Generic Noetherian Theorems
Defining Noetherian Topologies Through Iterations

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1. WQOs and Beyond

2. Minimal Bad Sequence Arguments

3. Applications

4. Open Questions
1. WQOs and Beyond
   Well-quasi-orders
   Noetherian Spaces

2. Minimal Bad Sequence Arguments

3. Applications

4. Open Questions
WQOs and Beyond

Well-quasi-orders
<table>
<thead>
<tr>
<th>Quasi-Order</th>
<th>Sequence</th>
<th>Good</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\mathbb{N}, =) )</td>
<td>( i \mapsto i )</td>
<td>✗</td>
</tr>
<tr>
<td>( (\mathbb{N}, \leq) )</td>
<td>( i \mapsto i )</td>
<td>✓</td>
</tr>
<tr>
<td>( {a, b}, \sqsubseteq )</td>
<td>( i \mapsto a^i )</td>
<td>✓</td>
</tr>
<tr>
<td>( {a, b}, \sqsubseteq )</td>
<td>( i \mapsto ba^i )</td>
<td>✗</td>
</tr>
<tr>
<td>( {a, b}, \leq_* )</td>
<td>( i \mapsto ba^i )</td>
<td>✓</td>
</tr>
<tr>
<td>( G, \subseteq_i )</td>
<td>( i \mapsto C_i )</td>
<td>✗</td>
</tr>
<tr>
<td>( G, \leq_{\text{minor}} )</td>
<td>( i \mapsto C_i )</td>
<td>✓</td>
</tr>
</tbody>
</table>
Algebra of WQOs

\[ D ::= (F, =) \]

- \((\mathbb{N}, \leq)\)  
  natural numbers
- \(\Sigma_{i=1}^n D_i\)  
  finite disjoint sums
- \(\Pi_{i=1}^n D_i\)  
  finite products
- \(D^*\)  
  finite words, subword embedding
- \(D^\circ\)  
  finite multisets, multiset embedding
- \(\mathcal{P}_f(D)\)  
  finite sets, Hoare embedding
- \(T(D)\)  
  finite trees, Kruskal embedding
Keep in mind

How do we choose the order?

- Lattice of WQOs
- Not complete
Ordering on Words: suffix, pointwise & substructures

\[
\begin{align*}
  u_0 & \quad b_1 & \quad u_1 & \quad b_2 & \quad u_2 & \quad b_3 & \quad u_3 & \quad b_4 & \quad u_4 & \quad b_5 & \quad u_5 & \quad b_6 & \quad u_6 & \quad b_7 & \quad u_7 \\
  \leq & \quad \leq & \quad \leq & \quad \leq & \quad \leq & \quad \leq & \quad \leq & \quad \leq \\
  a_1 & \quad a_2 & \quad a_3 & \quad a_4 & \quad a_5 & \quad a_6 & \quad a_7 &
\end{align*}
\]
Ordering on Words: suffix, pointwise & substructures

\[ u_0 \ b_1 \ u_1 \ b_2 \ u_2 \ b_3 \ u_3 \ b_4 \ u_4 \ b_5 \ u_5 \ b_6 \ u_6 \ b_7 \ u_7 \]

\[ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \]
Ordering on Words: suffix, pointwise & substructures

\[ u_0 \begin{array}{cccccccccc}
  b_1 & u_1 & b_2 & u_2 & b_3 & u_3 & b_4 & u_4 & b_5 & u_5 & b_6 & u_6 & b_7 & u_7 \\
\end{array} \]

\[ \leq \]

\[ \begin{array}{ccccccc}
  a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\
\end{array} \]
Ordering on Words: suffix, pointwise & substructures

\[ u_0 \]  
\[ b_1 \]  
\[ u_1 \ b_2 \ u_2 \ b_3 \ u_3 \ b_4 \ u_4 \ b_5 \ u_5 \ b_6 \ u_6 \ b_7 \ u_7 \]

\[ a_1 \ a_2 \ a_3 \ a_4 \ a_5 \ a_6 \ a_7 \]
Some constructions fail to preserve WQOs

- $X^*$: prefix $X$, suffix $X$, factor $X$, pointwise $X$, subword $✓$
- $X^\omega$: infinite words with subword $X$
- $\mathcal{P}(X)$: powerset with embedding $X$

The Powerset Problem

Rado’s structure Rado (1954)
WQOs and Beyond

Noetherian Spaces
<table>
<thead>
<tr>
<th>Pre-order $\leq$</th>
<th>Topology $\text{Alex}(\leq)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U$ is upwards-closed</td>
<td>$U$ is open</td>
</tr>
<tr>
<td>$f$ is monotone</td>
<td>$f$ is continuous</td>
</tr>
<tr>
<td>$E$ has finitely many minimal elements</td>
<td>$E$ is compact</td>
</tr>
<tr>
<td>wqo</td>
<td>Noetherian</td>
</tr>
</tbody>
</table>
Recap of Jean’s talk

- Posets are topological spaces with the Alexandroff topology
Better than posets?

