From the motion group of the trivial link to the homology of the hypertree poset

Bérénice Oger

Institut Camille Jordan (Lyon)

Friday January, 10th 2014
Seminar of the ANR HOGT project
From the motion group of the $a$ \textit{n-component} trivial link to the homology of the $a$ hypertree poset on $n$ vertices

Bérénice Oger

Institut Camille Jordan (Lyon)

Friday January, 10th 2014
Seminar of the ANR HOGT project
Motivation : $P\Sigma_n$

- $F_n$ generated by $(x_i)_i^{n}$
- $P\Sigma_n$, pure symmetric automorphism group
  - group of automorphisms of $F_n$ which send each $x_i$ to a conjugate of itself,
  - group of motions of a collection of $n$ coloured unknotted, unlinked circles in 3-space.
- It seems that their cohomology groups are not Koszul (A. Conner and P. Goetz).
Use of the hypertree poset for the computation of the $l^2$-Betti numbers of $P\Sigma_n$ by C. Jensen, J. McCammond and J. Meier.

Action of $P\Sigma_n$ on a contractible complex $MM_n$ defined by McCullough and Miller in 1996 in terms of marking of hypertrees, whose fundamental domain is the hypertree poset on $n$ vertices,

$P\Sigma_n \triangleright Inn(F_n) \implies OP\Sigma_n = P\Sigma_n / Inn(F_n)$

$OP\Sigma_n$ acts cocompactly on $MM_n$

Use of a theorem by Davis, Januszkiewicz and Leary to obtain the expression of $l^2$-cohomology of the group in term of the cohomology of the fundamental domain of the complex.
Summary

1. The hypertree poset
   - Hypertrees
   - Hypertree poset
   - Homology of the hypertree poset

2. Computation of the homology of the hypertree poset
   - Species
   - Counting strict chains using large chains
   - Pointed hypertrees
   - Relations between chains of hypertrees
   - Dimension of the homology

3. From the hypertree poset to rooted trees
   - PreLie species
   - Character for the action of the symmetric group on the homology of the poset
Hypergraphs and hypertrees

Definition ([Ber89])

A hypergraph (on a set $V$) is an ordered pair $(V, E)$ where:

- $V$ is a finite set (vertices)
- $E$ is a collection of subsets of cardinality at least two of elements of $V$ (edges).

Example of a hypergraph on $[1; 7]$
Walk on a hypergraph

Definition

Let $H = (V, E)$ be a hypergraph.

A walk from a vertex or an edge $d$ to a vertex or an edge $f$ in $H$ is an alternating sequence of vertices and edges beginning by $d$ and ending by $f$:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all $i$, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.

The length of a walk is the number of edges and vertices in the walk.

Examples of walks
Hypertrees

Definition

A hypertree is a non-empty hypergraph $H$ such that, given any distinct vertices $v$ and $w$ in $H$,

- there exists a walk from $v$ to $w$ in $H$ with distinct edges $e_i$, ($H$ is connected),
- and this walk is unique, ($H$ has no cycles).

Example of a hypertree
The hypertree poset

**Definition**

Let $I$ be a finite set of cardinality $n$, $S$ and $T$ be two hypertrees on $I$.

\[ S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T \]

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

**Example with hypertrees on four vertices**

\[
\begin{array}{c}
\diamondsuit \\
\heartsuit \\
\spadesuit \\
\clubsuit
\end{array}
\preceq
\begin{array}{c}
\diamondsuit \\
\heartsuit \\
\spadesuit \\
\clubsuit
\end{array}
\]

but not

\[
\begin{array}{c}
\diamondsuit \\
\heartsuit \\
\spadesuit \\
\clubsuit
\end{array}
\prec
\begin{array}{c}
\diamondsuit \\
\heartsuit \\
\spadesuit \\
\clubsuit
\end{array}
\]
• Graded poset by the number of edges [McCullough and Miller 1996],
• There is a unique minimum 0,
• HT(I) = hypertree poset on I,
• HT_n = hypertree poset on n vertices.
• Möbius number : (n - 1)^{n-2} [McCammond and Meier 2004]

Goal:
• New computation of the homology dimension
• Computation of the action of the symmetric group on the homology
  (Conjecture of Chapoton 2007)
Homology of the poset

To every poset $P$, one can associate a simplicial complex (nerve of the poset seen as a category) whose
- vertices are elements of $P$,
- faces are the chains of $P$.

