Homology of the hypertree poset

Bérénice Oger

Institut Camille Jordan (Lyon)

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Algebra and Computation (Lyon)

Motivation : $P\Sigma_n$

- F_n generated by $(x_i)_{i=1}^n$
- $P\Sigma_n$, pure symmetric automorphism group / McCool group
  - group of automorphisms of F_n which send each x_i to a conjugate of itself,
  - group of motions of a collection of n coloured unknotted, unlinked circles in 3-space.
- Koszulness of their cohomology groups?

- Use of the hypertree poset for the computation of the l^2-Betti numbers of $P\Sigma_n$ by C. Jensen, J. McCammond and J. Meier.
- Action of $P\Sigma_n$ on a contractible complex MM_n defined by McCullough and Miller in 1996 in terms of marking of hypertrees, whose fundamental domain is the hypertree poset on n vertices,
- $P\Sigma_n \triangleright Inn(F_n) => OP\Sigma_n = P\Sigma_n / Inn(F_n)$
- $OP\Sigma_n$ acts cocompactly on MM_n
- Use of a theorem by Davis, Januszkiewicz and Leary to obtain the expression of l^2-cohomology of the group in term of the cohomology of the fundamental domain of the complex.

Summary

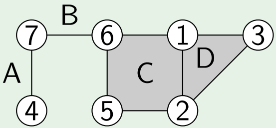
Hypergraphs and hypertrees

Definition (Berge, 1989)

A hypergraph (on a set V) is an ordered pair (V, E) where:

- *V is a finite set (vertices)*
- *E is a collection of subsets of cardinality at least two of elements of V (edges).*

Example of a hypergraph on $[1; 7]$



Walk on a hypergraph

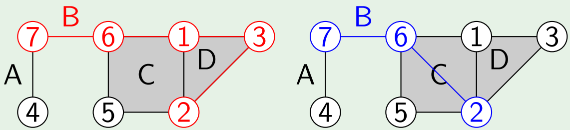
Definition

Let $H = (V, E)$ be a hypergraph.
A walk from a vertex or an edge d to a vertex or an edge f in H is an alternating sequence of vertices and edges beginning by d and ending by f:

$$(d, \ldots, e_i, v_i, e_{i+1}, \ldots, f)$$

where for all i, $v_i \in V$, $e_i \in E$ and $\{v_i, v_{i+1}\} \subseteq e_i$.
The length of a walk is the number of edges and vertices in the walk.

Examples of walks



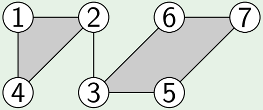
Hypertrees

Definition

A hypertree *is a non-empty hypergraph H such that, given any distinct vertices v and w in H,*

- *there exists a walk from v to w in H with distinct edges e_i, (H is* connected*),*
- *and this walk is unique, (H has* no cycles*).*

Example of a hypertree



The hypertree poset

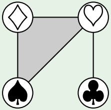
Definition

Let I be a finite set of cardinality n, S and T be two hypertrees on I.

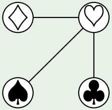
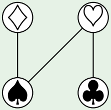
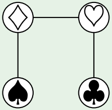
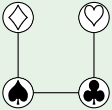
$$S \preceq T \iff \text{Each edge of } S \text{ is the union of edges of } T$$

We write $S \prec T$ if $S \preceq T$ but $S \neq T$.

Example with hypertrees on four vertices



$\preceq$

   but not  .

- Graded poset by the number of edges [McCullough and Miller 1996],
- There is a unique minimum $\hat{0}$,
- $\mathrm{HT(I)}$ = hypertree poset on I,
- $\mathrm{HT_n}$ = hypertree poset on n vertices.
- Möbius number : $(n-1)^{n-2}$ [McCammond and Meier 2004]

Goal:

- New computation of the homology dimension
- Computation of the action of the symmetric group on the homology (Conjecture of Chapoton 2007)

Homology of the poset

To every poset P, one can associate a simplicial complex (nerve of the poset seen as a category) whose

- vertices are elements of P,
- faces are the chains of P.

Definition

A strict k-chain of hypertrees on I is a k-tuple $(a_1, \ldots, a_k)$, where a_i are hypertrees on I different from the minimum $\hat{0}$ and $a_i \prec a_{i+1}$.

