# PhD Defense

First Order Preservation Theorems in Finite Model Theory: Locality, Topology, and Limit Constructions

Aliaume Lopez

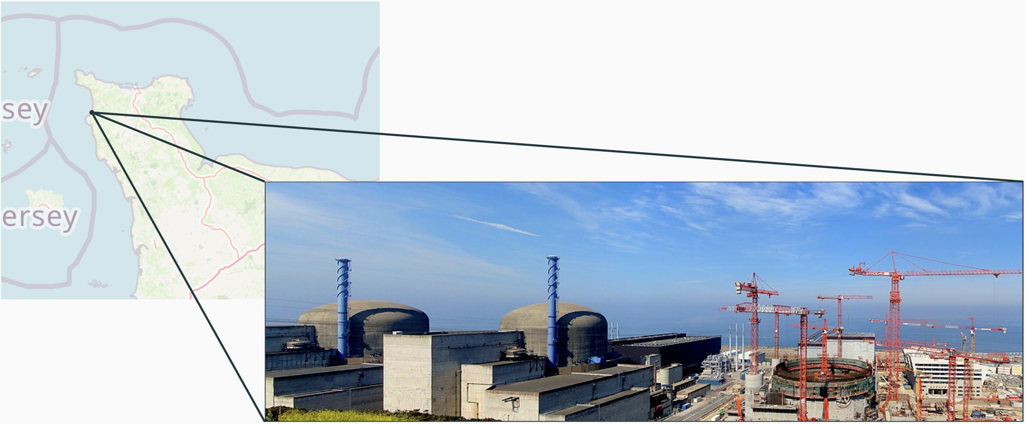
September 12, 2023, IRIF, France

Under the supervision of **Jean Goubault-Larrecq**, and **Sylvain Schmitz**.

# Introduction

## A Nuclear Question

# Velo Trip Time



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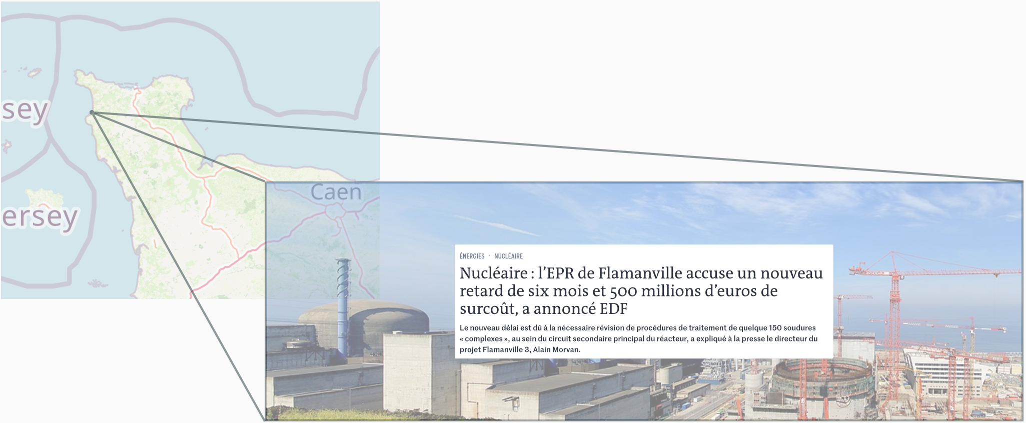
# Velo Trip Time

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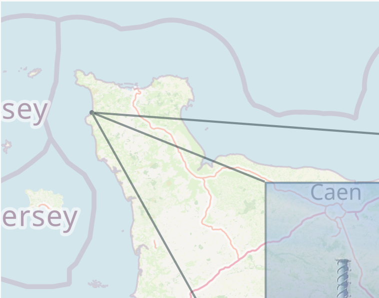


| Now | Then |
|---|---|
| ☢ 2 Reactors | ☢ 3 Reactors |
| ⚡ 2660 MWe | ⚡ 4290 MWe |


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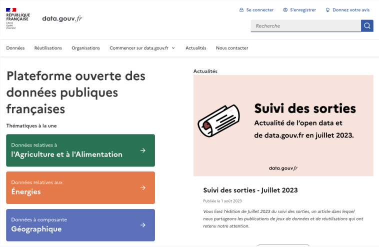
“Is the maximal installed power obtained with the highest number of reactors?”

# POV: YOU ARE A DATA ANALYST



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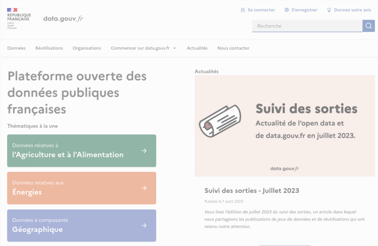
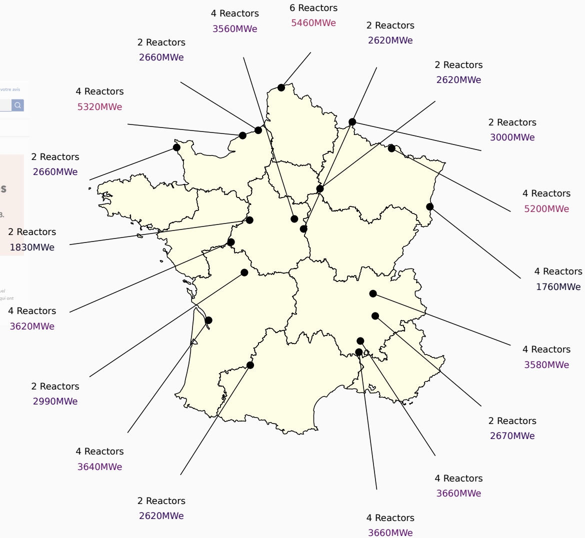



liste-centrales-nucleaires.csv

# POV: YOU ARE A DATA ANALYST





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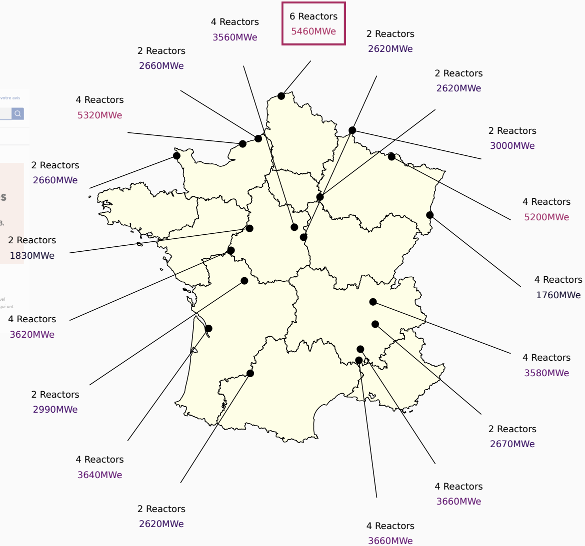


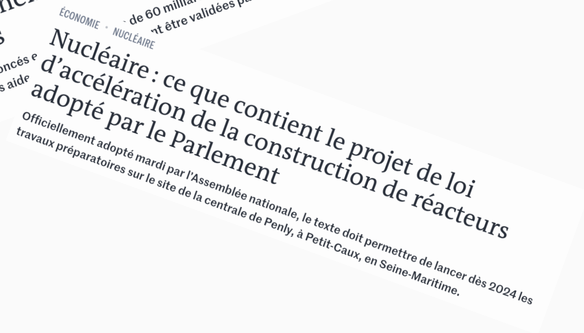
“What about the new ones?”

# Our theorem might become false!

ÉCONOMIE • NUCLÉAIRE

Nucléaire : le gouvernement au défi de financer de nouveaux réacteurs

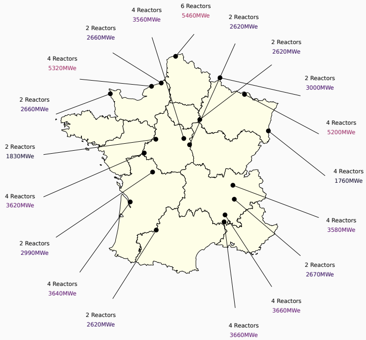
La construction des six réacteurs annoncés e… de 60 milliards d'euros. Un budget qui reste à trouver, sachant que les éventuelles aide… …nt être validées par la Commission européenne.



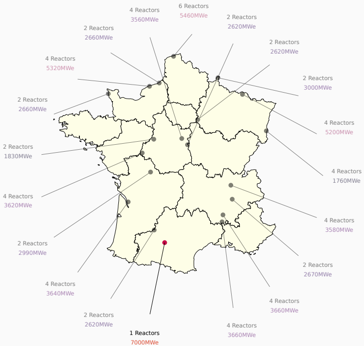
ÉCONOMIE • NUCLÉAIRE

Nucléaire : ce que contient le projet de loi d'accélération de la construction de réacteurs adopté par le Parlement

Officiellement adopté mardi par l'Assemblée nationale, le texte doit permettre de lancer dès 2024 les travaux préparatoires sur le site de la centrale de Penly, à Petit-Caux, en Seine-Maritime.

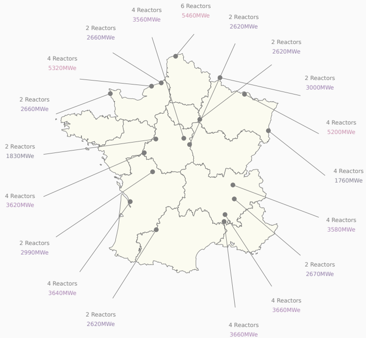

4 Reactors
3560MWe
6 Reactors
5460MWe
2 Reactors
2620MWe
2 Reactors
2660MWe
2 Reactors
2620MWe
4 Reactors
5320MWe
2 Reactors
3000MWe
2 Reactors
2660MWe
4 Reactors
5200MWe
2 Reactors
1830MWe
4 Reactors
1760MWe
4 Reactors
3620MWe
4 Reactors
3580MWe
2 Reactors
2990MWe
2 Reactors
2670MWe
4 Reactors
3640MWe
4 Reactors
3660MWe
2 Reactors
2620MWe
4 Reactors
3660MWe

Current ✅

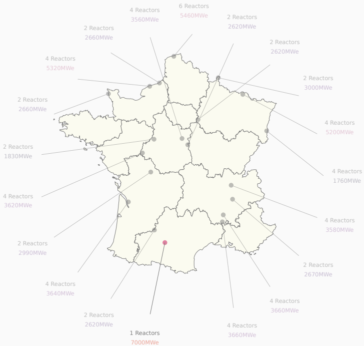



Scenario A ❌

1 Reactor / 7000 MWe




Current ✔

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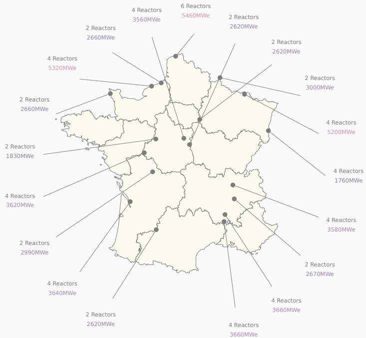


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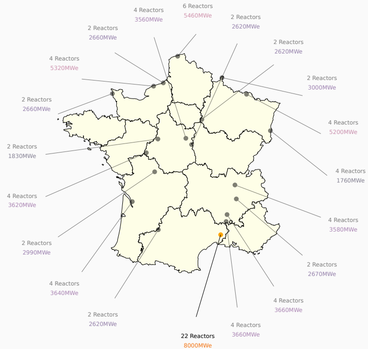
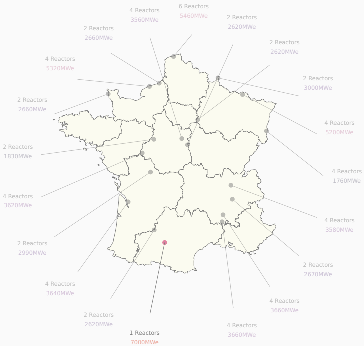