Definition

A strict $k$-chain of hypertrees on $I$ is a $k$-tuple $(a_1, \ldots, a_k)$, where $a_i$ are hypertrees on $I$ different from the minimum $\hat{0}$ and $a_i \prec a_{i+1}$.
Homology of the poset

To every poset $P$, one can associate a simplicial complex (nerve of the poset seen as a category) whose

- vertices are elements of $P$,
- faces are the chains of $P$.

**Definition**

A strict $k$-chain of hypertrees on $I$ is a $k$-tuple $(a_1, \ldots, a_k)$, where $a_i$ are hypertrees on $I$ different from the minimum $\hat{0}$ and $a_i < a_{i+1}$.

Let $C_k$ be the vector space generated by strict $k$-chains. We define $C_{-1} = \mathbb{C}e$. We define the following linear map on the set $(C_k)_{k \geq -1}$:

$$\partial_k(a_1 < \ldots < a_{k+1}) = \sum_{i=1}^{k} (-1)^i (a_1 < \ldots < \hat{a}_i < \ldots < a_k),$$

$$(a_1 < \ldots < a_{k+1}) \in C_k.$$
Homology of the poset

To every poset $P$, one can associate a simplicial complex (nerve of the poset seen as a category) whose

- vertices are elements of $P$,
- faces are the chains of $P$.

**Definition**

A **strict $k$-chain of hypertrees on $I$** is a $k$-tuple $(a_1, \ldots, a_k)$, where $a_i$ are hypertrees on $I$ different from the minimum $\hat{0}$ and $a_i \prec a_{i+1}$.

Let $C_k$ be the vector space generated by strict $k$-chains. We define $C_{-1} = \mathbb{C}$, e. We define the following linear map on the set $(C_k)_{k \geq -1}$:

$$\partial_k(a_1 \prec \ldots \prec a_{k+1}) = \sum_{i=1}^{k} (-1)^i (a_1 \prec \ldots \prec \hat{a}_i \prec \ldots \prec a_k),$$

$(a_1 \prec \ldots \prec a_{k+1}) \in C_k$.

The homology is defined by:

$$\tilde{H}_j = \ker \partial_j / \text{im} \partial_{j+1}.$$
Theorem ([MM04])

*The homology of $\hat{HT}_n$ is concentrated in maximal degree $(n - 3)$.**
**Theorem ([MM04])**

The homology of $\widehat{HT}_n$ is concentrated in maximal degree $(n - 3)$.

**Corollary**

The character for the action of the symmetric group on $\tilde{H}_{n-3}$ is given in terms of characters for the action of the symmetric group on $C_k$ by:

$$\chi_{\tilde{H}_{n-3}} = (-1)^{n-3} \sum_{k=-1}^{n-3} (-1)^k \chi_{C_k}, \text{ where } n = \#I.$$
What are species?

**Definition**

A species $F$ is a functor from the category of finite sets and bijections to itself. To a finite set $I$, the species $F$ associates a finite set $F(I)$ independent from the nature of $I$. 

Counterexamples

The following sets are not obtained using species:

- $\{(1,3,2), (2,1,3)\}$ (set of permutations of $\{1,2,3\}$ with exactly 1 descent)
- $\{(2,3,1), (3,1,2)\}$ (graph of divisibility of $\{1,2,3,4,5,6\}$)
What are species?

Definition

A species $F$ is a functor from the category of finite sets and bijections to itself. To a finite set $I$, the species $F$ associates a finite set $F(I)$ independent from the nature of $I$.

Species $= \text{Construction plan, such that the set obtained is invariant by relabelling}$
What are species?

**Definition**

A *species* $F$ is a functor from the category of finite sets and bijections to itself. To a finite set $I$, the species $F$ associates a finite set $F(I)$ independent from the nature of $I$.

Species = Construction plan, such that the set obtained is invariant by relabelling
What are species?

Definition

A species \( F \) is a functor from the category of finite sets and bijections to itself. To a finite set \( I \), the species \( F \) associates a finite set \( F(I) \) independent from the nature of \( I \).

Species = Construction plan, such that the set obtained is invariant by relabelling
What are species?

**Definition**

A species $F$ is a functor from the category of finite sets and bijections to itself. To a finite set $I$, the species $F$ associates a finite set $F(I)$ independent from the nature of $I$.

