mathfrak{A} \models \varphi$, and consider $\psi_{\mathfrak{A}}$ that defines $\uparrow \mathfrak{A}$.

$$\varphi \equiv \bigvee_{\mathfrak{A} \models \varphi} \psi_{\mathfrak{A}}$$

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**Definition ([Lop21, Definition 3.2])**

A logically presented pre-spectral space is a triple $\langle\langle X, \tau, \mathcal{B}\rangle\rangle$ such that

1. $(X, \tau)$ is a topological space — notion of completions
2. $(X, \mathcal{B})$ is a boolean subalgebra of $\mathcal{P}(X)$ — notion of queries,
3. $\langle \tau \cap \mathcal{B} \rangle_{\text{top}} = \tau$ — enough queries exist (think Chandra and Merlin),
4. $\tau \cap \mathcal{B} = \mathcal{K}^{\circ}(\tau)$ — definable and open subsets of X are compact open.

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**Compact open in the ordered case**

compact open subsets are the upwards closed subsets that have finitely many minimal elements.

# A completeness result (specialised to Łoś-Tarski)

Let $\mathcal{C} \subseteq \mathsf{Fin}$.

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Let $\mathcal{C} \subseteq \mathrm{Fin}$. We consider $\mathcal{B}$ to be the FO-definable subsets of $\mathcal{C}$, and $\tau$ to be the collection of upwards closed subsets of $\mathcal{C}$ (for extensions).

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**Theorem ([Lop21, Theorem 3.4], specialised to the Łoś-Tarski Theorem and the finite setting)**

1. The Łoś-Tarski Theorem relativises to $\mathcal{C}$,
   **and** existential sentences define compact open subsets.
2. The space $\langle\langle \mathcal{C}, \tau, \mathcal{B} \rangle\rangle$ is an LPPS.

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**Remarks**

- LPPS captures a subset of preservation theorems.
- The two coincide on hereditary classes of finite structures.
- LPPS will be stable under composition (finite sums, finite products, etc.)

## Example and non examples

We are working with the Łoś-Tarski Theorem for simplicity.
That is, ordering structures with $\subseteq_i$ and using the fragment $\exists$FO.

| **Class** | **Relativisation?** | **LPPS?** |
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| Struct | $\top$ | – |

LPPS captures “reasonable” preservation theorems.

**Generalises Already Known Spaces**

- $\langle\!\langle \mathcal{C}, \tau, \mathcal{P}(\mathcal{C}) \rangle\!\rangle$ is an LPPS $\leftrightsquigarrow$ $(\mathcal{C}, \tau)$ is a Noetherian space (see [GL13])
- $\langle\!\langle \mathcal{C}, \tau, \langle \mathcal{K}^{\circ}(\tau) \rangle_{\mathrm{bool}} \rangle\!\rangle$ is an LPPS $\leftrightsquigarrow$ $(\mathcal{C}, \tau)$ is a Spectral space (see [DST19])

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**Compositional?**

Both spectral and Noetherian spaces can be composed!

# What are the compositions?

**Theorem (LPPS stabilty)**

| Operation | Symbol | Extra Hypothesis |
|---|---|---|
| sum | $\mathcal{C} + \mathcal{C}'$ | - |
| product | $\mathcal{C} \times \mathcal{C}'$ | - |
| inner product | $\mathcal{C} \otimes \mathcal{C}'$ | - |
| finite words | $\mathcal{C}^\star$ | - |
| wreath product | $\mathcal{C} \rtimes \mathcal{C}'$ | $\mathcal{C}$ is $\infty$-wqo |

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**Lemma (Other stability results)**

- Surjective continuous and definable maps $f\colon \mathcal{C} \twoheadrightarrow \mathcal{C}'$.
- Boolean combinations of compact open subsets.

# Composing LPPS

## Subsets and maps

$$f\colon (X, \tau, \mathcal{B}) \to (X_2, \tau_2, \mathcal{B}_2)$$

1. $f^{-1}\colon \tau_2 \to \tau$ (continuous), and
2. $f^{-1}\colon \mathcal{B}_2 \to \mathcal{B}$ (definable).

**Lemma**

If $f$ is surjective and $\langle\langle X, \tau, \mathcal{B}\rangle\rangle$ is an lpps, then $\langle\langle X_2, \tau_2, \mathcal{B}_2\rangle\rangle$ is also an lpps.