• Recap of Jean’s talk
  - Posets are topological spaces with the Alexandroff topology
  - There are topological equivalent to the constructions on posets (products, sums, ...)

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Recap of Jean’s talk

- Posets are topological spaces with the **Alexandroff topology**
- There are topological equivalent to the constructions on posets (products, sums, ...)
- **wqo** is a way of stating **compactness**
Recap of Jean’s talk

- Posets are topological spaces with the **Alexandroff topology**
- There are topological equivalent to the constructions on posets (products, sums, ...)
- **wqo** is a way of stating **compactness**

▲ **Different approach than BQO**

We broaden the notion of “well-behaved” space.
Noetherian space

A topological space \((X, \tau)\) is **Noetherian** if every subset of \(X\) is compact.
<table>
<thead>
<tr>
<th>Space</th>
<th>Topology</th>
<th>Compact</th>
<th>Noetherian</th>
<th>WQO</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{N} )</td>
<td>Alex(( \leq ))</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \mathbb{N} )</td>
<td>cofinite</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>( \mathbb{N} )</td>
<td>discrete</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>( \Sigma^* )</td>
<td>Upper(( \sqsubseteq ))</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>( \Sigma^* )</td>
<td>Alex(( \sqsubseteq ))</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>( \Sigma^* )</td>
<td>Alex(( \leq_* ))</td>
<td>✓</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>( \mathbb{R} )</td>
<td>metric</td>
<td>✗</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>([0, 1])</td>
<td>metric</td>
<td>✓</td>
<td>✗</td>
<td>✗</td>
</tr>
<tr>
<td>( \mathbb{C} )</td>
<td>Zariski</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
<tr>
<td>( \Sigma^* )</td>
<td>regular subword</td>
<td>✓</td>
<td>✓</td>
<td>✗</td>
</tr>
</tbody>
</table>

The table shows the properties of various spaces and topologies, with '✓' indicating the property holds and '✗' indicating it does not. The 'WQO' column indicates whether the space is well-quasi-ordered.
\( D ::= (X, \text{Alex}(\leq)) \)

- \( \sum_{i=1}^{n} D_i \) finite disjoint sums
- \( \prod_{i=1}^{n} D_i \) finite products
- \( D^* \) finite words, regular subword topology
- \( D^\circ \) finite multisets, multiset topology
- \( \mathcal{P}(D) \) arbitrary subsets, lower Vietoris topology
- \( T(D) \) finite trees, regular subtree topology
- \( S(D) \) sobrification
- \( X^\omega \) infinite words, regular subword topology
Focus on Finite Words

Orders on Finite Words

- prefix

Topologies on Finite Words
Focus on Finite Words

Orders on Finite Words
- prefix
- suffix

Topologies on Finite Words

Goubault-Larrecq (2013)
Focus on Finite Words

- Orders on Finite Words
  - prefix
  - suffix
  - factor

- Topologies on Finite Words
  - prefix topology
  - \( \sum_{k=0}^{X} \)
  - regular subword topology

Goubault-Larrecq (2013)
Focus on Finite Words

**Orders on Finite Words**
- prefix $\times$
- suffix $\times$
- factor $\times$
- subword $\checkmark$

**Topologies on Finite Words**

Goubault-Larrecq (2013)
Focus on Finite Words

**Orders on Finite Words**
- prefix ✗
- suffix ✗
- factor ✗
- subword ✓

**Topologies on Finite Words**
- prefix topology ✗

Goubault-Larrecq (2013)
Focus on Finite Words

Orders on Finite Words

- prefix
- suffix
- factor
- subword

Topologies on Finite Words

- prefix topology
- $\sum_{k \geq 0} X^k$
### Orders on Finite Words
- prefix ✗
- suffix ✗
- factor ✗
- subword ✓