**Counterexamples**

The following sets are not obtained using species:

- $\{(1, 3, 2), (2, 1, 3), (2, 3, 1)(3, 1, 2)\}$ (set of permutations of $\{1, 2, 3\}$ with exactly 1 descent)
- (graph of divisibility of $\{1, 2, 3, 4, 5, 6\}$)

![Graph of divisibility of \{1, 2, 3, 4, 5, 6\}](image-url)
Examples of species

- \{ (1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2), (3, 2, 1) \} (Species of lists Assoc on \{1, 2, 3\})
- \{ \{1, 2, 3\} \} (Species of non-empty sets Comm)
- \{ \{1\}, \{2\}, \{3\} \} (Species of pointed sets Perm)
- These sets are the image by species of the set \{1, 2, 3\}.
Examples of species

- \{ (♥, ♠, ♣), (♥, ♣, ♠), (♠, ♥, ♣), (♠, ♣, ♥) \} (Species of lists Assoc on \{♠, ♥, ♣\})
- \{\{♥, ♠, ♣\}\} (Species of non-empty sets Comm)
- \{\{♥\}, \{♠\}, \{♣\}\} (Species of pointed sets Perm)
- \{\text{rooted trees PreLie}\}

These sets are the image by species of the set \{♠, ♥, ♣\}.
Proposition

Let $F$ and $G$ be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)
Proposition

Let $F$ and $G$ be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)

Example: Derivative of the species of cycles on $I = \{\heartsuit, \spadesuit, \clubsuit\}$
Proposition

Let $F$ and $G$ be two species. The following operations can be defined on them:

1. $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)
2. $(F + G)(I) = F(I) \sqcup G(I)$, (addition)

where $P(I)$ runs on the set of partitions of $I$. 
Proposition

Let $F$ and $G$ be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)
- $(F + G)(I) = F(I) \sqcup G(I)$, (addition)
- $(F \times G)(I) = \sum_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$, (product)
Proposition

Let $F$ and $G$ be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)
- $(F + G)(I) = F(I) \sqcup G(I)$, (addition)
- $(F \times G)(I) = \sum_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$, (product)
- $(F \circ G)(I) = \bigsqcup_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J)$, (substitution) where $\mathcal{P}(I)$ runs on the set of partitions of $I$. 

Proposition

Let $F$ and $G$ be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)
- $(F + G)(I) = F(I) \sqcup G(I)$, (addition)
- $(F \times G)(I) = \sum_{I_1 \sqcup I_2 = I} F(I_1) \times G(I_2)$, (product)
- $(F \circ G)(I) = \bigsqcup_{\pi \in \mathcal{P}(I)} F(\pi) \times \prod_{J \in \pi} G(J)$, (substitution) where $\mathcal{P}(I)$ runs on the set of partitions of $I$.

Example of substitution: Rooted trees of lists on $I = \{1, 2, 3, 4\}$

```
(2, 4, 3), (4, 3, 2), (1), (3, 4), (3), (2), (1), (4, 2, 3), (1, 2), (4, 1), ...
```
Definition

To a species $F$, we associate its **generating series**:

$$C_F(x) = \sum_{n\geq 0} \# F(\{1, \ldots, n\}) \frac{x^n}{n!}.$$ 

Examples of generating series:

- The generating series of the species of lists is $C_{\text{Assoc}} = \frac{1}{1-x}$.
- The generating series of the species of non-empty sets is $C_{\text{Comm}} = \exp(x) - 1$.
- The generating series of the species of pointed sets is $C_{\text{Perm}} = x \cdot \exp(x)$.
- The generating series of the species of rooted trees is $C_{\text{PreLie}} = \sum_{n\geq 0} n^{n-1} \frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{\text{Cycles}} = -\ln(1-x)$. 

Bérénice Oger (ICJ - Lyon)

From $P\Sigma_n$ to the homology of $HT_n$

January, 10th 2014 18 / 36
Definition

The cycle index series of a species $F$ is the formal power series in an infinite number of variables $p = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(p) = \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{\sigma \in S_n} F^\sigma p_{1}^{\sigma_1} p_{2}^{\sigma_2} p_{3}^{\sigma_3} \ldots \right),$$

- with $F^\sigma$, is the set of $F$-structures fixed under the action of $\sigma$,
- and $\sigma_i$, the number of cycles of length $i$ in the decomposition of $\sigma$ into disjoint cycles.

Examples

- The cycle index series of the species of lists is $Z_{\text{Assoc}} = \frac{1}{1 - p_1}$.
- The cycle index series of the species of non empty sets is $Z_{\text{Comm}} = \exp(\sum_{k \geq 1} \frac{p_k}{k}) - 1$. 