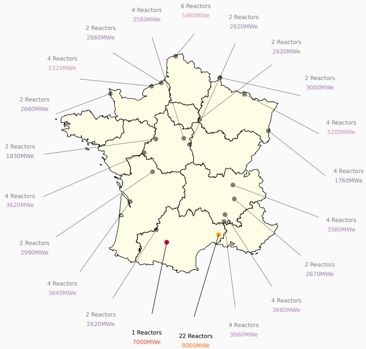

Scenario B ✔

22 Reactors / 8000 MWe



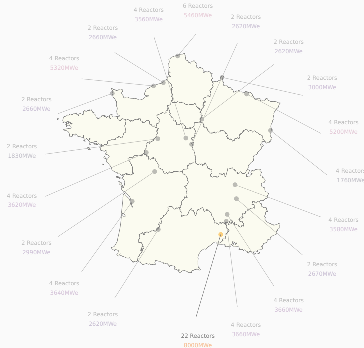

Scenario A ❌

1 Reactor / 7000 MWe



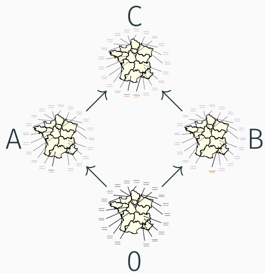

Scenario C ✔

22 Reactors / 8000 MWe

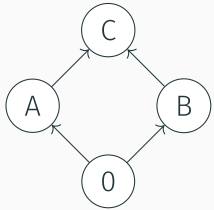


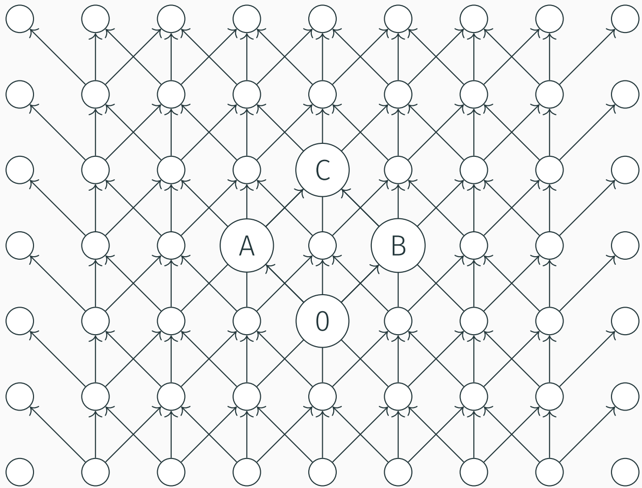

Scenario B ✔

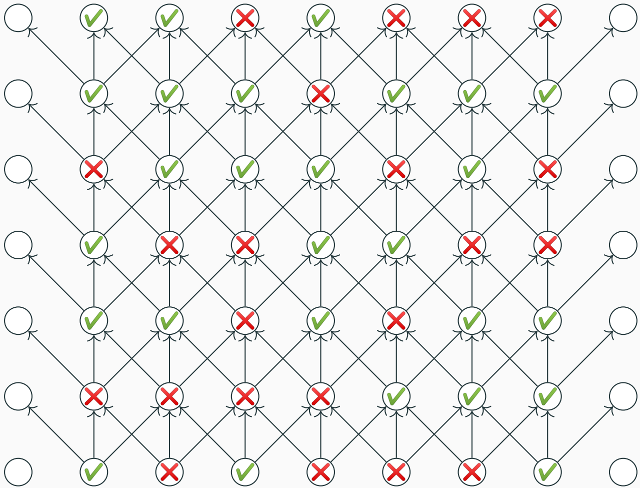
22 Reactors / 8000 MWe

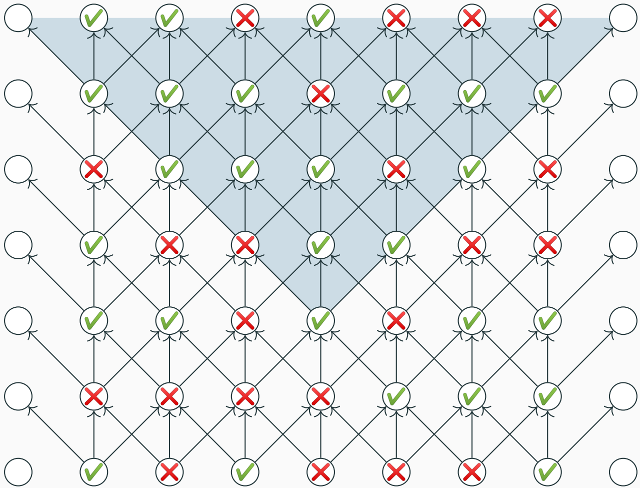

C
A
B
0

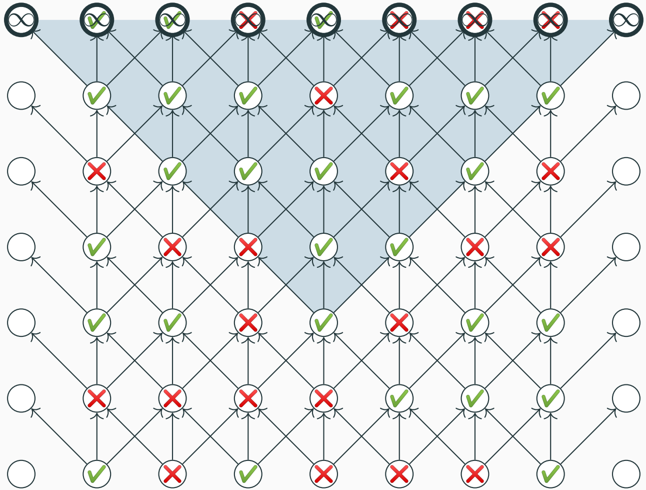
# Possible Scenarii Map (and extensions)

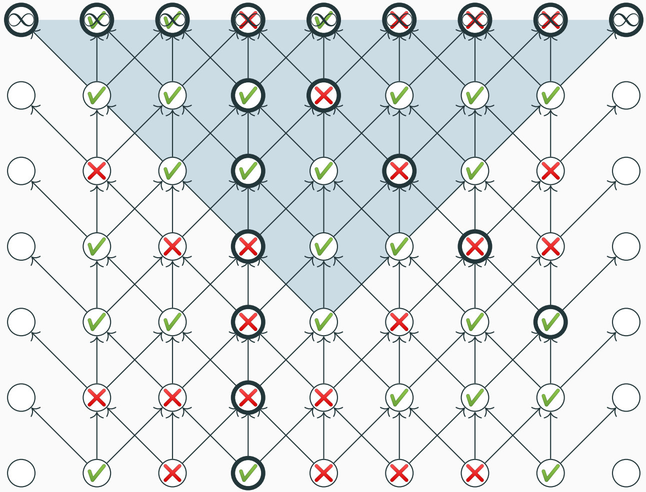


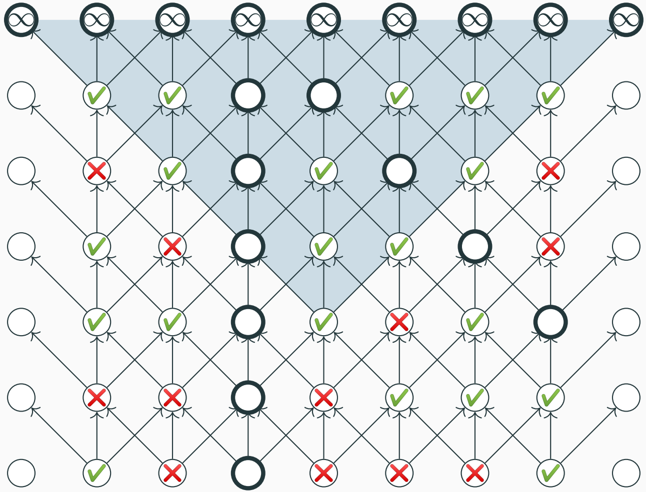

C
A
B
0











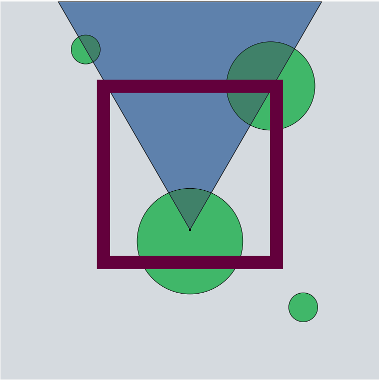
# Possible Scenarii Map From a Distance

Universe: Struct($\sigma$)

Query result: $[\![\varphi]\!]$

Upward closure: $\uparrow\mathfrak{A}$ (cone)

Restricted universe: $\mathcal{C}$



# Do we have to think before tweeting?

# Introduction

## Not All Queries are Born Equal

In First Order Logic (Theoretical Computer Science)

$$
\begin{aligned}
&\exists c.\exists n.\exists p. \\
&\text{IsAPowerplant}(c, n, p) \wedge \\
&(\forall c', n', p'. \\
&\quad \text{IsAPowerplant}(c', n', p') \\
&\quad \implies (p' \leq p \wedge n' \leq n)) \quad .
\end{aligned}
$$

# "Does more reactors mean more power?"

In First Order Logic (Theoretical Computer Science)

$$
\begin{aligned}
&\exists c.\exists n.\exists p. \\
&\mathrm{IsAPowerplant}(c, n, p) \wedge \\
&(\forall c', n', p'. \\
&\quad \mathrm{IsAPowerplant}(c', n', p') \\
&\quad \implies (p' \leq p \wedge n' \leq n)) \quad .
\end{aligned}
$$

SQL (Applied Computer Science)

```sql
SELECT sc.num_reactor, sc.installed_power
FROM scenario AS sc
WHERE sc.num_reactor =
      (SELECT MAX(scm.num_reactor)
       FROM scenario AS scm)
AND   sc.installed_power =
      (SELECT MAX(scm.installed_power)
       FROM scenario AS scm)
```

Existential Formulas (EFO)

$$\varphi := \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg R(x_1, \dots, x_n) \mid R(x_1, \dots, x_n) \mid \exists x . \varphi$$

## Existential Formulas (EFO)

$$\varphi := \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg R(x_1, \dots, x_n) \mid R(x_1, \dots, x_n) \mid \exists x. \varphi$$

## Lemma (folklore)

*These can be evaluated naïvely, in any context: for every existential sentence $\varphi$, $[\![\varphi]\!] = \uparrow [\![\varphi]\!]$. Equivalently, $[\![\varphi]\!]$ is upwards closed, or $\varphi$ is preserved under extensions (i.e., injective strong homomorphisms).*

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$$\varphi := \top \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg R(x_1, \ldots, x_n) \mid R(x_1, \ldots, x_n) \mid \exists x.\varphi$$

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*These can be evaluated naïvely, in any context: for every existential sentence $\varphi$, $[\![\varphi]\!] = \uparrow[\![\varphi]\!]$. Equivalently, $[\![\varphi]\!]$ is upwards closed, or $\varphi$ is preserved under extensions (i.e., injective strong homomorphisms).*

## Existential Formulas (EFO)

$$\varphi := \top \mid \bot \mid \varphi \land \varphi \mid \varphi \lor \varphi \mid \neg R(x_1, \ldots, x_n) \mid R(x_1, \ldots, x_n) \mid \exists x.\varphi$$

## Lemma (folklore)

*These can be evaluated naïvely, in any context: for every existential sentence $\varphi$, $[\![\varphi]\!] = \uparrow[\![\varphi]\!]$. Equivalently, $[\![\varphi]\!]$ is upwards closed, or $\varphi$ is preserved under extensions (i.e., injective strong homomorphisms).*

**Theorem (Łoś-Tarski)**

*For every first order sentence $\varphi$, the following are equivalent*

1. $[\![\varphi]\!] = \uparrow[\![\varphi]\!]$ *($[\![\varphi]\!]$ is upwards closed, $\varphi$ is preserved under extensions), and*
2. *there exists an existential sentence $\psi$ such that $[\![\varphi]\!] = [\![\psi]\!]$.*

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$$\text{“}[\![\mathsf{FO}]\!] + \uparrow \leftrightsquigarrow [\![\mathsf{EFO}]\!]\text{”}$$

A non-existential query

$$
\begin{aligned}
&\exists c.\exists n.\exists p.\\
&\text{IsAPowerplant}(c, n, p)\wedge\\
&(\forall c', n', p'.\\
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\end{aligned}
$$

A non-existential query

$$\exists c.\exists n.\exists p.\ \mathrm{IsAPowerplant}(c,n,p)\wedge (\forall c',n',p'.\ \mathrm{IsAPowerplant}(c',n',p') \implies (p' \leq p \wedge n' \leq n)) \quad .$$

**Łoś-Tarski does not relativise! (e.g., finite models)**

- Can be naïvely evaluated *in the subclass* $\mathcal{C}$: $[\![\varphi]\!] \cap \mathcal{C} = \uparrow[\![\varphi]\!] \cap \mathcal{C}$
- Is equivalent to an existential sentence *in the subclass* $\mathcal{C}$: $[\![\varphi]\!] \cap \mathcal{C} = [\![\psi]\!] \cap \mathcal{C}$.