# Morphisms of LPPS?

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**Ex: Continuous first order interpretations**

- Selecting subsets
- Defining new relations using CQs

# Composing LPPS

## The example of a product

# What is the product of two spaces?

Let $\langle\!\langle \mathcal{C}, \tau, \mathcal{B} \rangle\!\rangle$ and $\langle\!\langle \mathcal{C}', \tau', \mathcal{B}' \rangle\!\rangle$ be LPPS.

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**The elements of** $\mathcal{C} \times \mathcal{C}'$

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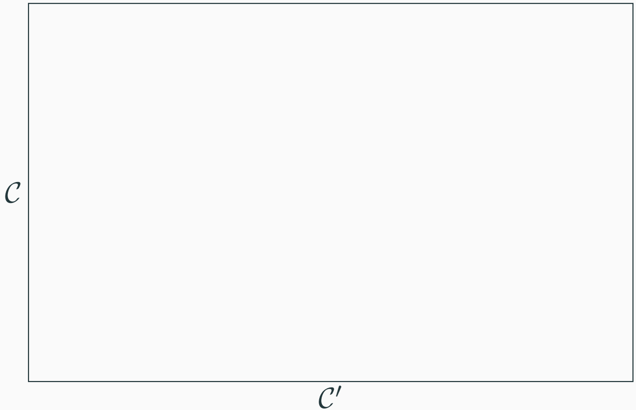
**Theorem ([Lop21, Proposition 5.8])**

$$\langle\!\langle \mathcal{C} \times \mathcal{C}', \tau^{\times}, \mathcal{B}^{\times} \rangle\!\rangle \text{ is an LPPS.}$$

**Let us prove:**

$$\tau^\times \cap \mathcal{B}^\times \subseteq \mathcal{K}^\circ(\tau^\times) \quad .$$

Let $U \in \tau^\times \cap \mathcal{B}^\times$.



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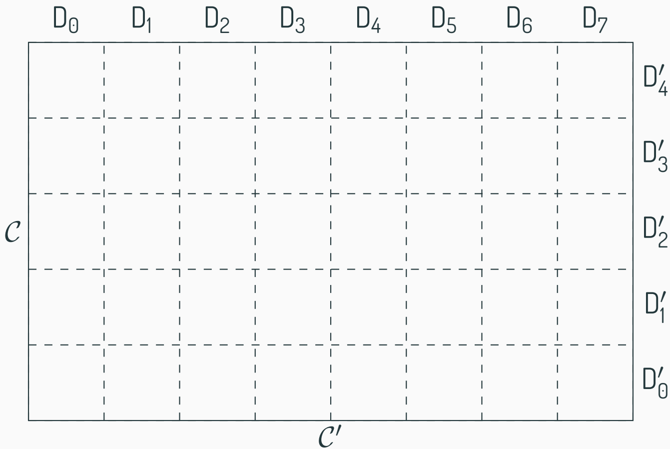

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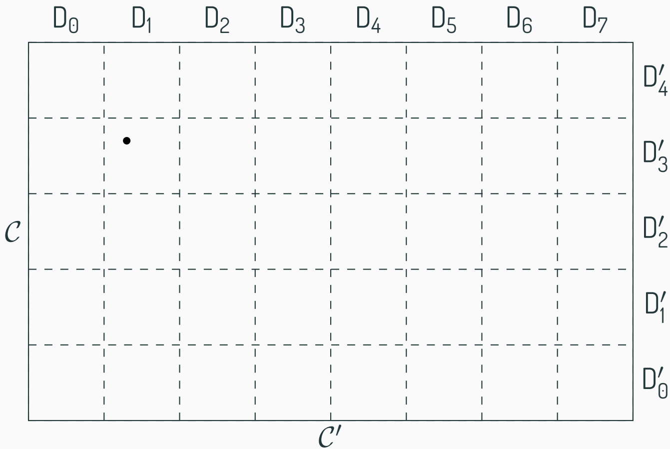