Bérénice Oger (ICJ - Lyon)
Operations on cycle index series

Operations on species give operations on their cycle index series:

**Proposition**

Let $F$ and $G$ be two species. Their cycle index series satisfy:

- $Z_{F + G} = Z_F + Z_G$
- $Z_{F \times G} = Z_F \times Z_G$
- $Z_{F \circ G} = Z_F \circ Z_G$
- $Z_{F'} = \frac{\partial Z_F}{\partial p_1}$

**Definition**

The *suspension* $\Sigma$ of a cycle index series $f(p_1, p_2, p_3, \ldots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \ldots).$$
Counting strict chains using large chains

Let $I$ be a finite set of cardinality $n$.

**Definition**

A large $k$-chain of hypertrees on $I$ is a $k$-tuple $(a_1, \ldots, a_k)$, where $a_i$ are hypertrees on $I$ and $a_i \preceq a_{i+1}$.
Counting strict chains using large chains

Let $I$ be a finite set of cardinality $n$.

**Definition**

A large $k$-chain of hypertrees on $I$ is a $k$-tuple $(a_1, \ldots, a_k)$, where $a_i$ are hypertrees on $I$ and $a_i \preceq a_{i+1}$.

Let $M_{k,s}$ be the set of words on $\{0, 1\}$ of length $k$, with $s$ letters "1". The species $M_{k,s}$ is defined by:

\[
\begin{align*}
\emptyset & \mapsto M_{k,s}, \\
V \neq \emptyset & \mapsto \emptyset.
\end{align*}
\]
Counting strict chains using large chains
Let \( I \) be a finite set of cardinality \( n \).

**Definition**

A *large \( k \)-chain of hypertrees on \( I \) is a \( k \)-tuple \((a_1, \ldots, a_k)\), where \( a_i \) are hypertrees on \( I \) and \( a_i \preceq a_{i+1} \).

Let \( M_{k,s} \) be the set of words on \( \{0, 1\} \) of length \( k \), with \( s \) letters ”1”. The species \( M_{k,s} \) is defined by:

\[
\begin{align*}
\emptyset & \mapsto M_{k,s}, \\
V \neq \emptyset & \mapsto \emptyset.
\end{align*}
\]

**Proposition**

The species \( \mathcal{H}_k \) of large \( k \)-chains and \( \mathcal{H}S_i \) of strict \( i \)-chains are related by:

\[
\mathcal{H}_k \cong \sum_{i \geq 0} \mathcal{H}S_i \times M_{k,i}.
\]
Proposition

The species $\mathcal{H}_k$ of large $k$-chains and $\mathcal{HS}_i$ of strict $i$-chains are related by:

$$\mathcal{H}_k \cong \sum_{i \geq 0} \mathcal{HS}_i \times \mathcal{M}_{k,i}.$$  

Proof.

Deletion of repetitions

$$(a_1, \ldots, a_k) \rightarrow (a_{j_1}, \ldots, a_{j_i})$$

$u_j = 0$ if $a_j = a_{j-1}$, 1 otherwise

$$(u_1, \ldots, u_k) \text{ with } a_0 = \hat{0}.$$
The previous proposition gives, for all integer $k > 0$:

$$
\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s.
$$
The previous proposition gives, for all integer $k > 0$:

$$
\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s.
$$

$\chi_k$ is a polynomial $P(k)$ in $k$ which gives, once evaluated in $-1$, the character:

**Corollary**

$$
\chi_{\tilde{H}_{n-3}} = (-1)^n P(-1) =: (-1)^n \chi_{-1}
$$

The hypertrees will now be on $n$ vertices.
Pointed hypertrees

**Definition**

Let $H$ be a hypertree on $I$. $H$ is:

- **rooted** in a vertex $s$ if the vertex $s$ of $H$ is distinguished,
- **edge-pointed** in an edge $a$ if the edge $a$ of $H$ is distinguished,
- **rooted edge-pointed** in a vertex $s$ in an edge $a$ if the edge $a$ of $H$ and a vertex $s$ of $a$ are distinguished.

**Example of pointed hypertrees**
Proposition: Dissymmetry principle

The species of hypertrees and of rooted hypertrees are related by:

\[ \mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}. \]

We write:

- \( \mathcal{H}_k \), the species of large \( k \)-chains of hypertrees,
- \( \mathcal{H}_k^{pa} \), the species of large \( k \)-chains of hypertrees whose minimum is rooted edge-pointed,
- \( \mathcal{H}_k^{p} \), the species of large \( k \)-chains of hypertrees whose minimum is rooted,
- \( \mathcal{H}_k^{a} \), the species of large \( k \)-chains of hypertrees whose minimum is edge-pointed.