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Let C_k be the vector space generated by strict k-chains. We define $C_{-1} = \mathbb{C}.e$. We define the following linear map on the set $(C_k)_{k \geq -1}$:

$$\partial_k(a_1 \prec \ldots \prec a_{k+1}) = \sum_{i=1}^{k} (-1)^i (a_1 \prec \ldots \prec \hat{a}_i \prec \ldots \prec a_k),$$

$(a_1 \prec \ldots \prec a_{k+1}) \in C_k$.

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$(a_1 \prec \ldots \prec a_{k+1}) \in C_k$.
The homology is defined by:

$$\tilde{H}_j = ker\partial_j / im\partial_{j+1}.$$

Theorem (McCammond and Meier, 2004)

The homology of $\widehat{\mathrm{HT}_\mathrm{n}}$ *is concentrated in maximal degree* $(n-3)$.

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Corollary

The character for the action of the symmetric group on $\tilde{H}_{n-3}$ is given in terms of characters for the action of the symmetric group on C_k by:

$$\chi_{\tilde{H}_{n-3}} = (-1)^{n-3} \sum_{k=-1}^{n-3} (-1)^k \chi_{C_k}, \text{ where } n = \#I.$$

What are species?

Definition

A species F *is a functor from the category of finite sets and bijections to itself. To a finite set* I*, the species* F *associates a finite set* $\mathrm{F}(I)$ *independent from the nature of* I*.*

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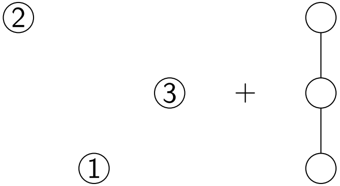

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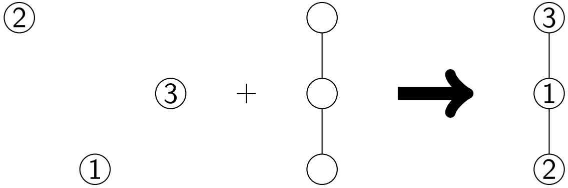

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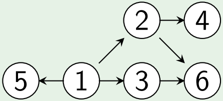
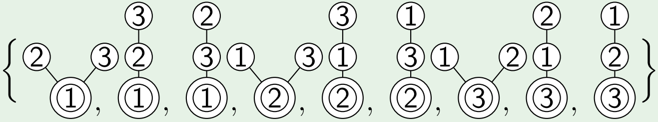
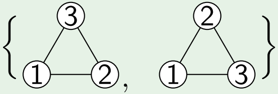
Counterexamples

The following sets are not obtained using species:

- $\{(1,\mathbf{3},2),(2,1,\mathbf{3}),(2,\mathbf{3},1)(3,1,\mathbf{2})\}$(set of permutations of $\{1,2,3\}$ with exactly 1 descent)
- (graph of divisibility of $\{1,2,3,4,5,6\}$)

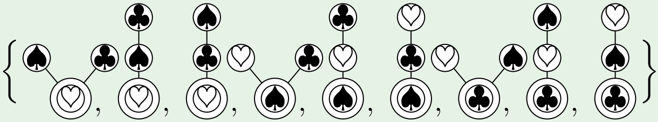
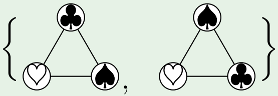


Examples of species

- $\{(1,2,3),(1,3,2),(2,1,3),(2,3,1),(3,1,2),(3,2,1)\}$ (Species of lists Assoc on $\{1,2,3\}$)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

These sets are the image by species of the set $\{1,2,3\}$.

Examples of species

- $\{(\heartsuit,\spadesuit,\clubsuit),(\heartsuit,\clubsuit,\spadesuit),(\spadesuit,\heartsuit,\clubsuit),(\spadesuit,\clubsuit,\heartsuit),(\clubsuit,\heartsuit,\spadesuit),(\clubsuit,\spadesuit,\heartsuit)\}$ (Species of lists Assoc on $\{\clubsuit,\heartsuit,\spadesuit\}$)
-  (Species of rooted trees PreLie)
-  (Species of cycles)

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Operations on species and associated series

Proposition

Let F and G be two species. The following operations can be defined on them:

- $F'(I) = F(I \sqcup \{\bullet\})$, (derivative)