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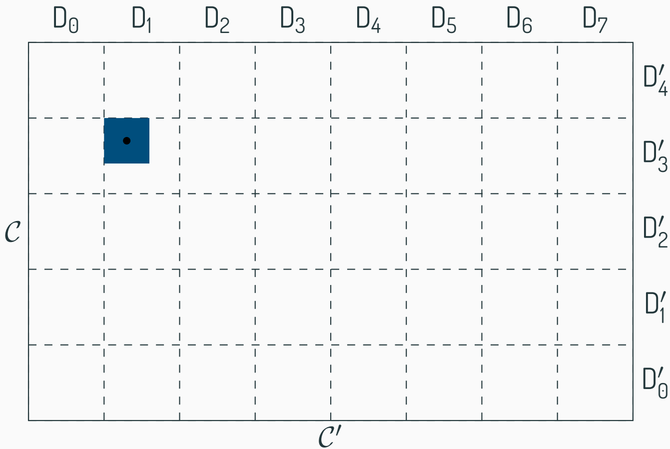

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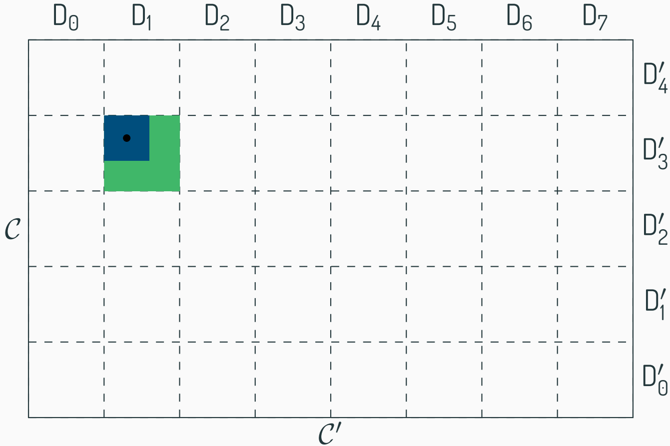

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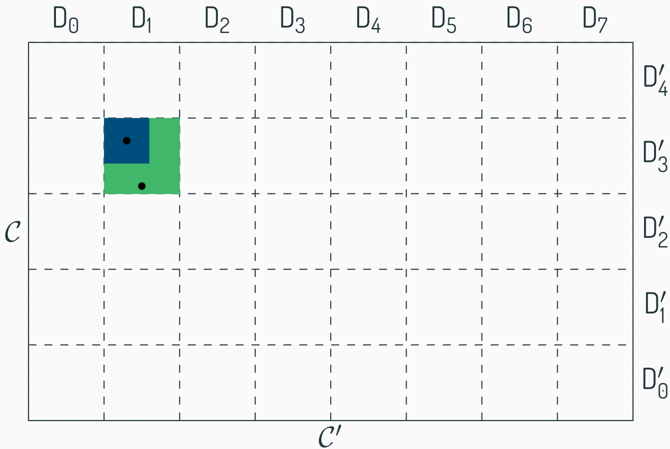

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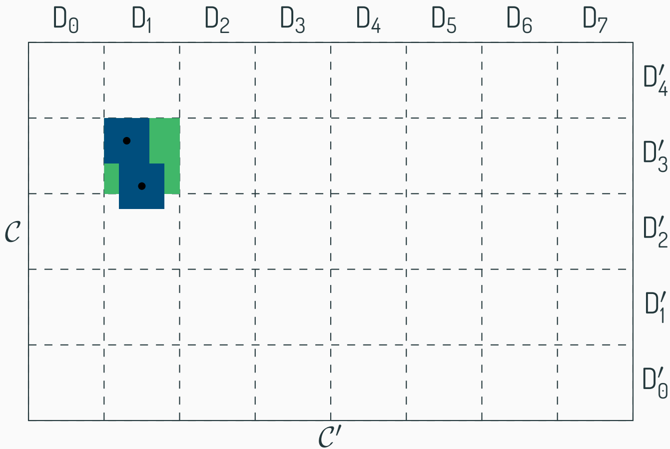

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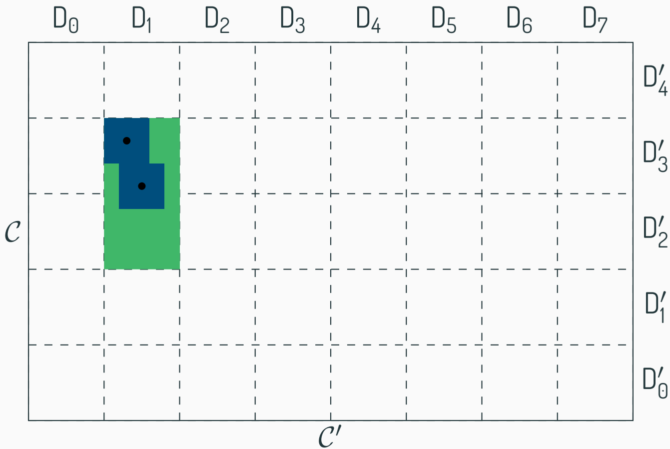