### Topologies on Finite Words
- prefix topology ✗
- $\sum_{k \geq 0} X^k$ ✗
- regular subword topology $[U_1, \ldots, U_n]$ ✓
- Goubault-Larrecq (2013) ✓
Intuition

topology \simeq \text{logic with infinite disjunction}
Regular Subword Topology

◆ **Intuition**

topology \(\simeq\) logic with infinite disjunction

◆ **Basic LTL Formulas** \(\text{PLTL}\)

\[ P_U \text{ for } U \text{ open} \]
\[ \diamond \varphi \text{ for } \varphi \text{ open} \]

Regular subword topology \(= \langle \diamond \varphi \mid \varphi \in \text{PLTL} \rangle\)
Keep in mind
How do we choose the order topology?
Getting back to preorders

Assume we know the preorder we want. Assert that the topology “corresponds”.

Definition (Specialisation Preorder)

\( a \leq_{\tau} b \iff \forall U \in \tau, a \in U \implies b \in U. \)
1. WQOs and Beyond

2. Minimal Bad Sequence Arguments
   For WQOs
   Refinement Functions
   Topology Expanders

3. Applications

4. Open Questions
The following are WQOs

- finite words $X^*$ with the Higman’s word embedding
- finite trees $T(X)$ with Kruskal’s tree embedding
- finite 2-trees $\mu Y. X \times T(Y)$
- finite $n$-trees...
- Graphs generated by a totally ordered monoid with induced subgraph (Daligault et al., 2010)

Using a minimal bad sequence argument.
Minimal Bad Sequence Arguments

For WQOs
• Consider a bad sequence \((w_n)_{n \in \mathbb{N}}\) that is minimal for \(\sqsubseteq\).
• It cannot contain \(\varepsilon\).
• Hence \(w_n = a_n \nu_n\).
• The set \(S \triangleq \{ \nu_n \mid n \in \mathbb{N} \}\) is a wqo.
• Hence \(X \times S\) is a wqo.
• But \(\{w_n \mid n \in \mathbb{N}\}\) reflects in \(X \times S\).
• Consider a bad sequence \((w_n)_{n \in \mathbb{N}}\) that is minimal for \(\subseteq\).
• It cannot contain \(\varepsilon\).
• Hence \(w_n = a_n v_n\) \(\text{decompose using } F(Y) = 1 + X \times Y\).
• The set \(S \triangleq \{v_n \mid n \in \mathbb{N}\}\) is a wqo \(\text{minimality}\).
• Hence \(X \times S\) is a wqo \(\text{stability through } F(Y)\).
• But \(\{w_n \mid n \in \mathbb{N}\}\) reflects in \(X \times S\) \(\text{recompose via } F\).
The proof in full generality

▲ Sufficient conditions?

- Studied by Hasegawa (2002) and Freund (2020)
- Lots of category theory

◆ Basic idea: order on inductive constructions

- Finite words: \( \mu Y.1 + X \times Y \)
- Finite trees: \( \mu Y.X \times Y^* \)
- Finite 2-trees: \( \mu Y.X \times T(Y) \)
- ...
What is the connection with our question?

It answers our question on wqos!
And provides a “canonical” ordering on the fixed points.

◆ Goals of this talk

• Adapt to a topological setting
• Justify existing constructions
• Provide new Noetherian spaces
Minimal Bad Sequence Arguments

Refinement Functions
Strategy over Topological Spaces

◆ **Idea: avoid categories**
  
  • The property of being *Noetherian* does not depend on the points
  
  • We can iteratively refine the topology

▲ **Consequence**

We decouple the construction of the space and of its topology.
Refinement Function

Fix $X$ a set, a refinement function $F$ maps topologies over $X$ to topologies over $X$ and

- Preserve Noetherian topologies
- Is monotone: $\tau \subseteq \tau' \implies F(\tau) \subseteq F(\tau')$

⚠️ Limit topology

One can iterate transfinitely $F$. One can define its least fixed point.
Iterating does not always work out

Building the prefix topology

\[ F(\tau) \triangleq \langle \{ T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]

\[ \bigcup_{i \in \mathbb{N}} a^i b \Sigma^* \text{ does not stabilise} \]
Iterating does not always work out

⚠️ Building the prefix topology

\[ F(\tau) \triangleq \langle \{ T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]

\[ \sum^* \]

\[ a\sum^* \quad b\sum^* \]

\[ \emptyset \]

\[ \bigcup_{i \in \mathbb{N}} a^i b\sum^* \] does not stabilise
Iterating does not always work out

**Building the prefix topology**

\[ F(\tau) \triangleq \langle \{ T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]

\[ \bigcup_{i \in \mathbb{N}} a^i b \Sigma^* \text{ does not stabilise} \]
Recall why the subword embedding works

We lack the notion of substructures. For $\mu Y.1 + X \times Y$, we have to consider suffixes.
What is a topology expander?