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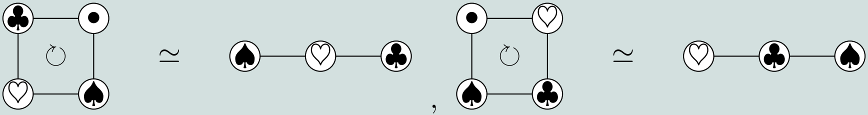
Example: Derivative of the species of cycles on $I = \{\heartsuit, \spadesuit, \clubsuit\}$



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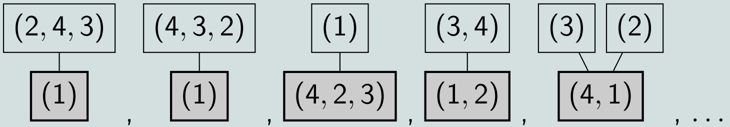
Example of substitution: Rooted trees of lists on $I = \{1, 2, 3, 4\}$



Definition

To a species F, we associate its generating series:

$$C_F(x) = \sum_{n \geq 0} \#F(\{1, \dots, n\}) \frac{x^n}{n!}.$$

Examples of generating series:

- The generating series of the species of lists is $C_{\mathrm{Assoc}} = \frac{1}{1-x}$.
- The generating series of the species of rooted trees is $C_{\mathrm{PreLie}} = \sum_{n \geq 0} n^{n-1} \frac{x^n}{n!}$.
- The generating series of the species of cycles is $C_{Cycles} = -\ln(1-x)$.

Definition

The cycle index series of a species F is the formal power series in an infinite number of variables $\mathfrak{p} = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(\mathfrak{p}) = \sum_{n \geq 0} \frac{1}{n!} \left(\sum_{\sigma \in \mathfrak{S}_n} F^\sigma p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- *with F^σ, is the set of F-structures fixed under the action of σ,*
- *and σ_i, the number of cycles of length i in the decomposition of σ into disjoint cycles.*

Examples

- The cycle index series of the species of lists is $Z_{\text{Assoc}} = \frac{1}{1-p_1}$.

Operations on cycle index series

Operations on species give operations on their cycle index series:

Proposition

Let F and G be two species. Their cycle index series satisfy:

$$\begin{aligned} Z_{F+G} &= Z_F + Z_G, & Z_{F\times G} &= Z_F \times Z_G, \\ Z_{F\circ G} &= Z_F \circ Z_G, & Z_{F'} &= \frac{\partial Z_F}{\partial p_1}. \end{aligned}$$

Definition

The suspension Σ of a cycle index series $f(p_1, p_2, p_3, \ldots)$ is defined by:

$$\Sigma f = -f(-p_1, -p_2, -p_3, \ldots).$$

Counting strict chains using large chains

Let I be a finite set of cardinality n.

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Let $M_{k,s}$ be the set of words on $\{0, 1\}$ of length k, with s letters "1". The species $\mathcal{M}_{k,s}$ is defined by:

$$\begin{cases} \emptyset & \mapsto & M_{k,s}, \\ V \neq \emptyset & \mapsto & \emptyset. \end{cases}$$

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The species $\mathcal{H}_k$ of large k-chains and $\mathcal{HS}_i$ of strict i-chains are related by:

$$\mathcal{H}_k \cong \sum_{i \geq 0} \mathcal{HS}_i \times \mathcal{M}_{k,i}.$$

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Proof.

$$\begin{tikzcd} & \boxed{(a_{j_1}, \dots, a_{j_i})} \\ \boxed{(a_1, \dots, a_k)} \arrow[ur, "\text{Deletion of repetitions}"] \arrow[dr, "u_j = 0 \text{ if } a_j = a_{j-1}, \text{ 1 otherwise}"'] & \\ & \boxed{(u_1, \dots, u_k)} \end{tikzcd}$$

with $a_0 = \hat{0}$. □

The previous proposition gives, for all integer $k > 0$:

$$\chi_k = \sum_{i=0}^{n-2} \binom{k}{i} \chi_i^s.$$

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χ_k is a polynomial $P(k)$ in k which gives, once evaluated in -1, the character:

Corollary

$$\chi_{\tilde{H}_{n-3}} = (-1)^n P(-1) =: (-1)^n \chi_{-1}$$

The hypertrees will now be on n vertices.