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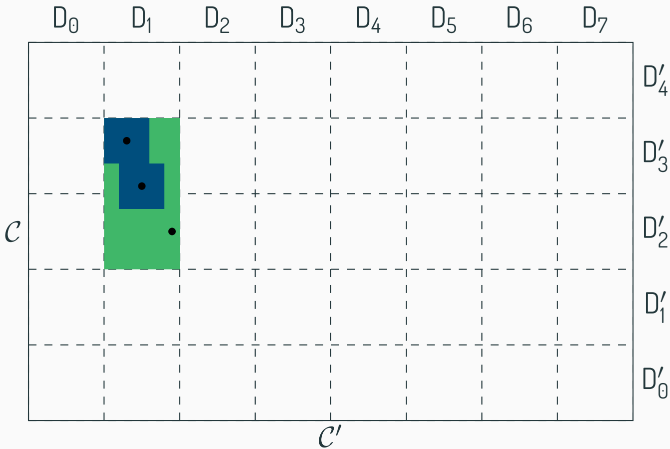

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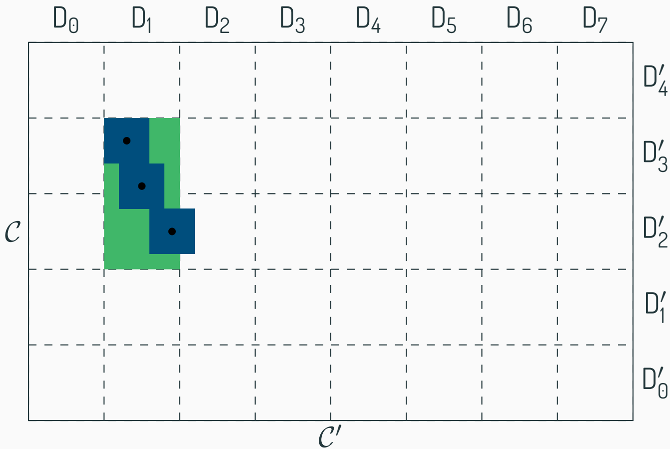

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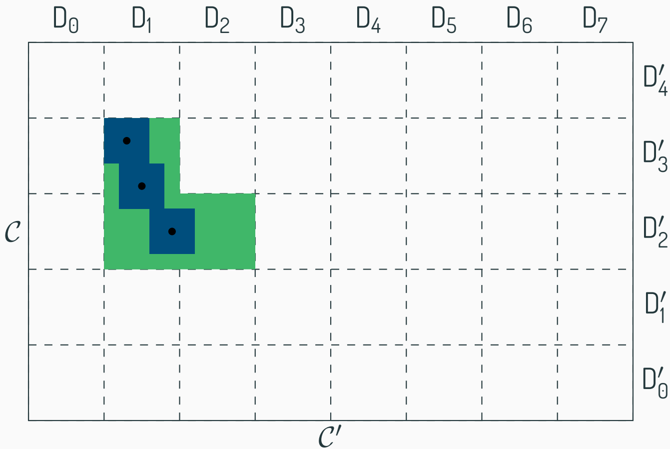

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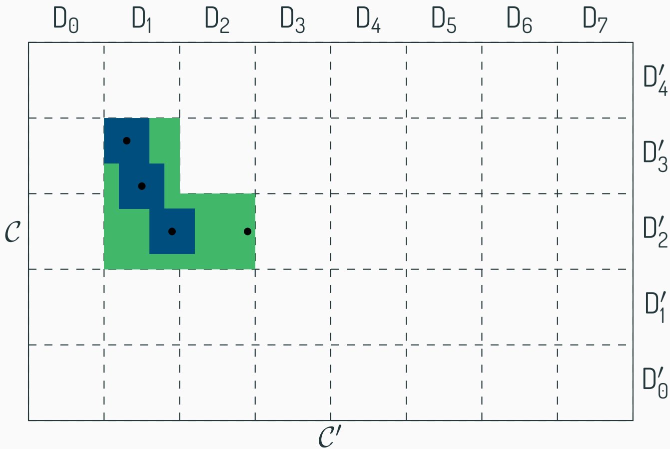