Preservation theorems: variations around Łoś-Tarski

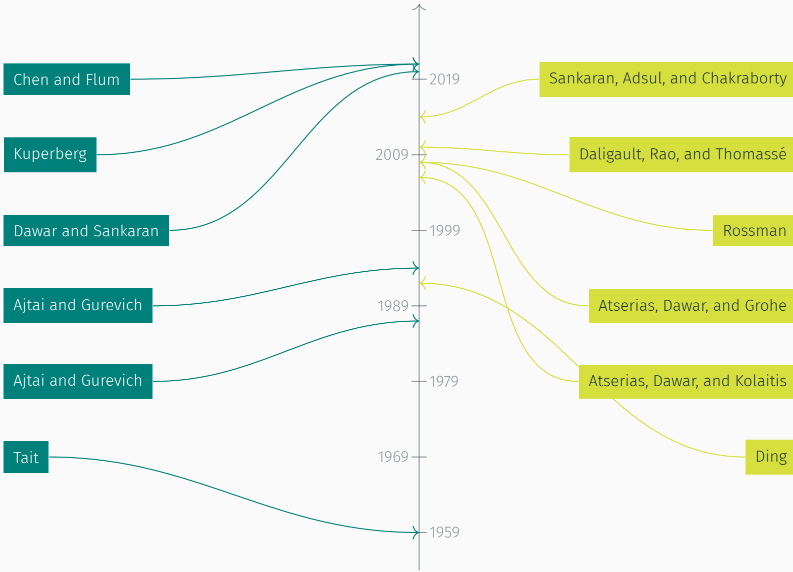
- Different possibilities to order structures: $\uparrow$,
- Different fragments of FO: EFO,
- Different subsets of interest: $\mathcal{C}$ (e.g., finite models).

| Preservation Under |
|---|
| homomorphisms |
| injective homomorphisms (Tarski-Lyndon) |
| strong injective homomorphisms (Łoś-Tarski) |
| surjective homomorphisms (Lyndon) |
| strong surjective homomorphism |
| $\forall$FO-embeddings (dual Chang-Łoś-Suszko) |

# Most variants do not relativise to the finite

| Preservation Under | Relativises to Fin(σ) |
|---|---|
| homomorphisms | ✔ [Ros08] |
| injective homomorphisms (Tarski-Lyndon) | ❌ [AG94a, Theorem 10.2] |
| strong injective homomorphisms (Łoś-Tarski) | ❌ [Tai59; Gur84; DS21] |
| surjective homomorphisms (Lyndon) | ❌ [AG87a; Sto95] |
| strong surjective homomorphism | ❌ [Cap+20] |
| ∀FO-embeddings (dual Chang-Łoś-Suszko) | ❌ [San+12] |

# Positive and Negative Results in the Finite



# Why is this theorem useful?

**Not used to rewrite queries!**

- Better understand Finite Model Theory (compared to Model Theory),
- Provide completeness of proof techniques ([Lib11; DNR08]).

Understand how and why preservation theorems relativise to some classes of (finite) structures.

# Diving in

## Three Specific Examples Among Classes of Finite Undirected Graphs

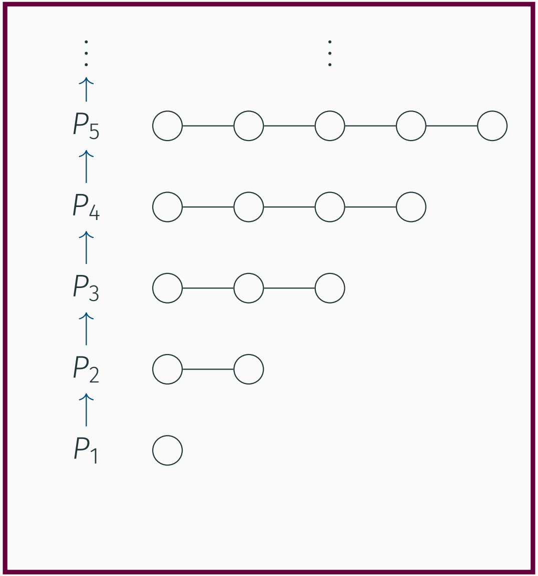
Universe: Undirected Graphs

Query: $[\![\varphi]\!]$

Order: extensions

Restriction: Finite Paths (Paths)

The Łoś-Tarski Theorem *relativises* to Paths!



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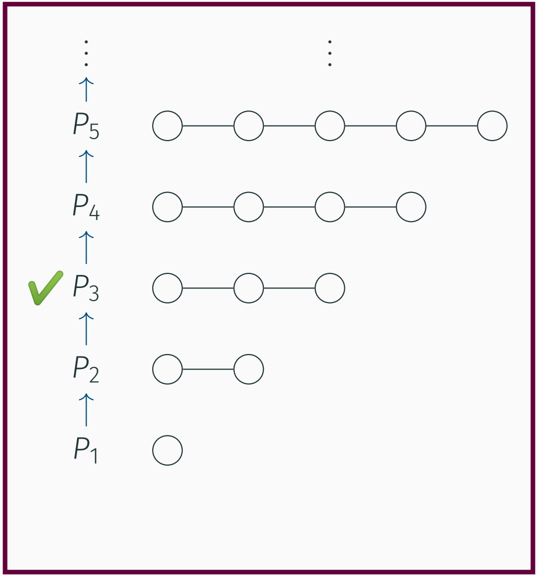

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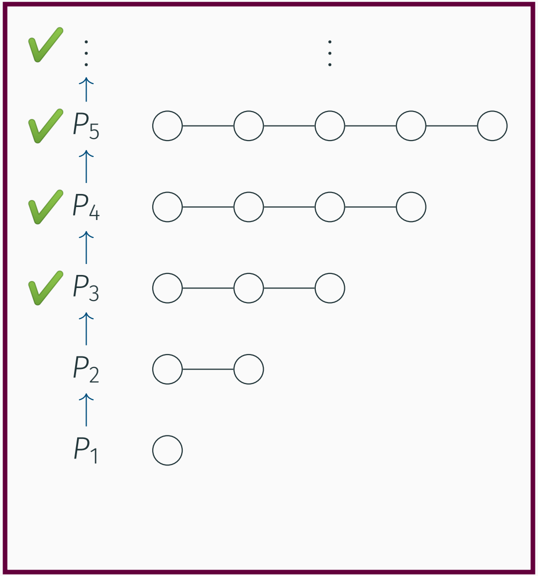

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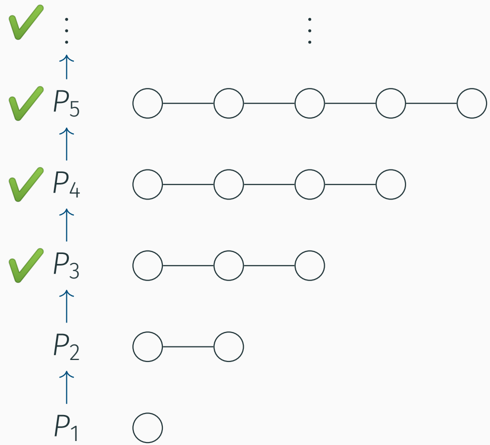
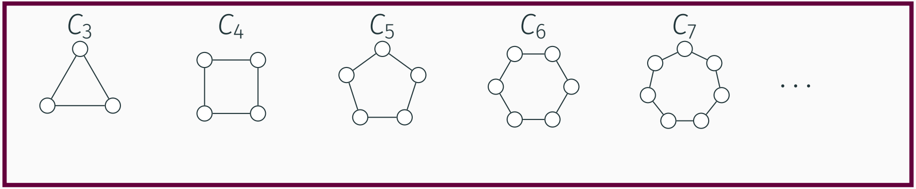



$[\![\exists^{\geq 3} x. \top]\!] \cap \text{Paths} = [\![\varphi]\!] \cap \text{Paths}$

Universe: Undirected Graphs
Order: extensions

Query: $[\![\varphi]\!]$
Restriction: Finite Cycles (Cycles)



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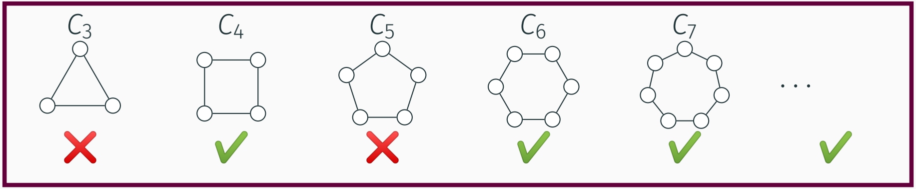

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Order: extensions

Query: $[\![\varphi]\!]$
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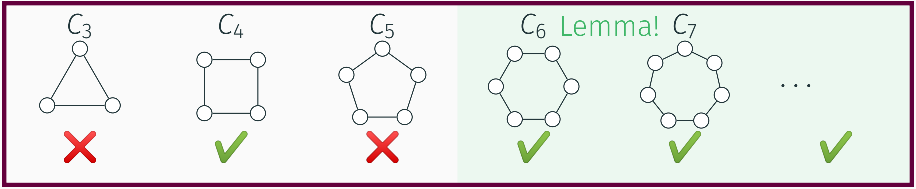


**Lemma (folklore)**

*For every $\varphi \in$ FO, there exists $N_0$, such that for all $n, m \geq N_0$, $C_m \in [\![\varphi]\!] \iff C_n \in [\![\varphi]\!]$.*

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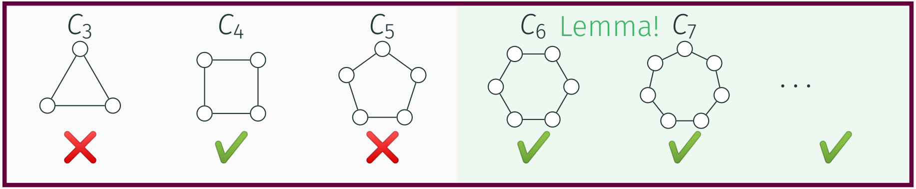


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$$[\![\varphi]\!] \cap \text{Cycles} = [\![\exists^{=4} x.\top \vee \exists^{\geq 6} x.\top]\!] \cap \text{Cycles}$$