Corollary ([Oge13])

The species of large \( k \)-chains of hypertrees are related by:

\[ \mathcal{H}_k + \mathcal{H}_k^{pa} = \mathcal{H}_k^{p} + \mathcal{H}_k^{a}. \]
Last but not least type of hypertrees

**Definition**

A *hollow hypertree* on $n$ vertices ($n \geq 2$) is a hypertree on the set \{#, 1, \ldots, n\}, such that the vertex labelled by #, called the gap, belongs to one and only one edge.

**Example of a hollow hypertree**

![Graph](image)

We denote by $\mathcal{H}_k^c$, the species of large $k$-chains of hypertrees whose minimum is a hollow hypertree.
Relations between species of hypertrees

Theorem

The species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

\begin{align*}
\mathcal{H}_k^p &= X \times \mathcal{H}_k' \\
\mathcal{H}_k^p &= X \times \text{Comm} \circ \mathcal{H}_k^c + X, \\
\mathcal{H}_k^c &= \text{Comm} \circ \mathcal{H}_{k-1}^c \circ \mathcal{H}_k^p, \\
\mathcal{H}_k^a &= (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p, \\
\mathcal{H}_k^{pa} &= (\mathcal{H}_{k-1}^p - x) \circ \mathcal{H}_k^p.
\end{align*}
Relations between species of hypertrees

Theorem

The species $\mathcal{H}_k$, $\mathcal{H}^p_k$ and $\mathcal{H}^c_k$ satisfy:

$\mathcal{H}^p_k = X \times \mathcal{H}'_k$ \hspace{1cm} (1)

$\mathcal{H}^p_k = X \times \text{Comm} \circ \mathcal{H}^c_k + X$, \hspace{1cm} (2)

$\mathcal{H}^c_k = \text{Comm} \circ \mathcal{H}^c_{k-1} \circ \mathcal{H}^p_k$, \hspace{1cm} (3)

$\mathcal{H}^a_k = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}^p_k$, \hspace{1cm} (4)

$\mathcal{H}^{pa}_k = (\mathcal{H}^p_{k-1} - x) \circ \mathcal{H}^p_k$. \hspace{1cm} (5)

Proof.

1. Rooting a species $F$ is the same as multiplying the singleton species $X$ by the derivative of $F$,
Second part of the proof.

We separate the root and every edge containing it, putting gaps where the root was,

\[ \mathcal{H}_k^p = X \times \text{Comm} \circ \mathcal{H}_k^c + X, \]
Hollow case:

\[ \mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \]  \hspace{1cm} (6)

\[ \mathcal{H}_k^{cm} = \text{Comm} \circ \mathcal{H}_k^c. \]  \hspace{1cm} (7)
Dimension of the homology

Proposition

The generating series of the species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$C_k^p = x \cdot \exp \left( \frac{C_{k-1}^p \circ C_k^p}{C_k^p} - 1 \right),$$  \hspace{1cm} (8)

$$C_k^a = (C_{k-1} - x)(C_k^p),$$  \hspace{1cm} (9)

$$C_k^{pa} = (C_{k-1}^p - x)(C_k^p),$$  \hspace{1cm} (10)

$$x \cdot C_k' = C_k^p,$$  \hspace{1cm} (11)

$$C_k + C_k^{pa} = C_k^p + C_k^a.$$  \hspace{1cm} (12)
Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

\[ C_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1, \]

\[ C_0^p = xe^x. \]
Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}^p_0$ are given by:

$$C_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$C_0^p = xe^x.$$

This implies with the previous theorem:

Theorem ([MM04])

The dimension of the only homology group of the hypertree poset is $(n - 1)^{n-2}$.

This dimension is the dimension of the vector space $\text{PreLie}(n-1)$ whose basis is the set of rooted trees on $n - 1$ vertices.
From the hypertree poset to rooted trees

1. This dimension is the dimension of the vector space $\text{PreLie}(n-1)$ whose basis is the set of rooted trees on $n - 1$ vertices. The operad (a species with more properties on substitution) whose vector space are $\text{PreLie}(n)$ is $\text{PreLie}$.

2. This operad is anticyclic ([Cha05]): There is an action of the symmetric group $\mathfrak{S}_n$ on $\text{PreLie}(n - 1)$ which extends the one of $\mathfrak{S}_{n-1}$.