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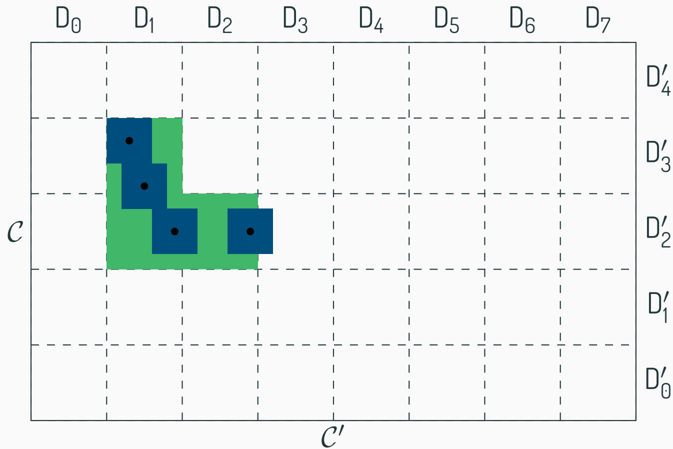

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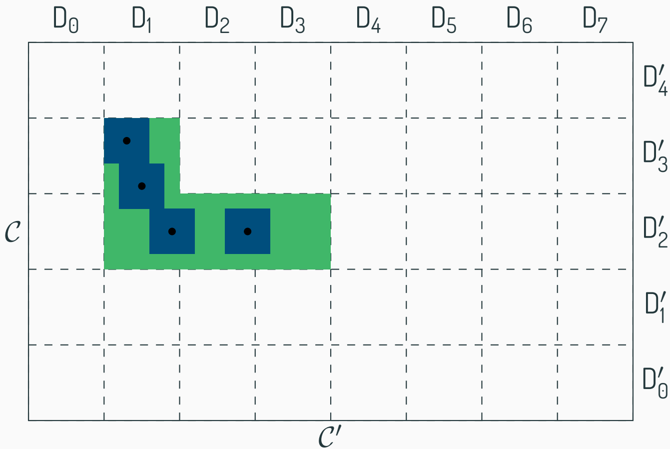

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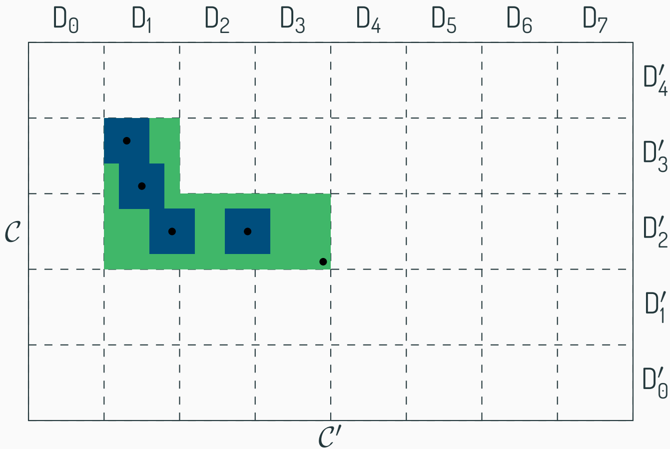

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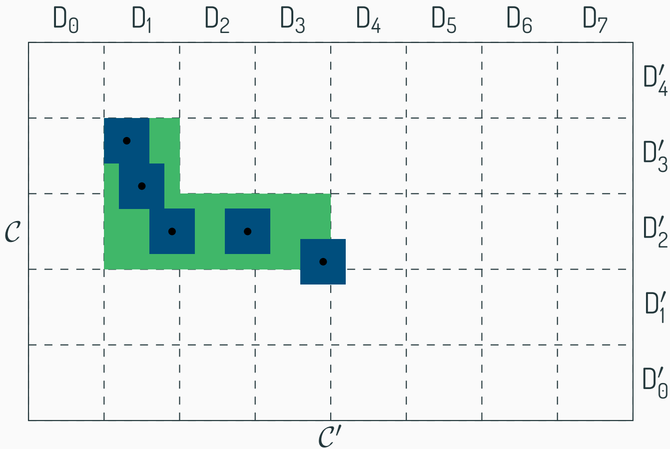