**Theorem ([ADG08, Theorem 4.3])**

*The Łoś-Tarski Theorem relativises to every class $\mathcal{C}$ of finite structures such that:*

1. *There exists a bound d on the maximal degree in the structures*
2. *The class is hereditary (neither Paths, nor Cycles)*
3. *The class is closed under disjoint unions (neither Paths, nor Cycles)*

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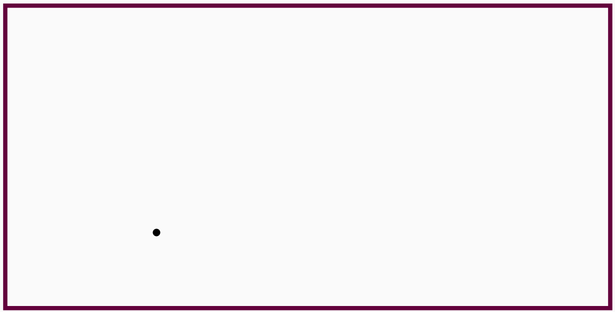

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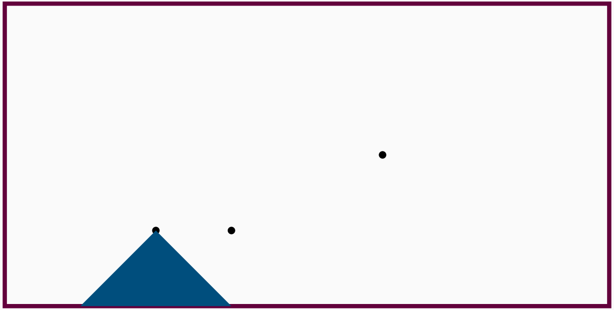

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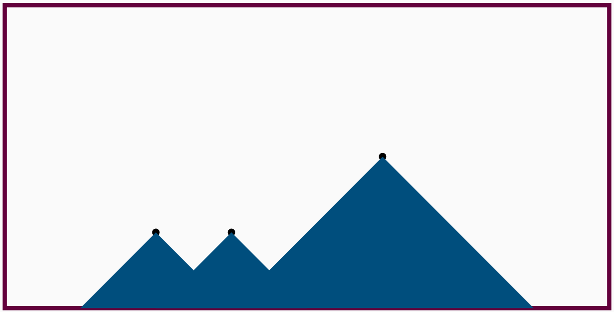

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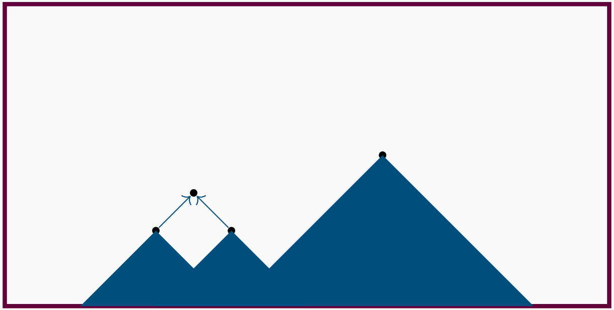

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*The Łoś-Tarski Theorem relativises to every class $\mathcal{C}$ of finite structures such that:*

1. *There exists a bound d on the maximal degree in the structures*
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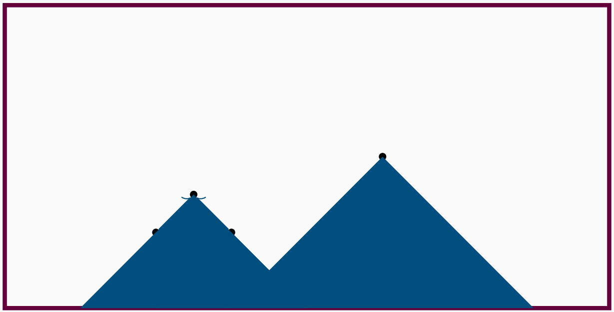

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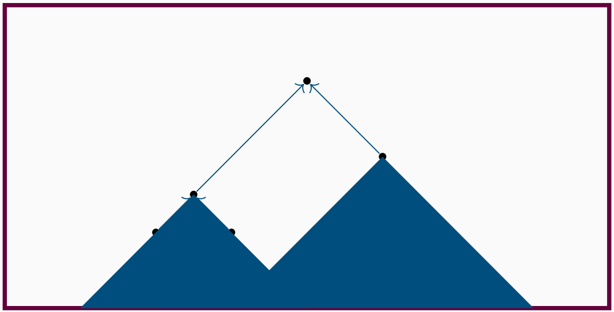

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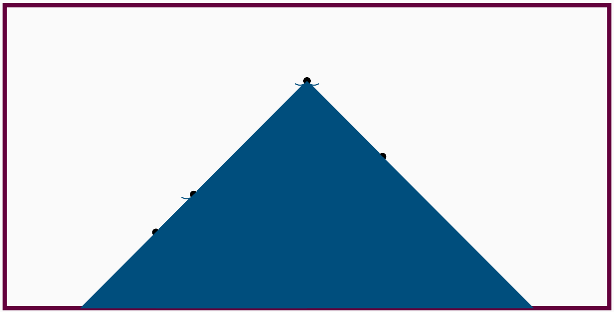

**Three Non Overlapping Internal Approaches**

1. Upwards closed subsets are "simple" (Paths) – $\uparrow E$ where $E$ is finite
2. Definable subsets are "simple" (Cycles) – (complements of) finite subsets
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**An external approach?**

Is it possible to avoid starting from scratch every time?

- Cycles $\cup$ Paths? None of the above apply!

# Diving in

## Expectations

### Definability

Local To Global
Łós-Tarski relativisation

Positive
Gaifman Normal Form

### External Approach

Logically presented
pre-spectral spaces

Composition theorems
for LPPS

### Topology

Topology Expanders
for Noetherian spaces

Limit Constructions
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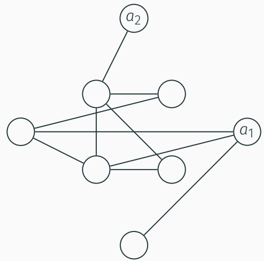
# Local Approach

## The locality Theorem

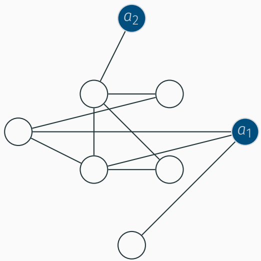
### Usage in Finite Model Theory

- It is a combinatorial tool that works in finite classes.
- Abstracts the low-level “game” arguments of first-order logic.
- Already has been used to prove the relativisation of preservation theorems [ADK06; ADG08, e.g.].

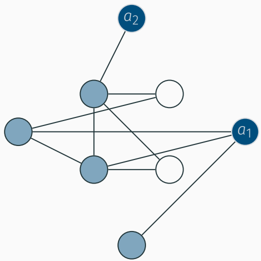
# Locality in a graph




A structure $\mathfrak{A}$.

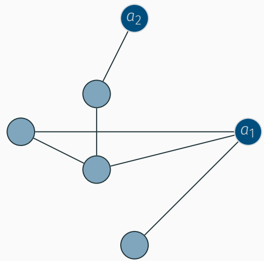



A structure $\mathfrak{A}$, with 2 selected nodes.



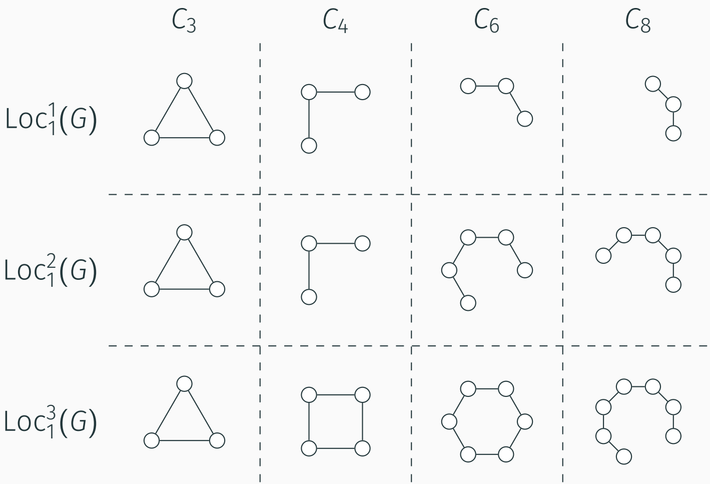

A structure $\mathfrak{A}$, with 2 selected nodes, and a 1-local neighborhood.

## Locality in a graph




$$\mathcal{N}_{\mathfrak{A}}(a_1 a_2, 1) \subseteq_i \mathfrak{A}.$$

# Local Neighbourhoods of Cycles



# A LOCALITY THEOREM

## Theorem ([Gai82])

*Every first order sentence (FO) is equivalent to a Boolean combination of basic local sentences.*

## Basic Local Sentence

$$\exists_r^{\geq n} x.\psi(x)$$

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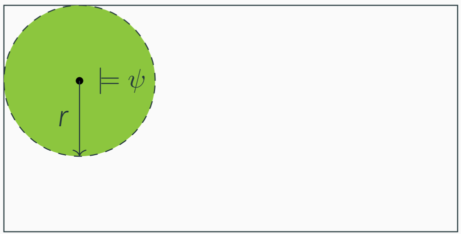

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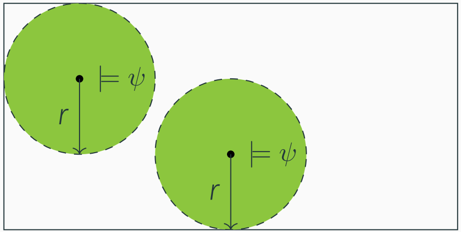

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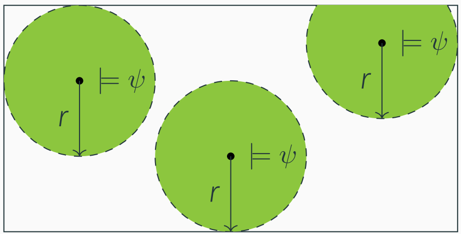

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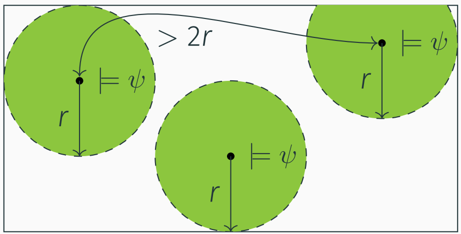

# Locally satisfying a property?

$$\mathrm{Loc}_k^r(\mathcal{C}) \stackrel{def}{=} \{\, \mathcal{N}_A(\vec{a}, r) \mid A \in \mathcal{C}, \vec{a} \in A^k \,\}$$

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**Localise Bounded Degree**

$\mathcal{C}$ is of bounded degree if and only if $\mathrm{Loc}_k^r(\mathcal{C})$ is finite for all $k, r \geq 0$, i.e., *locally* finite.

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*The Łoś-Tarski theorem relativises to hereditary classes of finite structures that are closed under* $\uplus$ *and* ***locally*** *finite.*

**Theorem ([Lop22, Theorem 6.7])**

*For a hereditary class of finite structures $\mathcal{C}$ that is closed under disjoint unions, the following are equivalent:*

1. *The Łoś-Tarski Theorem relativises to $\mathcal{C}$.*
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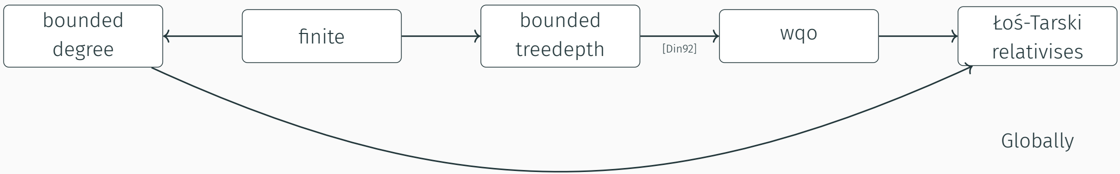
Full characterisation!