- **Correcting the prefix topology**

\[ F(\tau) \triangleq \langle \{ \uparrow \subseteq T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]
What is a topology expander?

Correcting the prefix topology

\[ F(\tau) \triangleq \langle \{ \uparrow \sqsubseteq T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]
What is a topology expander?

- Correcting the prefix topology

\[ F(\tau) \triangleq \langle \{ \uparrow \subseteq T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]
Correcting the prefix topology

\[ F(\tau) \triangleq \langle \{ \uparrow \subseteq T \cdot V \mid T \in \theta, V \in \tau \} \rangle \]

\[ \text{lfp } F \text{ is the regular subword topology.} \]
Minimal Bad Sequence Arguments

Topology Expanders
Can we replace “suffix” in general?

Definition

\[ \tau|_H \triangleq \{ U \cup H^c \mid U \in \tau \} \]

◆ In the case of \( \tau = \text{Alex}(\leq) \) and \( H \) downwards closed

\[ x \leq_{\tau|_H} y \iff \forall U \in \tau|_H, x \in U \Rightarrow y \in U \]

\[ \iff \forall U \in \tau, x \in U \cup H^c \Rightarrow y \in U \cup H^c \]

\[ \iff x \in \uparrow x \cup H^c \Rightarrow y \in \uparrow x \cup H^c \]

\[ \iff \begin{cases} x \leq y \in H \\ y \not\in H \end{cases} \]
Can we replace “suffix” in general? YES

\[ h_1, h_2, h_3, H, \chi \]
Can we replace “suffix” in general? YES
Topology Expander

$F$ is a refinement function and

$\forall \tau \subseteq F(\tau), \forall H \text{ closed in } \tau, F(\tau)|H \subseteq F(\tau|H)|H$
Translating and building intuition

The condition on topologies

\[ F(\tau)|H \subseteq F(\tau|H)|H \]

Specialisation preorders

\[ \leq_{F(\leq)|H} \supseteq \leq_{F(\leq|H)|H} \]

△ Refinement happens locally in closed sets

\[ x F(\leq|H) y \in H \implies x F(\leq) y \in H \]
Refinement happens “locally”
Refinement happens “locally”
Refinement happens “locally”
Theorem (Iteration)

If $\tau$ is Noetherian and $\tau \subseteq F(\tau)$ then $F^\alpha(\tau)$ is Noetherian for all $\alpha$. 
What is a minimal bad sequence?

**Definition (Good sequence)**

\((U_n)_{n \in \mathbb{N}}\) is **good** if \(\exists i. U_i \subseteq \bigcup_{j < i} U_j\)

A sequence that is not **good** is **bad**.

**Goubault-Larrecq (2013, Lemma 9.7.31)**

If \(\tau\) is generated by \(B\), \(\sqsubseteq\) is well-founded on \(B\), and \((X, \tau)\) is not **Noetherian**, then there exists a \(\sqsubseteq\)-minimal bad sequence of opens in \(B\).
Definition (Depth)

The first ordinal $\beta$ such that $U \in F^\beta(\tau)$.

Very Sketchy Sketch

- If $\alpha = \beta + 1$: automatic
Definition (Depth)
The first ordinal $\beta$ such that $U \in F^\beta(\tau)$.

**Very Sketchy Sketch**

- If $\alpha = \beta + 1$: automatic
- If $\alpha$ is limit: $F^\alpha$ is generated by opens of depth $\beta < \alpha$
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- If $\alpha = \beta + 1$: automatic
- If $\alpha$ is limit: $F^\alpha$ is generated by opens of depth $\beta < \alpha$
- In a minimal bad sequence $(U_n)$, $\beta = 0$ or $\beta = \beta' + 1$
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- In a minimal bad sequence $(U_n)$, $\beta = 0$ or $\beta = \beta' + 1$
- Let $V_n \triangleq \bigcup_{i<n} U_i$ and $H_n \triangleq X \setminus V_n$
Proof scheme

$F^\alpha(\tau)$ is Noetherian

**Definition (Depth)**

The first ordinal $\beta$ such that $U \in F^\beta(\tau)$.