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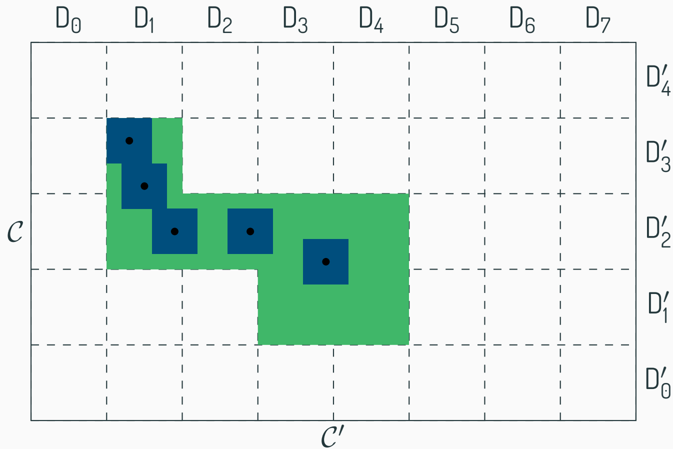

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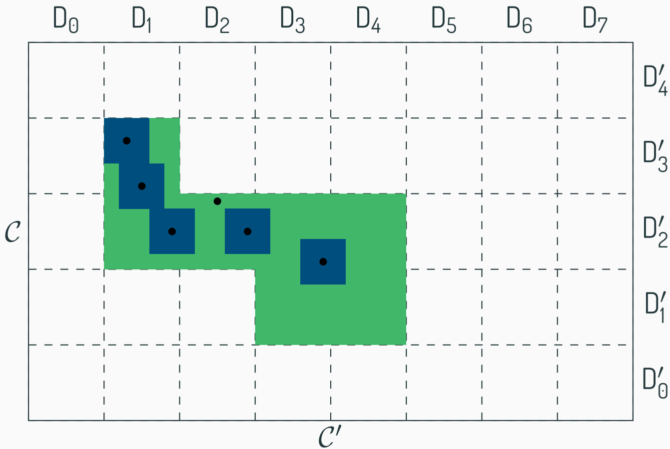

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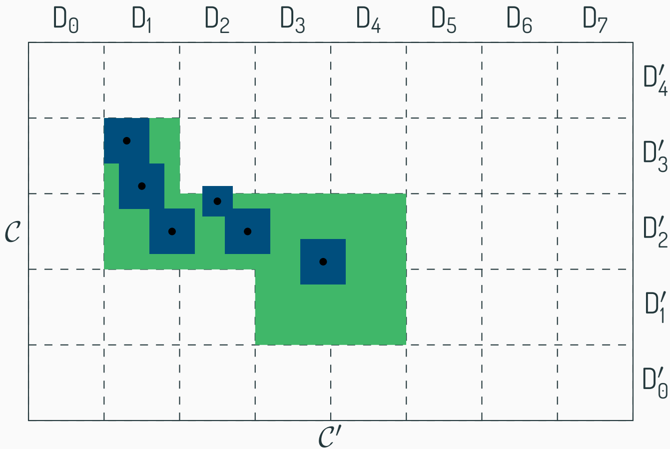

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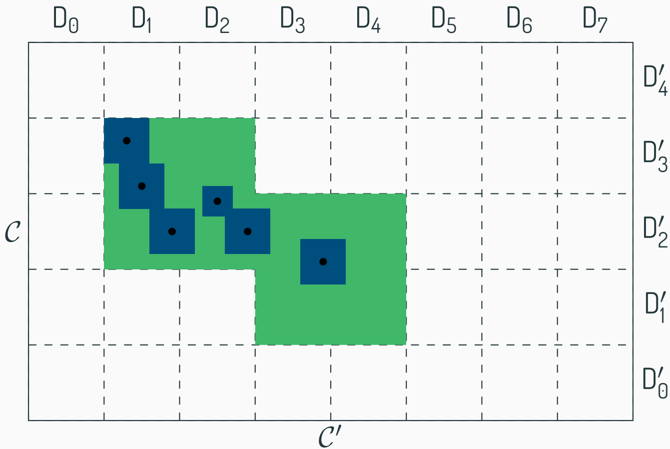