Agrandir les flèches dire que les inclusions sont strictes



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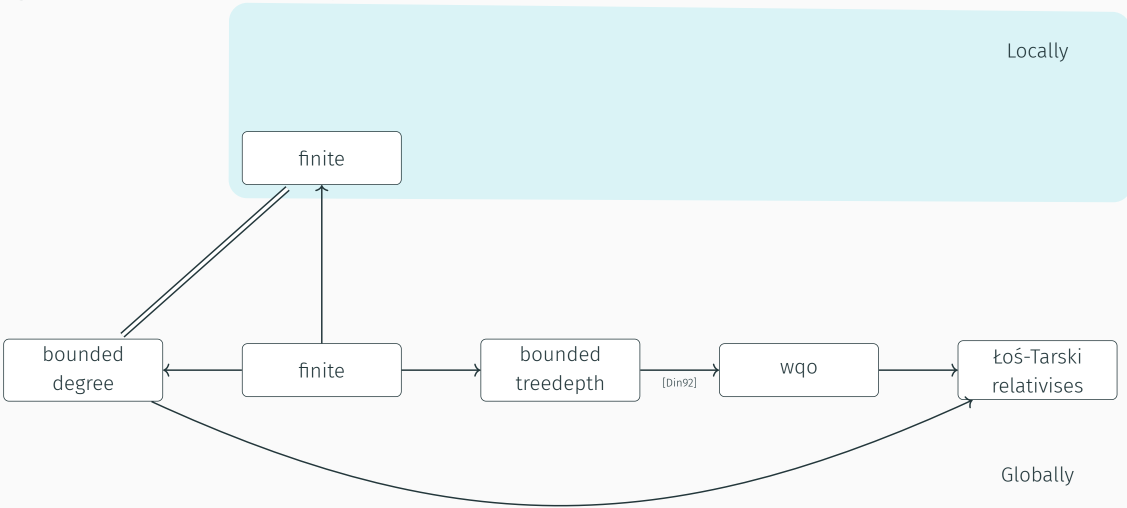

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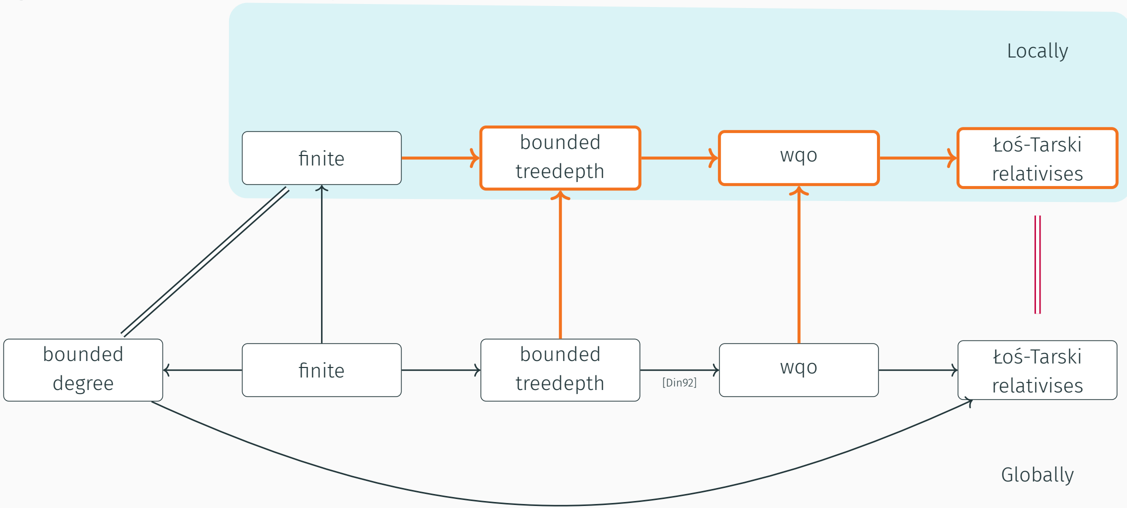

# Proof Sketch (for the difficult direction)

A Two Step Process

1. $\varphi_0$ + preserved under extensions $\rightsquigarrow$ $\varphi_1$ *existential-local*
2. $\varphi_1$ *existential-local* + preserved under extensions $\rightsquigarrow$ $\varphi_2$ *existential*

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**Existential, Existential Local, and Arbitrary Sentences**

- existential local sentences with $r = 0$ $\rightsquigarrow$ existential sentences
- existential local sentences over $\mathcal{C}$ $\leftrightsquigarrow$ sentences over $\mathrm{Loc}_k^r(\mathcal{C})$.

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### Core Combinatorial Argument

preserved under extensions $\rightsquigarrow$ minimal models are found in some $\mathrm{Loc}_k^r(\mathcal{C})$.

The usual approach: use the Gaifman Locality Theorem.

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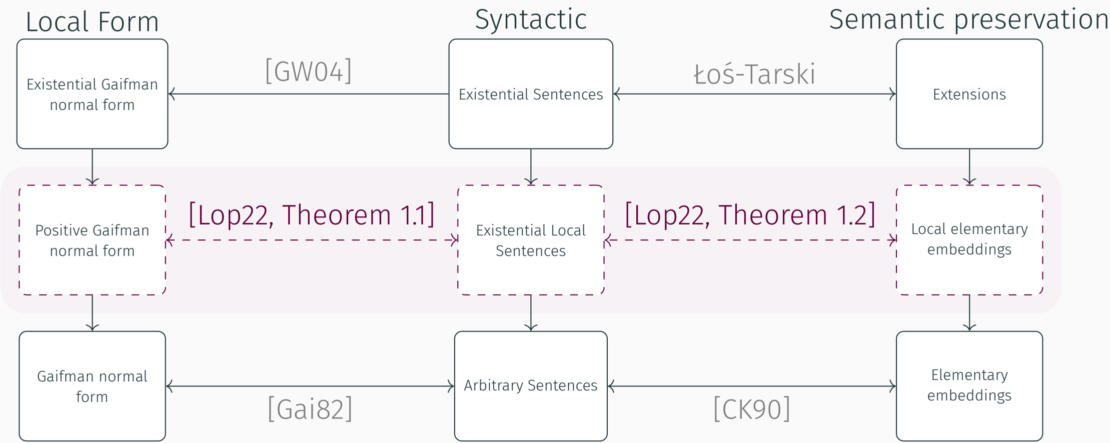
**Theorem ([Lop22, Theorem 1.1])**

*Let* $\mathcal{C} \subseteq \mathrm{Struct}(\sigma)$ *be a class of structures, and* $\varphi \in \mathrm{FO}[\sigma]$*. The following are equivalent*

1. $\varphi$ *is equivalent to an existential-local sentence, and*
2. $\varphi$ *is equivalent to a* ***positive*** *Boolean combination of basic local sentences.*

# Existential local sentences (in the general case)



| Definability | External Approach | Topology |
|---|---|---|
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# Compositional Approach

## The right abstraction

**Wishful conjecture**

Assume that the Łoś-Tarski relativises to $\mathcal{C}$ and $\mathcal{C}'$. Does the Łoś-Tarski theorem relativise to $\mathcal{C} \cup \mathcal{C}'$?

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Could we find a subset of preservation theorems that can be composed?

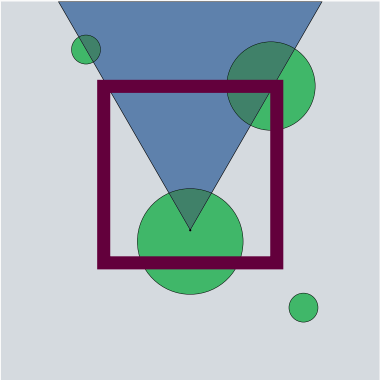
# A reasonable abstraction: LPPS

$$\langle\langle \mathcal{C}, \tau, \mathcal{B} \rangle\rangle$$

$\mathcal{C}$ is a class of structures

$\tau$ is a *topology* over $\mathcal{C}$

$\mathcal{B}$ is a *Boolean algebra* over $\mathcal{C}$



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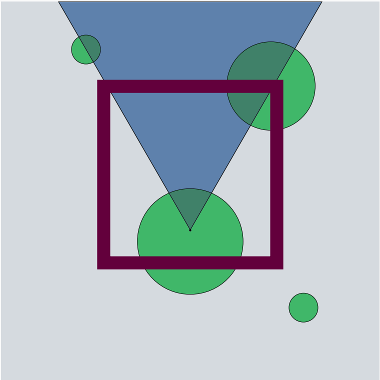

$$\langle \tau \cap \mathcal{B} \rangle_{\text{topo}} = \tau$$

$$\tau \cap \mathcal{B} = \mathcal{K}^{\circ}(\tau)$$

[Lop21, Definition 3.2]: logically presented pre-spectral space.

# What does $\mathcal{K}^\circ(\tau)$ means?

Definition: $\mathcal{K}^\circ(\tau)$ = compact open subsets

Typical example of compact open subset: $\uparrow\{\mathfrak{A}_1, \ldots, \mathfrak{A}_n\}$ (finite union of cones!)

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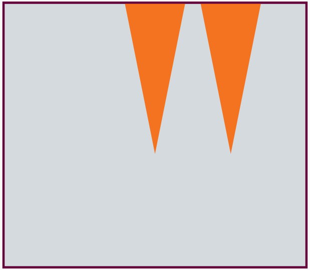

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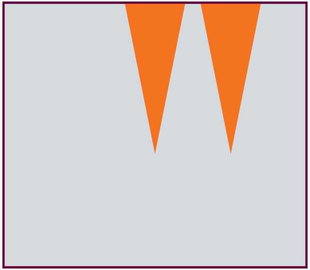

**Topological Property of Existential Sentences (in hereditary classes)**

They have finitely many minimal models, hence define compact open subsets!

Equation 1: enough subsets are definable and open (logically presented)

$$\langle \tau \cap \mathcal{B} \rangle_{\mathrm{topo}} = \tau$$

$\rightsquigarrow$ cones ($\uparrow \mathfrak{A}$) are first order definable!

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$\rightsquigarrow$ cones ($\uparrow\mathfrak{A}$) are first order definable!

Equation 2: definable and open subsets are compact open (pre-spectral)

$$\tau \cap \mathcal{B} = \mathcal{K}^{\circ}(\tau)$$

$\rightsquigarrow$ sentences preserved under extensions (in $\mathcal{C}$) define compact open subsets.

Let $\mathcal{C} \subseteq \mathrm{Fin}(\sigma)$.

# A completeness result (specialised to Łoś-Tarski)

Let $\mathcal{C} \subseteq \mathrm{Fin}(\sigma)$. We consider $\mathcal{B}$ to be the FO-definable subsets of $\mathcal{C}$, and $\tau$ to be the collection of upwards closed subsets of $\mathcal{C}$ (for extensions).

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**Theorem ([Lop21, Theorem 3.4], specialised to EFO and the finite setting)**

1. *The Łoś-Tarski Theorem relativises to $\mathcal{C}$,*
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2. *The space $\langle\langle \mathcal{C}, \tau, \mathcal{B} \rangle\rangle$ is an LPPS.*

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**Remarks**

- LPPS captures a subset of preservation theorems.
- The two coincide on *hereditary classes* of finite structures.
- LPPS will be stable under composition (finite sums, finite products, etc.)

LPPS captures “reasonable” preservation theorems.

