Parking trees

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1. *L’avventura*: Set and noncrossing partitions

2. *Michelle*: Parking functions

3. *Video killed the radio star*: Parking trees and species

4. *Blinding Lights*: Parking poset
L’avventura : Set and noncrossing partitions
Set partition and noncrossing partitions

**Definition**

A (set) partition of $E$ is $\pi = \{\pi_1, \ldots, \pi_k\}$ s.t.:

- $\pi_k \cap \pi_l \neq \emptyset \implies k = l$
- and $\bigcup_{i=1}^{k} \pi_i = E.$

$\Pi_E = \text{set of partitions of } E$

**Examples:**

**Definition (Kreweras, 1972)**

A partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of $\{1, \ldots, n\}$ is non-crossing iff

\[\begin{align*}
a < b < c < d \\
a, c \in \pi_i \\
b, d \in \pi_j
\end{align*}\]

\[\implies i = j\]

$NC_n = \text{set of non-crossing partitions of } \{1, \ldots, n\}$

$\rightarrow$ Catalan numbers $\frac{1}{n+1}{2n \choose n}$
Partitions and non-crossing partitions poset
Partitions and non-crossing partitions poset
Non-crossing 2-partitions

**Definition (Edelman, 1980)**

A n.c. 2-partition of $E$ is a triple $(\pi, \rho, \lambda)$ where:

- $\pi \in NC_{|E|}$ and $\rho \in \Pi_E$,
- $\lambda : \pi \leftrightarrow \rho$ s.t. $\forall B \in \pi, |\lambda(B)| = |B|$.

$\{\{1, 5, 6, 8\}, \{2, 4\}, \{3\}, \{7\}, \{9, 10, 12\}, \{11\}\}$

$\lambda$

$\{\{1\}, \{2, 9, 10, 11\}, \{3, 4, 8\}, \{5\}, \{6, 12\}, \{7\}\}$
First result: Bijection between 2NCP and labelled NCP

Labelled NCP: every element labelled so that labelling increases from left to right in the same part

\[\lambda\]

\[
\{\{1, 5, 6, 8\}, \{2, 4\}, \{3\}, \{7\}, \{9, 10, 12\}, \{11\}\}
\]

\[
\{\{1\}, \{2, 9, 10, 11\}, \{3, 4, 8\}, \{5\}, \{6, 12\}, \{7\}\}
\]
2NCP poset

Covering relation in $\Pi$: rearranging labels to respect the increasing condition

Example:

\[
\begin{array}{cccc}
3 & 4 & 1 & 2 \\
\end{array}
\]

\[\rightarrow (n + 1)^{n-1}\]
Michelle : Parking functions
Parking function [Konheim-Weiss, 1966]

1  2  3  4  5  6

Question:
How to park 6 cars in 6 parking spaces?

Easy answer: bijection between the parking spaces and cars.

What if you pick at random for each car its place? Can all cars park?

→ If yes, parking function!
Parking function: examples and counter-examples

136524 - 122333 - 416114 - 153436

| 1 | 2 | 3 | 4 | 5 | 6 |

\[ f : \{1, \ldots, n\} \rightarrow \{1, \ldots, n\} \]

\[ \text{s.t. } \bigcup_{j=1}^i |f^{-1}(j)| \geq i \]
Video killed the radio star: Parking trees and species
Species [Joyal, 1980]

Definition

Species $F : \text{FinSet} \to \text{FinSet}$ which associates to a finite set $I$, the finite set $F(I)$, only depending on the cardinality of $I$. 

```
2 3 1
```

```
1 3 2
```

```
2
1
3
```

```
1
2
3
```
Species [Joyal, 1980]

Definition

Species $F : \text{FinSet} \rightarrow \text{FinSet}$ which associates to a fin. set $I$, the fin. set $F(I)$, only depending on the cardinality of $I$.

Species encodes the action of the symmetric group.
A parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

- $V \in \Pi_L$,
- $v \in V$ has $|v|$ children.
Bijection between 2NCP $\leftrightarrow$ parking trees
Why do we need species?

Let $F$ and $G$ be two species.

- $(F + G)(I) = F(I) \sqcup G(I)$,

- $(F \times G)(I) = \bigsqcup_{l_1 \sqcup l_2 = I} F(l_1) \times G(l_2)$. 
Definition

The **cycle index series** of a species $F$ is the formal power series in an infinite number of variables $p = (p_1, p_2, p_3, \ldots)$ defined by:

$$Z_F(p) = \sum_{n \geq 0} \frac{1}{n!} \left( \sum_{\sigma \in S_n} F^{\sigma_1} p_1^{\sigma_1} p_2^{\sigma_2} p_3^{\sigma_3} \ldots \right),$$

- with $F^\sigma = |\{ x \in F(\{1, \ldots, n\}) | \sigma \cdot x = x \}|$
- and $\sigma$ has $\sigma_i$ cycles of length $i$.

Example:
Back to parking trees

Proposition (DO, Josuat-Vergès, Randazzo, 20+)

\[ P_f = \sum_{p \geq 1} \mathcal{E}_p \times (1 + P_f)^p \]
Blinding Lights : Parking poset
Order on parking trees
Results

- This poset is a lattice
- When restricting right combs, get the face poset of the permutohedron
- New criterion to prove shelling!
- Enumeration of (weak) k-chains
Proposition (DO, Josuat-Vergès, Randazzo, 20+)

\[ C_{k,t}^l = \sum_{p \geq 1} C_{k-1,t}^{l,p} \times (tC_{k,t}^l + 1)^p , \]
Proposition (DO, Josuat-Vergès, Randazzo, 20+)

Chains $\phi_1 \leq \cdots \leq \phi_k$ in $\Pi_n$ are in bijection with $k$-parking trees.

The number of chains $\phi_1 \leq \cdots \leq \phi_k$ in $\Pi_n$ where $\text{rk}(\phi_k) = \ell$ is:

$$\ell! \left( \frac{kn}{\ell} \right) S_2(n, \ell + 1).$$
Definition

A $k$-parking tree on a set $L$ is a rooted plane tree $T = (V, E, r)$ such that:

1. $V \in \Pi_L$
2. $v \in V$ has $k|v|$ children.
Proposition (DO, Josuat-Vergès, Randazzo, 20+)

Chains $\phi_1 \leq \cdots \leq \phi_k$ in $\mathfrak{R}_n$ are in bijection with $k$-parking trees.

The number of chains $\phi_1 \leq \cdots \leq \phi_k$ in $\mathfrak{R}_n$ where $\text{rk}(\phi_k) = \ell$ is:

$$\ell! \binom{kn}{\ell} S_2(n, \ell + 1).$$

Thank you!
Lemma

Let $x, y, y', z \in \mathbb{P}_n$ such that $x \preceq y \preceq z$, $x \preceq y'$, and $y' \prec_x y$. Then:

- either there exists $y'' \in \mathbb{P}_n$ such that $x \preceq y'' \preceq z$ and $y'' \prec_x y$,
- or there exists $z' \in \mathbb{P}_n$ such that $y \preceq z' \preceq y' \lor z$ and $z' \prec_y z$. 