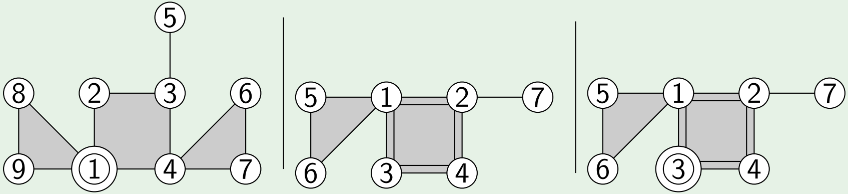
Pointed hypertrees

Definition

Let H be a hypertree on I. H is:

- *rooted in a vertex s if the vertex s of H is distinguished,*
- *edge-pointed in an edge a if the edge a of H is distinguished,*
- *rooted edge-pointed in a vertex s in an edge a if the edge a of H and a vertex s of a are distinguished.*

Example of pointed hypertrees



Proposition: Dissymmetry principle

The species of hypertrees and of pointed hypertrees are related by:

$$\mathcal{H} + \mathcal{H}^{pa} = \mathcal{H}^{p} + \mathcal{H}^{a}.$$

We write:

- $\mathcal{H}_k$, the species of large k-chains of hypertrees,
- $\mathcal{H}_k^{pa}$, the species of large k-chains of hypertrees whose minimum is rooted edge-pointed,
- $\mathcal{H}_k^{p}$, the species of large k-chains of hypertrees whose minimum is rooted,
- $\mathcal{H}_k^{a}$, the species of large k-chains of hypertrees whose minimum is edge-pointed.

Corollary (O., 2013)

The species of large k-chains of hypertrees are related by:

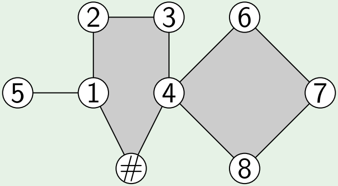
$$\mathcal{H}_k + \mathcal{H}_k^{pa} = \mathcal{H}_k^{p} + \mathcal{H}_k^{a}.$$

Last but not least type of hypertrees

Definition

A *hollow hypertree* *on n vertices ($n \geq 2$) is a hypertree on the set $\{\#, 1, \dots, n\}$, such that the vertex labelled by $\#$, called the gap, belongs to one and only one edge.*

Example of a hollow hypertree



We denote by $\mathcal{H}_k^c$, the species of large k-chains of hypertrees whose minimum is a hollow hypertree.

Relations between species of hypertrees

Theorem

The species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$\mathcal{H}_k^p = X \times \mathcal{H}_k' \tag{1}$$

$$\mathcal{H}_k^p = X \times \mathrm{Comm} \circ \mathcal{H}_k^c + X, \tag{2}$$

$$\mathcal{H}_k^c = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c \circ \mathcal{H}_k^p, \tag{3}$$

$$\mathcal{H}_k^a = (\mathcal{H}_{k-1} - x) \circ \mathcal{H}_k^p, \tag{4}$$

$$\mathcal{H}_k^{pa} = \left(\mathcal{H}_{k-1}^p - x\right) \circ \mathcal{H}_k^p. \tag{5}$$

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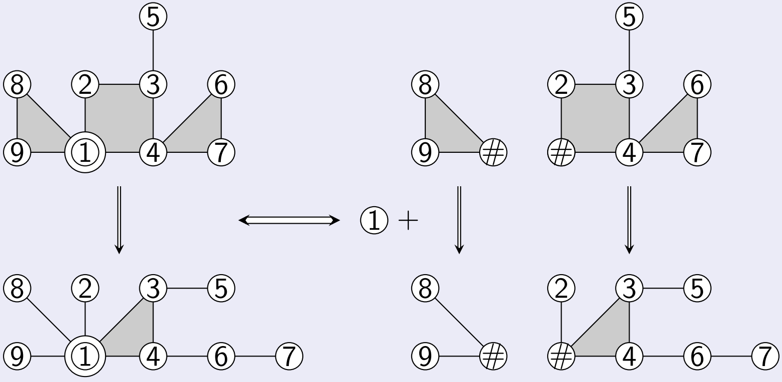
Proof.

1. Rooting a species F is the same as multiplying the singleton species X by the derivative of F,

Second part of the proof.