### Generalises Already Known Spaces

- $\langle\!\langle \mathcal{C}, \tau, \mathcal{P}(\mathcal{C}) \rangle\!\rangle$ is an LPPS $\leftrightsquigarrow$ $(\mathcal{C}, \tau)$ is a Noetherian space (see [Gou13])
- $\langle\!\langle \mathcal{C}, \tau, \langle \mathcal{K}^\circ(\tau) \rangle_{\text{bool}} \rangle\!\rangle$ is an LPPS $\leftrightsquigarrow$ $(\mathcal{C}, \tau)$ is a Spectral space (see [DST19])

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### Compositional?

Both spectral and Noetherian spaces can be composed!

LPPS are stable under the following operations

| Operation | Symbol | Extra Hypothesis |
|---|---|---|
| sum | $\mathcal{C} + \mathcal{C}'$ | - |
| product | $\mathcal{C} \times \mathcal{C}'$ | - |
| inner product | $\mathcal{C} \otimes \mathcal{C}'$ | - |
| finite words | $\mathcal{C}^\star$ | - |
| wreath product | $\mathcal{C} \rtimes \mathcal{C}'$ | $\mathcal{C}$ is $\infty$-wqo |

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**Other stability results:**

- Surjective continuous and definable maps $f : \mathcal{C} \twoheadrightarrow \mathcal{C}'$.
- Boolean combinations of compact open subsets.

# Compositional Approach

## A concrete example: The product

# What is the product of two spaces?

Let $\langle\!\langle \mathcal{C}, \tau, \mathcal{B} \rangle\!\rangle$ and $\langle\!\langle \mathcal{C}', \tau', \mathcal{B}' \rangle\!\rangle$ be LPPS.

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**The elements of $\mathcal{C} \times \mathcal{C}'$**

Pairs $(\mathfrak{A}, \mathfrak{A}')$, with $\mathfrak{A} \in \mathcal{C}$ and $\mathfrak{A}' \in \mathcal{C}'$.

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**The definable subsets of $\mathcal{C} \times \mathcal{C}'$** **(works for FO!)**

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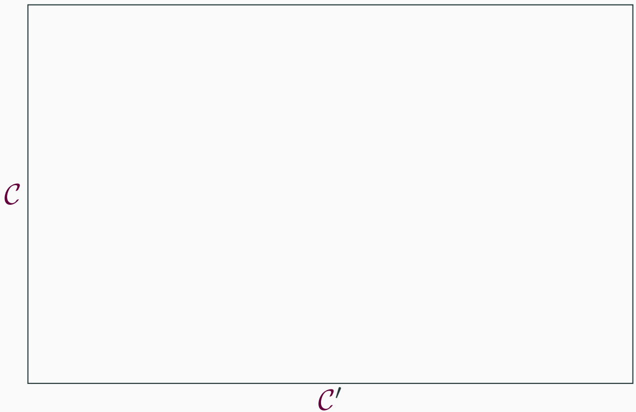
**Theorem ([Lop21, Proposition 5.8])**

$\langle\langle \mathcal{C} \times \mathcal{C}', \tau^{\times}, \mathcal{B}^{\times} \rangle\rangle$ *is an LPPS.*

Let us prove:

$$\tau^{\times} \cap \mathcal{B}^{\times} \subseteq \mathcal{K}^{\circ}(\tau^{\times}) \quad .$$

Let $U \in \tau^{\times} \cap \mathcal{B}^{\times}$.



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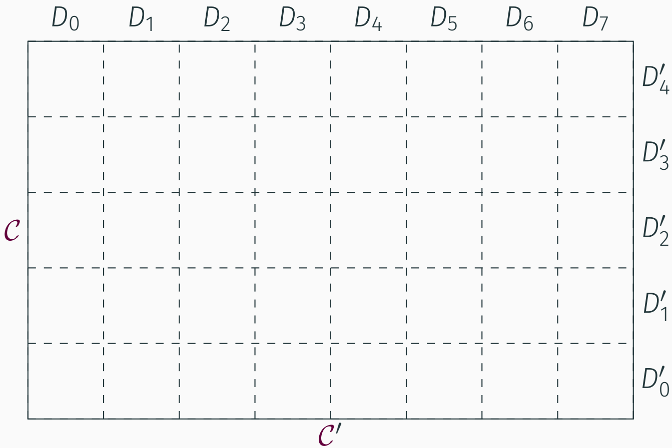

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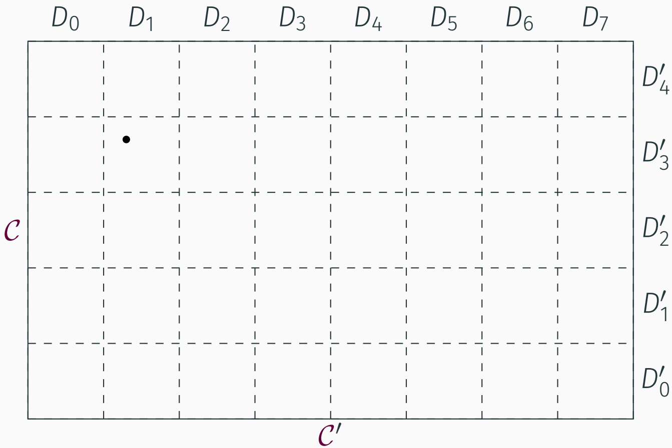

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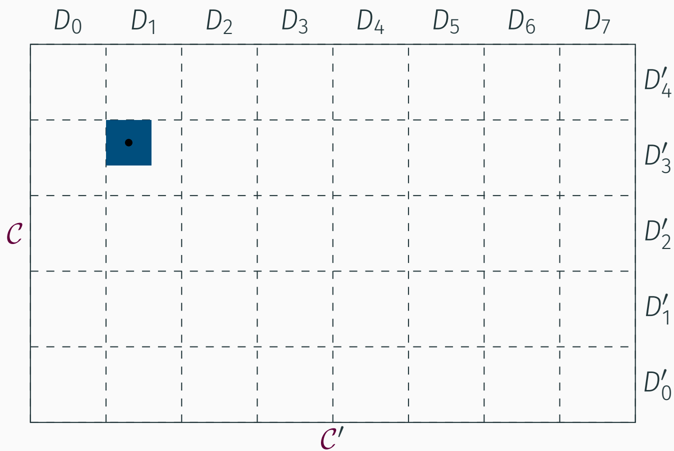

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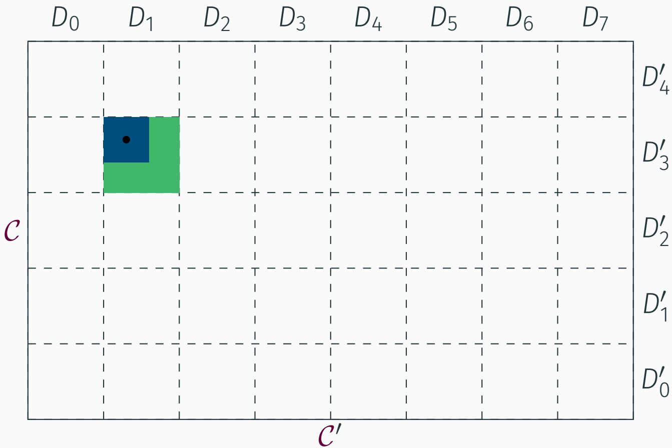

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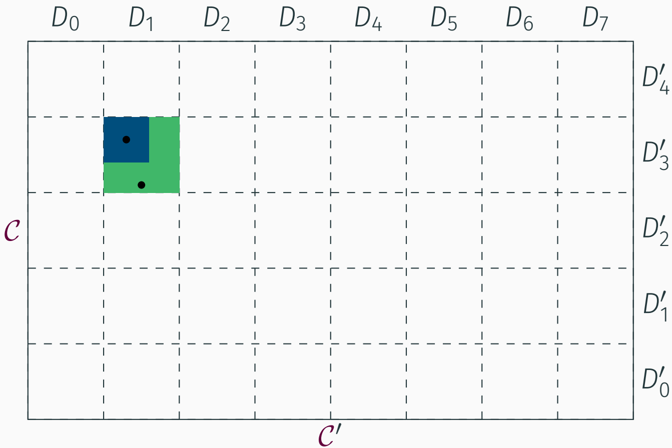

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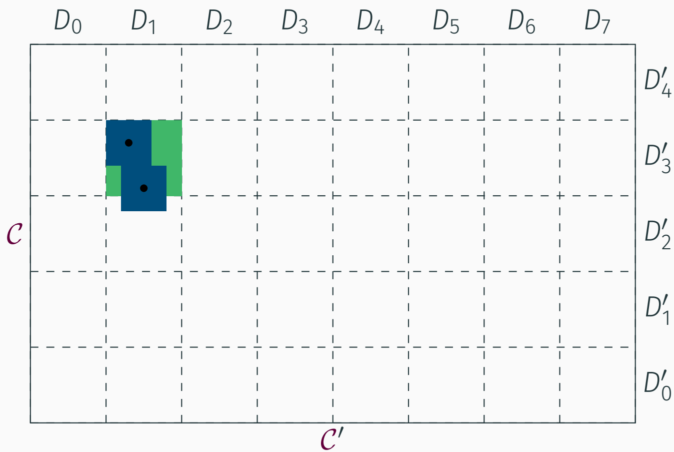

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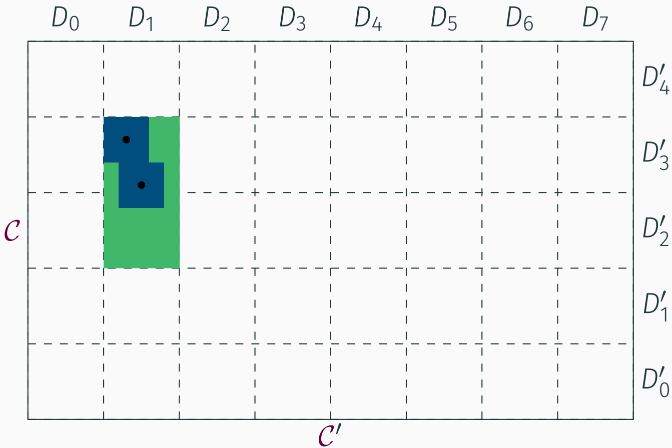

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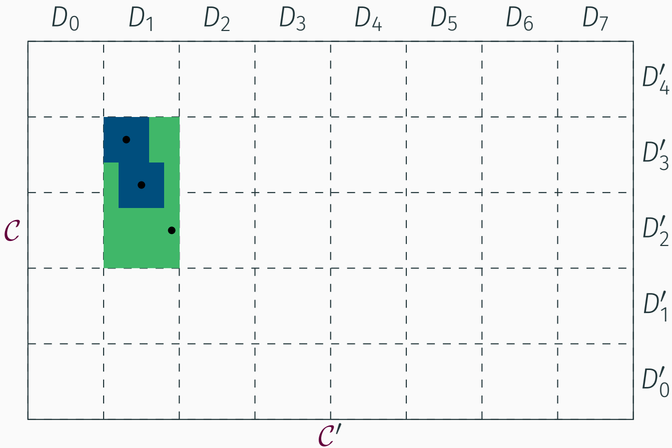

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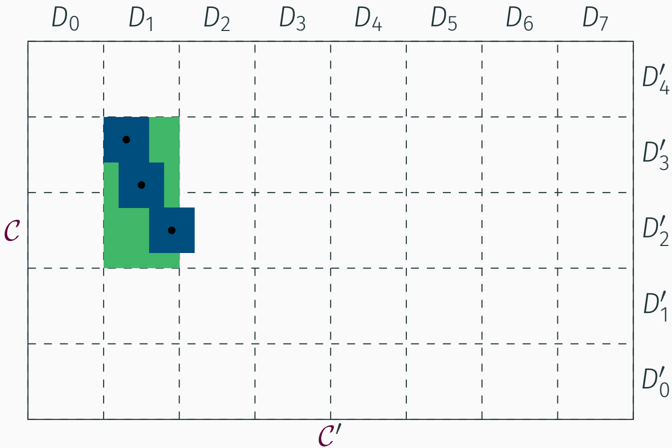