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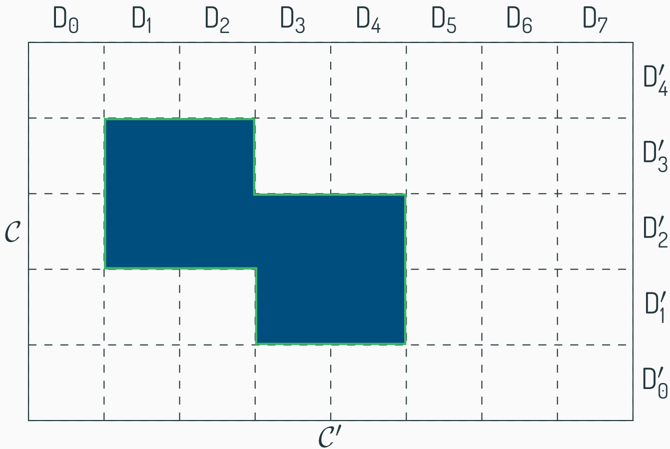

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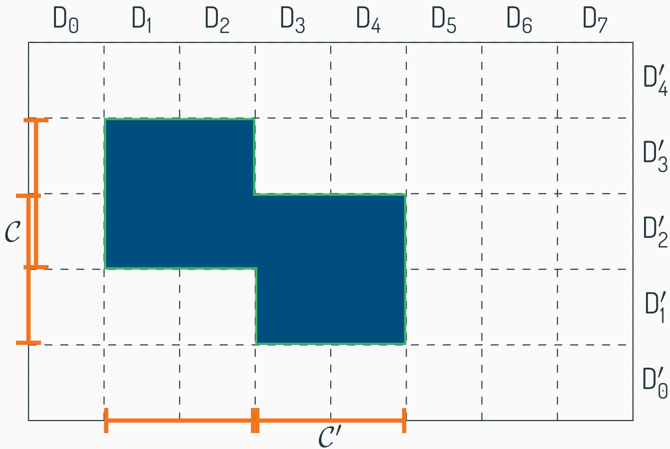

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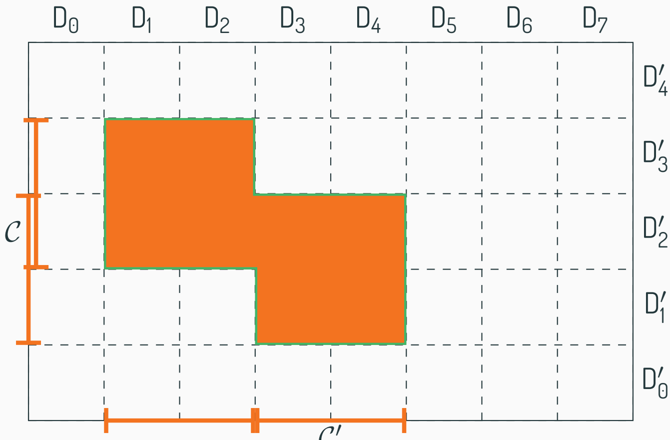

We composed two preservation theorems without knowing how they were obtained!

**The gap**

$$\langle\!\langle \mathsf{Paths} \times \mathsf{Cliques}, \subseteq_{\mathsf{i}}, \langle \mathsf{FO}[E] \times \mathsf{FO}[E'] \rangle_{\mathsf{bool}} \rangle\!\rangle \text{ is an LPPS.}$$

$$\langle \mathsf{FO}[A] \times \mathsf{FO}[B] \rangle_{\mathsf{bool}} \stackrel{?}{=} \mathsf{FO}[A \uplus B]$$

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**Theorem**

Yes, using compositional techniques à la Feferman and Vaught [FV59].

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**Theorem**

Yes, using compositional techniques à la Feferman and Vaught [FV59].

**Lemma**

$$\begin{aligned} f \colon \langle\!\langle \mathrm{Paths} \times \mathrm{Cliques}, \subseteq_i, \mathrm{FO}[E \uplus E'] \rangle\!\rangle &\to \langle\!\langle \mathrm{Paths} \times \mathrm{Cliques}, \subseteq_i, \mathrm{FO}[E] \rangle\!\rangle \\ (P_n, K_m) &\mapsto P_n \uplus K_m[E' \mapsto E] \end{aligned}$$

is a surjective continuous and definable map.