We separate the root and every edge containing it, putting gaps where the root was,

$$\mathcal{H}_k^p = X \times \mathrm{Comm} \circ \mathcal{H}_k^c + X,$$



Dimension of the homology

Proposition

The generating series of the species $\mathcal{H}_k$, $\mathcal{H}_k^p$ and $\mathcal{H}_k^c$ satisfy:

$$\mathcal{C}_k^p = x \cdot \exp\left(\frac{\mathcal{C}_{k-1}^p \circ \mathcal{C}_k^p}{\mathcal{C}_k^p} - 1\right), \tag{6}$$

$$\mathcal{C}_k^a = (\mathcal{C}_{k-1} - x)(\mathcal{C}_k^p), \tag{7}$$

$$\mathcal{C}_k^{pa} = (\mathcal{C}_{k-1}^p - x)(\mathcal{C}_k^p), \tag{8}$$

$$x \cdot \mathcal{C}_k' = \mathcal{C}_k^p, \tag{9}$$

$$\mathcal{C}_k + \mathcal{C}_k^{pa} = \mathcal{C}_k^p + \mathcal{C}_k^a. \tag{10}$$

Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

$$\mathcal{C}_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$\mathcal{C}_0^p = xe^x.$$

Lemma

The generating series of $\mathcal{H}_0$ and $\mathcal{H}_0^p$ are given by:

$$\mathcal{C}_0 = \sum_{n \geq 1} \frac{x^n}{n!} = e^x - 1,$$

$$\mathcal{C}_0^p = xe^x.$$

This implies with the previous theorem:

Theorem (McCammond and Meier, 2004)

The dimension of the only homology group of the hypertree poset is $(n-1)^{n-2}$.

This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.

From the hypertree poset to rooted trees

1. This dimension is the dimension of the vector space PreLie(n-1) whose basis is the set of rooted trees on $n-1$ vertices.
   The operad (a species with more properties on substitution) whose vector space are PreLie(n) is PreLie.
2. This operad is anticyclic [Chapoton, 2005]: There is an action of the symmetric group $\mathfrak{S}_n$ on PreLie($n-1$) which extends the one of $\mathfrak{S}_{n-1}$.
3. Moreover, there is an action of $\mathfrak{S}_n$ on the homology of the poset of hypertrees on n vertices.

Question

Is the action of $\mathfrak{S}_n$ on PreLie(n-1) the same as the action on the homology of the poset of hypertrees on n vertices?

Character for the action of the symmetric group on the homology of the poset

Using relations on species established previously, we obtain:

Proposition

The series Z_k, Z_k^p, Z_k^a and Z_k^{pa} satisfy the following relations:

$$Z_k + Z_k^{pa} = Z_k^p + Z_k^a, \tag{11}$$

$$Z_k^p = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z_{k-1}^p \circ Z_k^p - Z_k^p}{Z_k^p} \right), \tag{12}$$

$$Z_k^a + Z_k^p = Z_{k-1} \circ Z_k^p, \tag{13}$$

$$Z_k^{pa} + Z_k^p = Z_{k-1}^p \circ Z_k^p, \text{ and } p_1 \frac{\partial Z_k}{\partial p_1} = Z_k^p. \tag{14}$$

Theorem (O. 2013, conjecture of [Chapoton, 2007)

] The cycle index series Z_{-1}, which gives the character for the action of $\mathfrak{S}_n$ on $\tilde{H}_{n-3}$, is linked with the cycle index series M associated with the anticyclic structure of PreLie *by:*

$$Z_{-1} = p_1 - \Sigma M = \mathrm{Comm} \circ \Sigma \, \mathrm{PreLie} + p_1 \left(\Sigma \, \mathrm{PreLie} + 1 \right). \tag{15}$$

The cycle index series Z^p_{-1} is given by:

$$Z^p_{-1} = p_1 \left(\Sigma \, \mathrm{PreLie} + 1 \right). \tag{16}$$

Theorem (O. 2013, conjecture of [Chapoton, 2007)

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The cycle index series Z^p_{-1} is given by:

$$Z^p_{-1} = p_1 \left(\Sigma\, \mathrm{PreLie} + 1\right). \tag{16}$$

Proof.

Sketch of the proof

1. Computation of $Z_0 = \mathrm{Comm}$ and $Z_0^p = \mathrm{Perm} = p_1 + p_1 \times \mathrm{Comm}$
2. Replaced in the formula giving Z_0^p in terms of itself and Z^p_{-1}

$$Z_0^p = p_1 + p_1 \times \mathrm{Comm} \circ \left(\frac{Z^p_{-1} \circ Z_0^p - Z_0^p}{Z_0^p} \right),$$

Second part of the proof.