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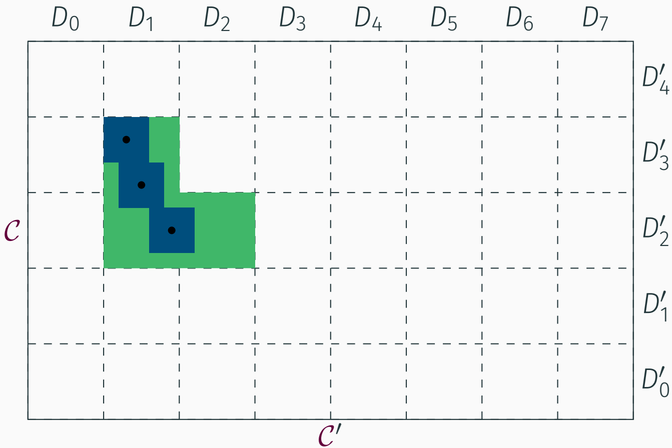

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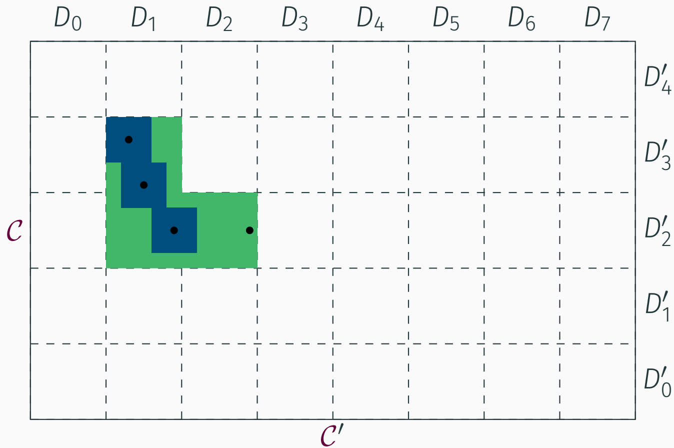

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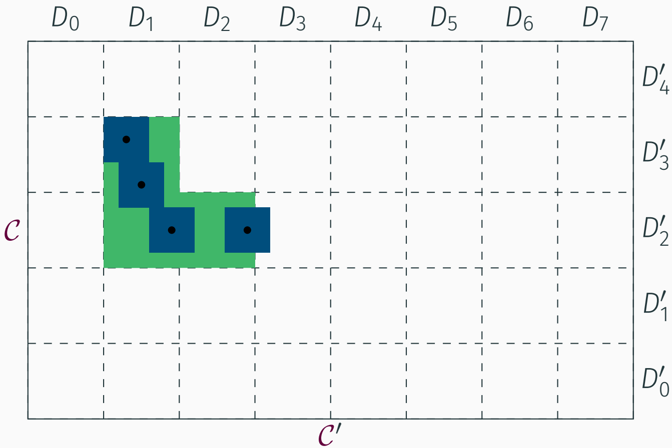

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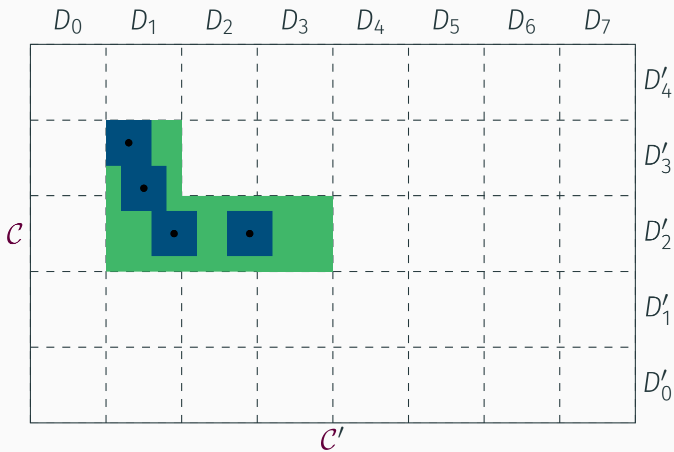

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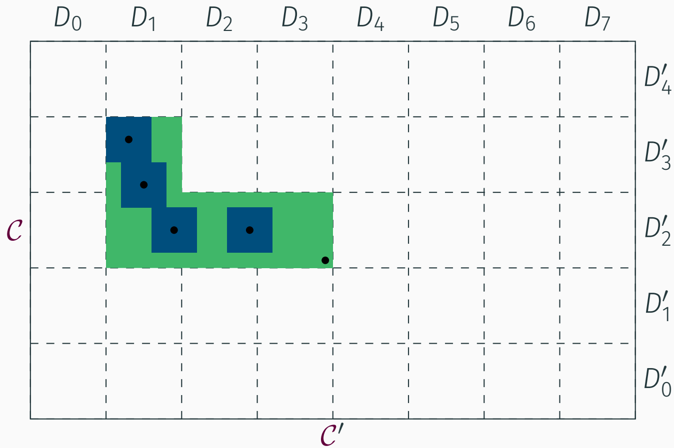

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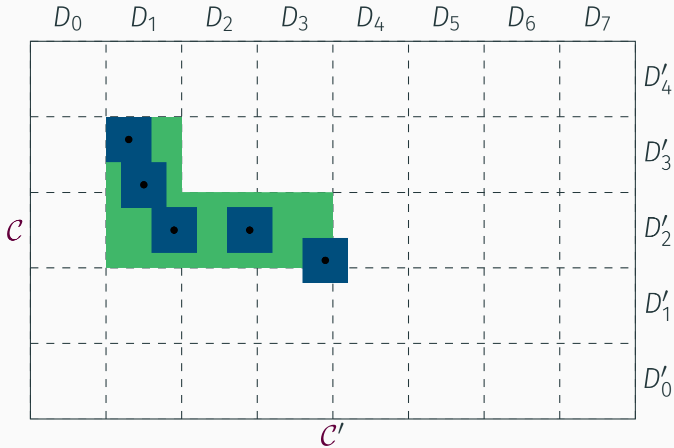

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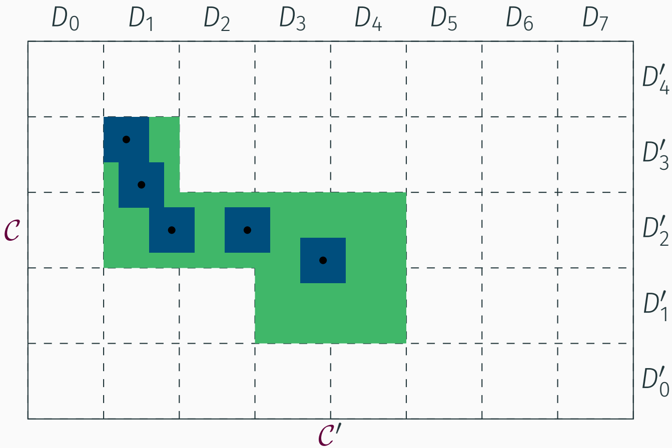

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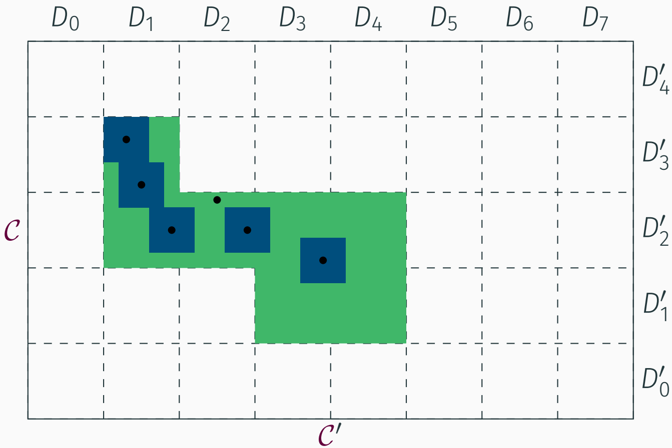

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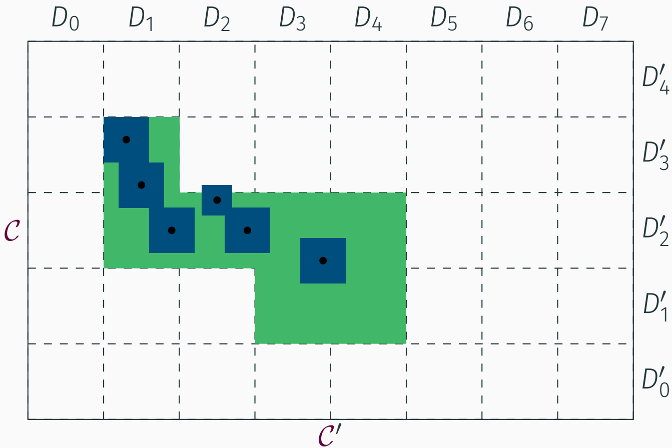

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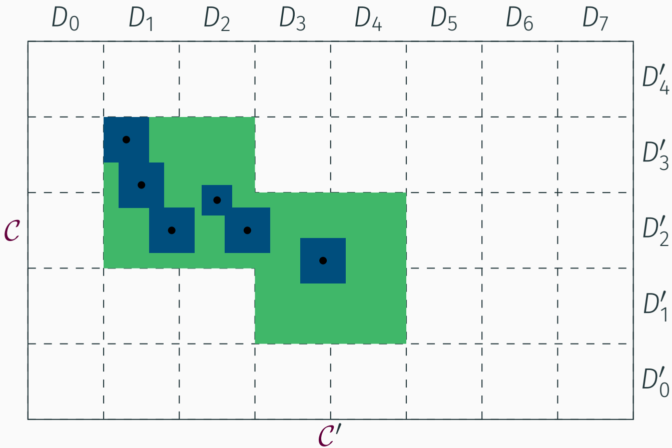

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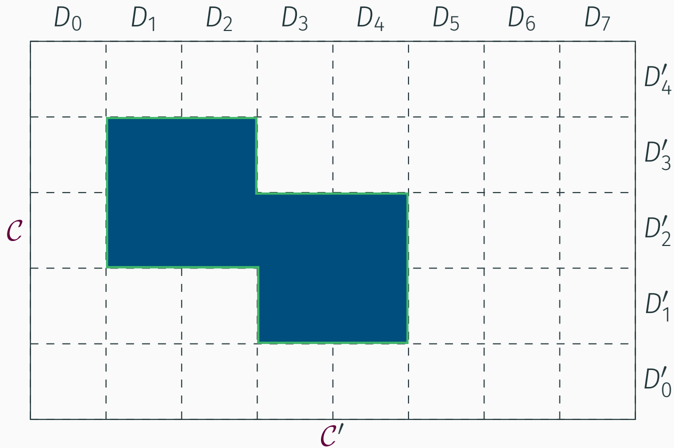

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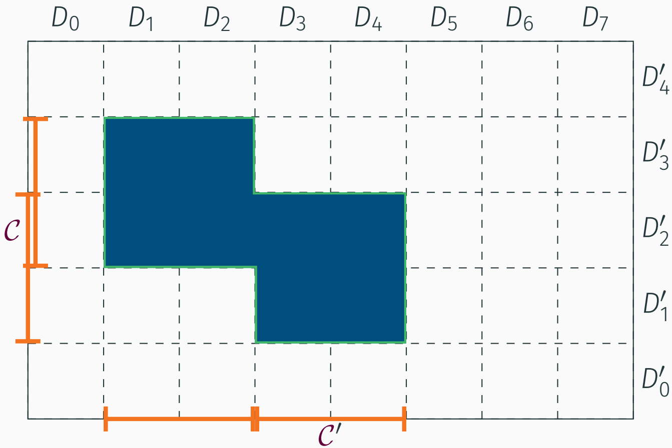