# Composing LPPS

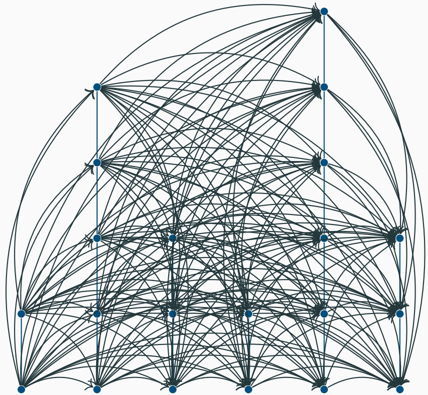
## Complicated examples

# Example using the Wreath Product

$$((\mathsf{Paths} \times \mathsf{Deg}^{\leq 2}) + \mathsf{Cliques})^{\star} \times \mathsf{Paths}$$

# Example using the Wreath Product

LinOrd $\rtimes$ Paths **(for free!)**



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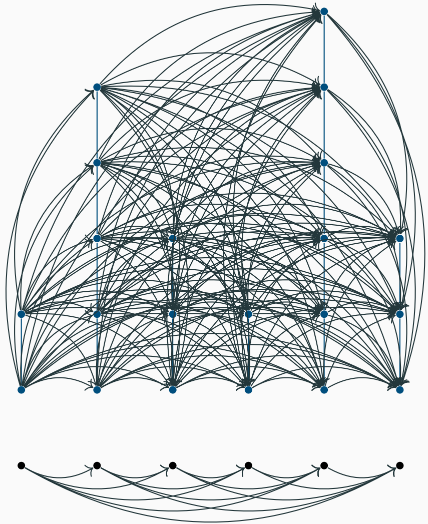

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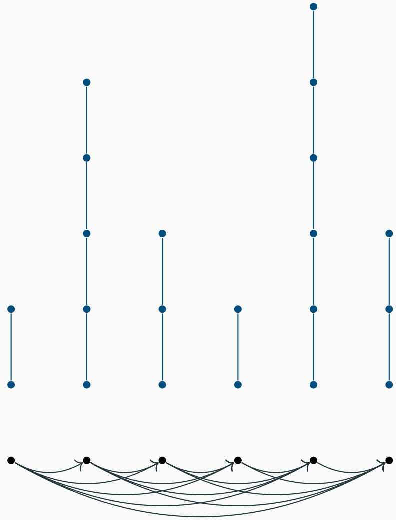

# Concluding Remarks

**Origin Story**

- Completion of databases
- Completeness of syntactic fragments
- Preservation theorems in classes of finite structures

**Contribution**

- A topological framework capturing preservation theorems
- For which compositionality is ensured
- With a nice theory behind it

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# What are world assumptions?

**Associates to a collection of completions to a point**

$$D \mapsto \{D_2 \mid D_2 \text{ completes } D\}$$

- Closed World Assumption: nulls are mapped to constants
- Open World Assumption: nulls are mapped to constants, and the database can be extended
- Weak Closed World Assumption: null are mapped to constants, the database can be extended, as long as its active domain does not change.

# Topologies arise through Kuratowski closure operators

**For our world assumptions**

$$\rho\colon X \mapsto \bigcup_{D \in X} \{D_2 \mid D_2 \text{ completes } D\}$$

- Closed subsets are upwards closed for a suitable family of homomorphisms.
- Open subsets are their complements.
- In this particular case, the collection of upwards closed susbets also forms a topology.

**Alexandroff topologies**

1. Are topologies where arbitrary unions of closed subsets are closed
2. Are uniquely determined by an ordering $D_1 \leq D_2$ if and only if $D_1 \in \rho(\{D_2\})$.

### Certain answers over the completion

- Consider databases $h : D \to_F D_2$
- But restrict our attention to databases $D_2$ such that $D_2 \models \Sigma$.

### Setting

- A set $\Sigma$ of constraints
- An incomplete database $D$
- A notion of completion $\to_F$

### The core chase algorithm

- Can be defined when $\Sigma$ has a certain shape
- May not terminate on some instances

**Theorem (Deutsch, Nash, and Remmel [DNR08])**

The core chase terminates for $Q, \Sigma, D$ if and only if $\uparrow(Q \cap \Sigma)$ is first order definable (in all structures).

| **Models** | **Theories** |
| --- | --- |
| Model $\mathfrak{A}$ | Complete Theory $T_{\mathfrak{A}}$ |
| $\mathfrak{A} \models \varphi$ | $\varphi \in T_{\mathfrak{A}}$ |

**Remark: spectral spaces**

Note that the original introduction of Stone defined what is now called Priestley spaces, which are more general. It was forgotten because the spaces were "non-standard" (understand, non hausdorff).