3. As $\Sigma\,\mathrm{PreLie} \circ \mathrm{Perm} = \mathrm{Perm} \circ \Sigma\,\mathrm{PreLie} = p_1$, according to [Chapoton, 2007], we get:
$$Z^{p}_{-1} = p_1\,(\Sigma\,\mathrm{PreLie} + 1)\,.$$
4. The dissymetry principle associated with the expressions gives:
$$\mathrm{Comm} + Z^{p}_{-1} \circ \mathrm{Perm} - \mathrm{Perm} = \mathrm{Perm} + Z_{-1} \circ \mathrm{Perm} - \mathrm{Perm}\,.$$
5. Thanks to equation [equation 50, Chapoton 2005], we conclude:
$$\Sigma M - 1 = -p_1(-1 + \Sigma\,\mathrm{PreLie} + \frac{1}{\Sigma\,\mathrm{PreLie}}).$$

$\square$

Back to the cohomology

Proposition (Jensen, McCammond and Meier, 2006)

The cohomology of $H^*(P\Sigma_n, \mathbb{Z})$ is generated by one-dimensional classes α^*_{ij}, where $1 \leq i \neq j \leq n$, subject to the relations :

1. $\alpha^*_{ij} \wedge \alpha^*_{ij} = 0$
2. $\alpha^*_{ij} \wedge \alpha^*_{ji} = 0$
3. $\alpha^*_{kj} \wedge \alpha^*_{ji} = (\alpha^*_{kj} - \alpha^*_{ij}) \wedge \alpha^*_{ki}$.

Question:

Are these algebras Koszul?

- $n = 2, 3$ not hard to prove it is Koszul.
- For $n \geq 4$: we consider $OP\Sigma_n$

Conditions on generators : $i \neq j$, $i \neq 1$ and $i \neq 2$ if $j = 1$

Back to the cohomology

- Dual presentation : Universal enveloping algebra of a Lie algebra
- Computation with Bergman : not Koszul (A. Conner and P. Goetz)
  The $8th$ term of the Hilbert series should be 589824, and it is 589834.

Thank you for your attention !

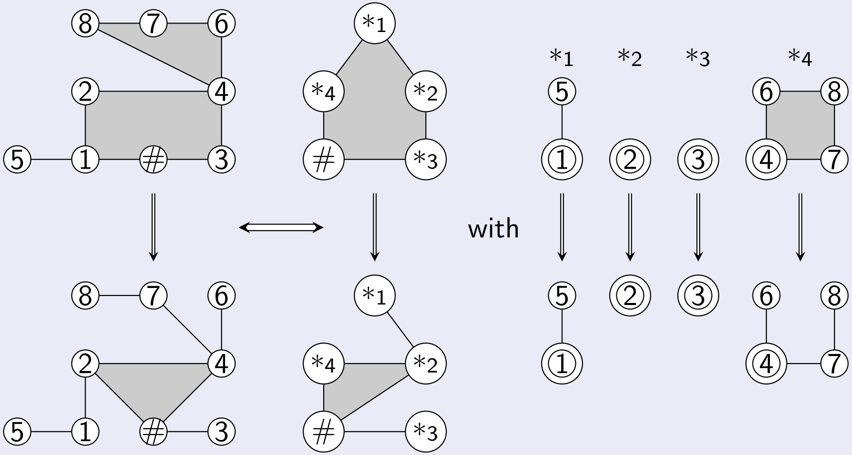
[O. 2013] Bérénice Oger Action of the symmetric groups on the homology of the hypertree posets. Journal of Algebraic Combinatorics, february 2013.

and End!

3. Hollow case:

$$\mathcal{H}_k^c = \mathcal{H}_k^{cm} \circ \mathcal{H}_k^p, \tag{17}$$

$$\mathcal{H}_k^{cm} = \mathrm{Comm} \circ \mathcal{H}_{k-1}^c. \tag{18}$$



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Eccentricity

Definition

The eccentricity *of a vertex or an edge is the maximal number of vertices on a walk without repetition to another vertex.*
The center *of a hypertree is the vertex or the edge with minimal eccentricity.*

Example of eccentricity