3. Moreover, there is an action of $\mathfrak{S}_n$ on the homology of the poset of hypertrees on $n$ vertices.

Question

Is the action of $\mathfrak{S}_n$ on $\text{PreLie}(n-1)$ the same as the action on the homology of the poset of hypertrees on $n$ vertices?
Character for the action of the symmetric group on the homology of the poset

Using relations on species established previously, we obtain:

**Proposition**

The series $Z_k$, $Z^p_k$, $Z^a_k$ and $Z^{pa}_k$ satisfy the following relations:

\[ Z_k + Z^{pa}_k = Z^p_k + Z^a_k, \]  
\[ (13) \]

\[ Z^p_k = p_1 + p_1 \times \text{Comm} \circ \left( \frac{Z^p_{k-1} \circ Z^p_k - Z^p_k}{Z^p_k} \right), \]  
\[ (14) \]

\[ Z^a_k + Z^p_k = Z_{k-1} \circ Z^p_k, \]  
\[ (15) \]

\[ Z^{pa}_k + Z^p_k = Z^p_{k-1} \circ Z^p_k, \text{ and } p_1 \frac{\partial Z_k}{\partial p_1} = Z^p_k. \]  
\[ (16) \]
Theorem ([Oge13], conjecture of [Cha07])

The cycle index series $Z_{-1}$, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}$, is linked with the cycle index series $M$ associated with the anticyclic structure of PreLie by:

$$Z_{-1} = p_1 - \Sigma M = \text{Comm} \circ \Sigma \text{PreLie} + p_1 (\Sigma \text{PreLie} + 1).$$

(17)

The cycle index series $Z_{-1}^p$ is given by:

$$Z_{-1}^p = p_1 (\Sigma \text{PreLie} + 1).$$

(18)
Theorem ([Oge13], conjecture of [Cha07])

The cycle index series $Z_{-1}$, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}$, is linked with the cycle index series $M$ associated with the anticyclic structure of PreLie by:

$$Z_{-1} = p_1 - \sum M = \text{Comm} \circ \sum \text{PreLie} + p_1 (\sum \text{PreLie} + 1). \quad (17)$$

The cycle index series $Z^p_{-1}$ is given by:

$$Z^p_{-1} = p_1 (\sum \text{PreLie} + 1). \quad (18)$$

Proof.

Sketch of the proof

1. Computation of $Z_0 = \text{Comm}$ and $Z^p_0 = \text{Perm} = p_1 + p_1 \times \text{Comm}$
2. Replaced in the formula giving $Z^p_0$ in terms of itself and $Z^p_{-1}$

$$Z^p_0 = p_1 + p_1 \times \text{Comm} \circ \left( \frac{Z^p_{-1} \circ Z^p_0 - Z^p_0}{Z^p_0} \right),$$
Second part of the proof.

3. As $\Sigma \text{PreLie} \circ \text{Perm} = \text{Perm} \circ \Sigma \text{PreLie} = p_1$, according to [Cha07], we get:

$$Z_{p_1} = p_1 (\Sigma \text{PreLie} + 1).$$

4. The dissymmetry principle associated with the expressions gives:

$$\text{Comm} + Z_{p_1} \circ \text{Perm} - \text{Perm} = \text{Perm} + Z_{-1} \circ \text{Perm} - \text{Perm}.$$

5. Thanks to equation [Cha05, equation 50], we conclude:

$$\Sigma M - 1 = -p_1 (-1 + \Sigma \text{PreLie} + \frac{1}{\Sigma \text{PreLie}}).$$
Thank you for your attention!

Claude Berge:
Combinatorics of finite sets, Translated from the French.

Frédéric Chapoton:

Frédéric Chapoton:
http://www.intlpress.com/hha/v9/n1/.
Jon McCammond et John Meier:
The hypertree poset and the $l^2$-Betti numbers of the motion group of the trivial link.

Bérénice Oger:
Action of the symmetric groups on the homology of the hypertree posets.
Eccentricity

Definition

The **eccentricity** of a vertex or an edge is the maximal number of vertices on a walk without repetition to another vertex. The **center** of a hypertree is the vertex or the edge with minimal eccentricity.

Example of eccentricity

\[ e = 4 \]
\[ e = 5 \]
\[ e = 6 \]
\[ e = 7 \]
\[ e = 5 \]
\[ e = 6 \]
\[ e = 7 \]
\[ e = 8 \]
\[ e = 9 \]