Transducers

Session 11: Monoids (finally)

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TA mail* Course page† Exercises page‡

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1 Monoids and MSO

Exercise 1 (Monoids and Regularity). We say that a language is recognized by a finite monoid M if there exists a morphism $\mu \colon A^* \to M$ and a subset $P \subseteq M$ such that $L = \mu^{-1}(P)$.

1. Prove that a language is regular if and only if it is recognized by a finite monoid.
2. Use the above result to conclude that regular languages are closed under the following operations:
   - union,
   - intersection,
   - complement,
   - reversal,
   - concatenation
3. Prove that the class of regular languages is closed under radicals $\sqrt{L} := \{w \in A^* \mid ww \in L\}$.
4. Prove that the class of regular language is closed under Kleene star.

Exercise 2 (From MSO to Monoids). Let φ be an $\mathbb{MSO}$ sentence over finite words. Prove that there exists a monoid M and a function $f : A^* \to M$ and a subset $P \subseteq M$ such that $w \models \varphi$ if and only if $f(w) \in P$.

Can you adapt the construction in the case of $\mathbb{MSO}$ formulas?

▷ Hint 1

Exercise 3 (From Monoids to MSO). Let M be a finite monoid and $m \in M$. Construct an $\mathbb{MSO}$ formula $\varphi_m(x, y)$ over M^* that accepts all pairs $x < y$ such that the factor $w[x : y]$ evaluates to m.

If the monoid is aperiodic, can you write this formula in $\mathbb{FO}$?

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†https://www.mimuw.edu.pl/~bojan/2023-2024/przeksztalcenia-automatowe-transducers
‡https://aliaumel.github.io/transducer-exercices/

2 Factorisation Forests

Exercise 4 (Baby Factorisation Forest)**.** Let M be a finite monoid. Prove that there exists a constant N such that for every $w \in M^*$ with $|w| \geq N$, there exist $v_0, v_1 \in M^*$, and $u \in M^+$ such that $w = v_0 u v_1$ and u is an idempotent element of M.

Exercise 5 (First-order Factorisation Forests)**.** Let M be a finite aperiodic monoid. Prove that there exists a constant N such that for every $w \in M^*$, one can build a factorisation of w of depth at most N, where idempotent products are replaced by constant products.

Exercise 6 (Pumping lemma for regular functions)**.** Let f be a regular function. Prove that there exists $N \geq 0$ such that for all $w \in A^*$ with $|w| \geq N$, there exist $v_0, v_1 \in A^*$, $u \in A^+$, $n \geq 0$, $\alpha_0, \ldots, \alpha_n \in B^*$, $\beta_1, \ldots, \beta_n \in B^+$ such that $w = v_0 u v_1$ and

$$f(v_0 u^{X+1} v_1) = \alpha_0 \beta_1^X \alpha_1 \ldots \beta_n^X \alpha_n \quad , \quad \text{for all } X \geq 0 \quad .$$

▷ Solution 1 (Solution)

Exercise 7 (Efficient Query Evaluation)**.** Let $q \in \mathbb{N}$. Provide a linear-time computation of a data-structure over a word w allowing for constant-time answer to $\mathbb{MSO}$ queries of quantifier depth at most q.

▷ Hint 2

A Hints

Hint 1 (Exercise 2 Use automata theory). At least for the first part, you can use the fact that φ defines a regular language.

Hint 2 (Exercise 7 Use factorisation forests). Construct a factorisation forest of the monoid of $\mathbb{MSO}^q$ types.

B Solutions

Solution 1 (Solution to Exercise 6). Without loss of generality, we assume that $Q = Q^{\leftarrow} \cup Q^{\rightarrow}$, where states in $Q^{\leftarrow}$ are always doing left transitions, while states in $Q^{\rightarrow}$ are always doing right transitions. We define the transition monoid of f as follows: $M := Q \to Q$. The intended semantics is that given a state $q \in Q$, and a word u, the transition performed by u is given by the first state reached by f outside of the word u.