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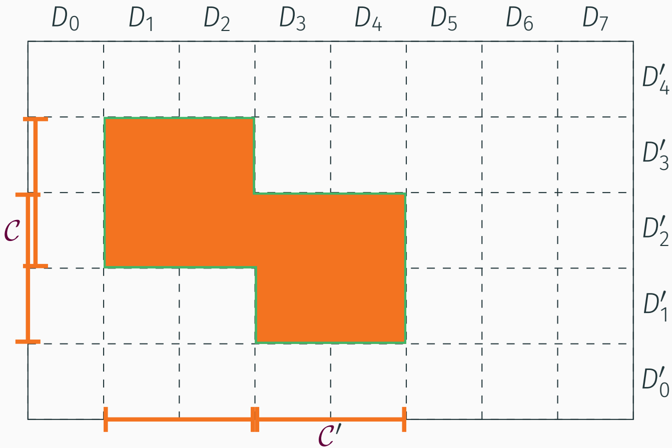
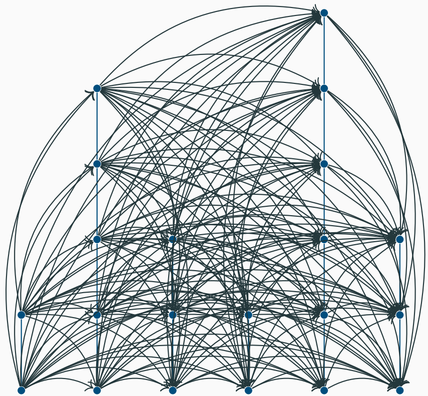


LinOrd $\rtimes$ Paths (for free!)



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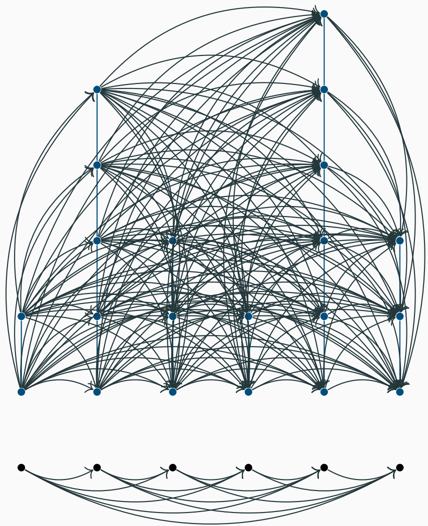
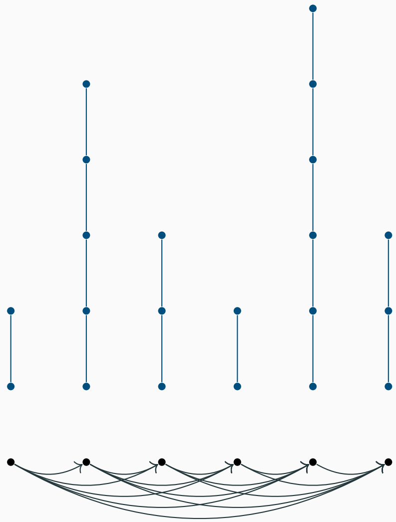


# Example

LinOrd ⋊ Paths (for free!)



# Concluding Remarks

## Contributions and Open Questions

| Definability | External Approach | Topology |
| --- | --- | --- |
| Local To Global Łós-Tarski relativisation | Logically presented pre-spectral spaces | Topology Expanders for Noetherian spaces |
| Positive Gaifman Normal Form | Composition theorems for LPPS | Limit Constructions of Noetherian spaces |

**Definability [Lop22]**

Local To Global
Łós-Tarski relativisation

Positive
Gaifman Normal Form

Twin-Width? [Bon+20]

**External Approach [Lop21]**

Logically presented
pre-spectral spaces

Composition theorems
for LPPS

Rossman's theorem?

**Topology [Lop23]**

Topology Expanders
for Noetherian spaces

Limit Constructions
of Noetherian spaces

Beyond Noetherian?

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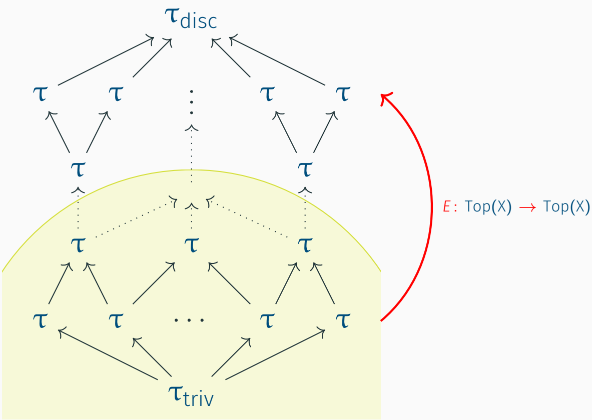



With $E$ monotone and fixing Noetherian topologies.

**Theorem ([Lop23, Theorem 3.21])**

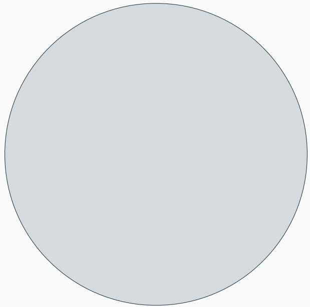
*If $E$ is monotone, fixes Noetherian topologies, and respects subsets, then the least fixed point of $E$ is a Noetherian topology.*

**Remarks**

- The extra condition is needed
- The proof uses a topological minimal bad sequence argument

For all $\tau$, $H$ closed subset of $\tau$,

$$E(\tau \mid H) \mid H = E(\tau) \mid H$$



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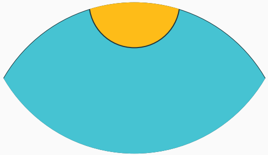

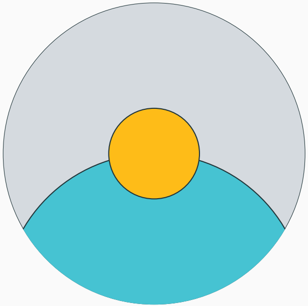

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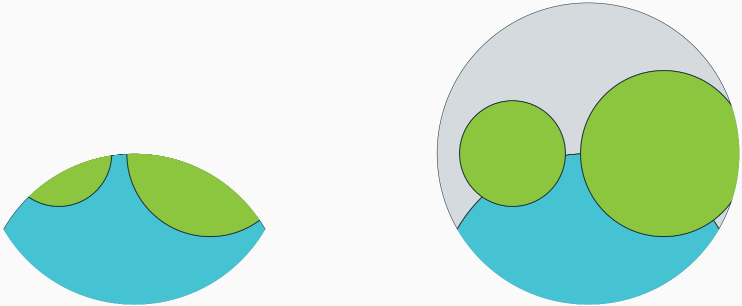

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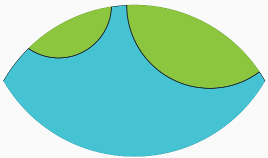


**Theorem ([Lop23, Theorem 5.13])**

*Given an inductively defined space* $X = F(X)$, *one can derive a generic topology expander*

## Remarks

- Gives back the previous topologies for finite words and finite trees!
- Correctly generalizes with what is done in the realm of well-quasi-orders, e.g., by [Has02].

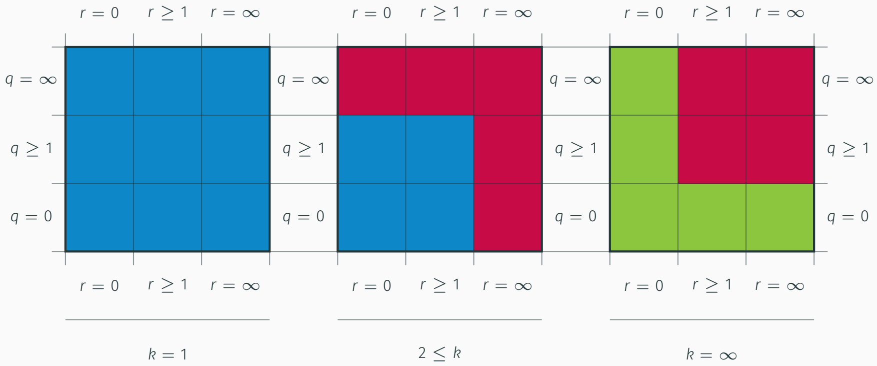
## Parameters of a local sentence

$$\exists x_1, \ldots, x_k . [Q_1 y_1 . Q_2 y_2 . \ldots . Q_q y_q . \theta(\vec{x}, \vec{y})]^{\vec{x}}_r$$

**Fixing all parameters...**

A sentence $\varphi$ preserved under $(r, q, k)$-local elementary embeddings is equivalent to an existential local sentence.

# Expanded Cube



# Representing relations with arity greater than 2.

The Gaifman Graph

$$\rightarrow (x, y) \stackrel{def}{=} \bigvee_{(R,n)\in\sigma} \exists z_1, \ldots, z_n, R(z_1, \ldots, z_n) \wedge \bigvee_{1\leq i,j\leq n}^{n} x = z_i \wedge y = z_j$$

# Representing relations with arity greater than 2.

## The Gaifman Graph

$$\rightarrow (x, y) \stackrel{def}{=} \bigvee_{(R,n)\in\sigma} \exists z_1, \ldots, z_n, R(z_1, \ldots, z_n) \wedge \bigvee_{1\leq i,j\leq n}^{n} x = z_i \wedge y = z_j$$

| | | |
|---|---|---|
| 2 | 2 | 1 |
| 2 | 1 | 0 |
| 2 | 0 | 2 |
| 1 | 2 | 0 |
| 1 | 1 | 2 |
| 1 | 0 | 1 |
| 0 | 2 | 2 |
| 0 | 1 | 1 |
| 0 | 0 | 0 |

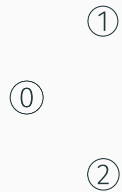



**Figure 1:** From a table to a graph.

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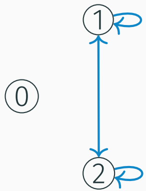
| $R$ | | | |
|---|---|---|---|
| | 2 | 2 | 1 |
| | 2 | 1 | 0 |
| | 2 | 0 | 2 |
| | 1 | 2 | 0 |
| | 1 | 1 | 2 |
| | 1 | 0 | 1 |
| | 0 | 2 | 2 |
| | 0 | 1 | 1 |
| | 0 | 0 | 0 |

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| | | | |
|---|---|---|---|
| R | 2 | 2 | 1 |
| | 2 | 1 | 0 |
| | 2 | 0 | 2 |
| R | 1 | 2 | 0 |
| | 1 | 1 | 2 |
| | 1 | 0 | 1 |
| | 0 | 2 | 2 |
| | 0 | 1 | 1 |
| | 0 | 0 | 0 |

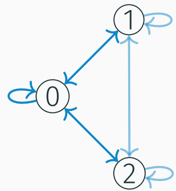


**Figure 1:** From a table to a graph.

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## The Gaifman Graph

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| | | | |
|---|---|---|---|
| $R$ | 2 | 2 | 1 |
| | 2 | 1 | 0 |
| | 2 | 0 | 2 |
| $R$ | 1 | 2 | 0 |
| | 1 | 1 | 2 |
| | 1 | 0 | 1 |
| | 0 | 2 | 2 |
| | 0 | 1 | 1 |
| | 0 | 0 | 0 |

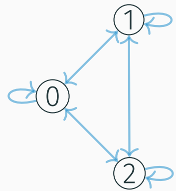


**Figure 1:** From a table to a graph.