**Very Sketchy Sketch**

- If $\alpha = \beta + 1$: automatic
- If $\alpha$ is limit: $F^\alpha$ is generated by opens of depth $\beta < \alpha$
- In a minimal bad sequence $(U_n)$, $\beta = 0$ or $\beta = \beta' + 1$
- Let $V_n \triangleq \bigcup_{i < n} U_i$ and $H_n \triangleq X \setminus V_n$
- $\{U \cup V_n \mid n \in \mathbb{N}, U < U_n\}$ generates a Noetherian topology $\mathcal{U}$
**Definition (Depth)**

The first ordinal \( \beta \) such that \( U \in F^\beta(\tau) \).

**Very Sketchy Sketch**

- If \( \alpha = \beta + 1 \): automatic
- If \( \alpha \) is limit: \( F^\alpha \) is generated by opens of depth \( \beta < \alpha \)
- In a minimal bad sequence \((U_n)\), \( \beta = 0 \) or \( \beta = \beta' + 1 \)
- Let \( V_n \triangleq \bigcup_{i < n} U_i \) and \( H_n \triangleq X \setminus V_n \)
- \( \{ U \cup V_n \mid n \in \mathbb{N}, U < U_n \} \) generates a Noetherian topology \( \mathcal{U} \)
- \( U_n \cup V_n \) open in \( F(\mathcal{U})|_{H_n} \) which is Noetherian
Proof scheme

$F^\alpha(\tau)$ is Noetherian

**Definition (Depth)**

The first ordinal $\beta$ such that $U \in F^\beta(\tau)$.

**Very Sketchy Sketch**

- If $\alpha = \beta + 1$: automatic
- If $\alpha$ is limit: $F^\alpha$ is generated by opens of depth $\beta < \alpha$
- In a minimal bad sequence $(U_n)$, $\beta = 0$ or $\beta = \beta' + 1$
- Let $V_n \triangleq \bigcup_{i < n} U_i$ and $H_n \triangleq X \setminus V_n$
- $\{U \cup V_n \mid n \in \mathbb{N}, U < U_n\}$ generates a Noetherian topology $\mathcal{U}$
- $U_n \cup V_n$ open in $F(\mathcal{U})|H_n$ which is Noetherian
- $U_{n_0} \subseteq U_{n_0} \cup V_{n_0} \subseteq \bigcup_{n < n_0} U_n \cup V_n = \bigcup_{n < n_0} U_n$ absurd
1. WQOs and Beyond

2. Minimal Bad Sequence Arguments

3. Applications
   - Divisibility Topology and Inductive Definitions
   - On big spaces

4. Open Questions
Applications

Divisibility Topology and Inductive Definitions
Definition (Divisibility topology)

Let $G(\mu G) \simeq_{\delta} \mu G$ and $\sqsubseteq$ the “substructure” ordering on $\mu G$. The divisibility topology is the least fixed point of

$$F_{\Diamond}(\tau) \triangleq \langle \{ \uparrow \sqsubseteq \delta(U) \mid U \text{ open in } G^T(\mu G, \tau) \} \rangle$$

Theorem (Coincidence)

The divisibility topology is Noetherian, and coincides with the Alexandroff topology of the divisibility preorder (Hasegawa, 2002, Def. 2.7) “when it makes sense”.
The topologies over trees and words from Goubault-Larrecq (2013) are the divisibility topologies of the appropriate functors

1. $X^* = \mu Y.1 + X \times Y$

2. $T(X) = \mu Y.X \times Y^*$

We justified their definition
Applications

On big spaces
Theorem (Recurrent Subword Topology)

$X^\alpha$ is Noetherian with the topology generated by the following closed sets

$$P ::= T^? \quad \text{T closed}$$

$$| P_1 \ldots P_n$$

$$| P^{<\beta}$$

$\beta \leq \alpha$

⚠️ This is clearly not WQO!!!

We took advantage of the refinement on a pre-existing space!
1. WQOs and Beyond

2. Minimal Bad Sequence Arguments

3. Applications

4. Open Questions
◆ Topological spaces and closed maps
Can we re-interpret the limits in this setting? What would be the relation with PO-dilators Girard (1981); Freund (2020)?

◆ Actions on Invariants
What is the effect of the fixed point on $\leq_\tau$? On the stature of the resulting space?

◆ Check that we can extend the algebra
Check that if $G(X, Y_1, \ldots Y_n)$ is good, so is $\mu X. G(X, Y_1, \ldots, Y_n)$. 
Thank You!


It is not that bad!