Three distance theorems and combinatorics on words

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Abstract

The aim of this paper is to investigate the connection between some generalizations of the three distance theorem and combinatorics on words for sequences defined as codings of irrational rotations on the unit circle. We also give some new results concerning the frequencies of factors for such sequences.

1 Introduction

For a given α in $]0,1[$, let us place the points $\{0\}, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ on the *unit circle* (we mean here the circle of *perimeter 1*), where $\{x\}$ denotes as usual, the fractional part of x (i.e., if $\lfloor x \rfloor$ denotes the largest integer not exceeding x, $\{x\} = x - \lfloor x \rfloor$). These points partition the unit circle into $n+1$ intervals having at most three lengths, one being the sum of the other two. This property is known as the *three distance theorem* and can be seen as a geometric interpretation of good approximation properties of the Farey partial convergents in the continued fraction expansion of α.

The connection between this classical theorem in diophantine approximation and combinatorics on words is particularly apparent in the following result, known as the *three gap theorem*, which is equivalent to the three distance theorem and can be seen as its "dual": assume we are given α and β in the interval $]0,1[$, the gaps between the successive n for which $\{\alpha n\} < \beta$ take at most three values, one being the sum of the other two. It is indeed natural to introduce the binary sequence with values 0 and 1, defined as the coding of the orbit of a point of the unit circle under the rotation by angle α with respect to the intervals $[0, \beta[, [\beta, 1[$ (in particular, if β equals $1-\alpha$ or α, this sequence is a Sturmian sequence): the lengths of strings consisting of 0's and 1's are thus directly connected with the three gaps. In fact, the three distance and the three gap theorems have deep relations with the measure-theoretic and topological properties of the dynamical systems associated with codings of rotations.

The aim of this paper is to review the different generalizations of the three distance and three gap theorems and to emphasize the relationships with combinatorics on words. This paper is organized as follows. We recall in Section 2 basic definitions and properties concerning codings of rotations. We emphasize the connections between frequencies of factors of given length for such sequences and the lengths of the intervals obtained by partitioning the unit circle by a set of points in arithmetical progression. We prove in particular that the three distance theorem is equivalent to the fact that the frequencies of factors of given length of a Sturmian sequence take at most three values. Furthermore, this last statement is easily proved by using the notion of graph of words, which gives us a very simple combinatorial proof of the three distance theorem. Section 3 is devoted to the study of the three distance theorem. We introduce the three gap theorem in Section 4. We will deduce from these two theorems in Section 5 the expression of the recurrence function of a Sturmian sequence, due to Hedlund and Morse [40]. Section 6 deals with generalizations of the three distance and the three gap theorems. We give in Section 7 a direct proof of a particular case of the two-dimensional version of the three distance theorem (i.e., that there are at most 5 lengths when the unit circle is partitioned by the points $\{i\alpha\}$ and $\{i\alpha+\beta\}$, for $0 \leq i \leq n$). In Section 8, we give a proof of the $3d$ distance theorem, proved by Chung and Graham [18, 37] (i.e., that there are at most $3d$ lengths when the unit circle is partitioned by the points $\{k_i\alpha+\gamma_i\}$, for $0 \leq i \leq d$ and $0 \leq k_i \leq n_i$).

In each case, we study the connection with frequencies of codings of rotations. More precisely, we prove that the frequencies of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values and we deduce from the two-dimensional version of the three distance theorem that the frequencies of a coding of an irrational rotation with respect to a partition in d intervals of the same length take ultimately at most $d+3$ values; more generally, we prove that the frequencies of a coding of an irrational rotation with respect to a partition into d intervals (not necessarily of the same length) take ultimately at most $3d$ values (this result corresponds to the $3d$ distance theorem).

Let us first review some of the many related results and applications of the three distance theorem. We will focus on the theorem itself and its different proofs in Section 3.

As one of the first applications the theorem of Hartman [33] (which answers an earlier question of Steinhaus concerning the circular dispersion spectrum) has been proved in [53].

Theorem 1 *Let $0 < \alpha < 1$ be an irrational number and let n be a positive integer. Let H_n (respectively h_n) denote the maximal (respectively the minimal) length of the $n+1$ intervals obtained by partitioning the unit circle by the points of the set $\{i\alpha,\ 0 \leq i \leq n-1\}$. If the partial quotients of the regular continued fraction expansion of α are unbounded, then*

$$\liminf_{n\to+\infty} n.h_n = 0,$$

$$\limsup_{n\to+\infty} n.h_n = 1,$$

$$\liminf_{n\to+\infty} n.H_n = 1,$$

$$\limsup_{n\to+\infty} n.H_n = +\infty.$$

In [21] Deléglise studies the length $L(h)$ of the smallest closed interval I of the unit circle such that $I, 2I, \ldots, hI$ cover the circle. More precisely, he shows the following result.

Theorem 2 *Let I be a closed interval of minimal length L such that $I, 2I, \ldots, hI$ cover the circle; we have, for $h \geq 3$*

$$L = \begin{cases} 3/(h(h+2)), & \textit{if } h \equiv 0 \quad \textit{or } 1 \textit{ mod } 3, \\ 3/(h(h+2)-2), & \textit{if } h \equiv 2 \textit{ mod } 3. \end{cases}$$

In particular, the function $L(h)$ is equivalent to $3/h^2$ when h tends towards infinity.

In [7], Bessi and Nicolas apply the three distance theorem to 2-highly composite numbers, i.e., if $\mathcal{N}_2$ denotes the set of integers having only 2 and 3 as prime factors, an integer n in $\mathcal{N}_2$ is said to be a 2-highly composite number if for any m in $\mathcal{N}_2$ such that $m < n$, then the number of divisors of m is strictly less than n. They prove, in particular, that there exists a constant c such that the number of 2-highly composite numbers smaller than X is larger than $c(\log X)^{4/3}$.

In [8] Boshernitzan extends the three distance theorem to the case of interval exchange maps in his proof of Keane's conjecture, which states the unique ergodicity of Lebesgue almost all minimal interval exchange maps. Let us recall briefly the definition of an interval exchange map. Assume we are given $\lambda = (\lambda_1, \ldots, \lambda_r)$ in the positive cone in $\mathbb{R}^r$, i.e., $\lambda_i > 0$, for $1 \leq i \leq r$; it defines a segment $I = [0, \sum_1^r \lambda_k[$ of $\mathbb{R}$ composed of r intervals $I_i = [\sum_0^i \lambda_k, \sum_1^{i+1} \lambda_k[$, for $0 \leq i \leq r-1$ and by taking $\lambda_0 = 0$. Let σ denote a permutation of $\{1, 2, \ldots, r\}$. The interval exchange map T associated with λ and σ is defined as the map from I to I which exchanges the intervals I_i according to the permutation σ:

$$T(x) = x + \left(\sum_{j<\sigma(i)} \lambda_{\sigma^{-1}(j)} - \sum_{j<i} \lambda_i \right), \text{ for } x \in I_i.$$

The n-fold iterate of T is also an interval exchange map of say $r(n)$ intervals $I_1, \ldots, I_{r(n)}$. Boshernitzan proved the following.

Theorem 3 *The number of intervals $I_1, \ldots, I_{r(n)}$ of different length is not greater than $3(r-1)$, for all $n \geq 1$.*

Let us note that a two interval exchange map is a rotation; hence when $r = 2$, this theorem reduces to the three distance theorem.

As another ergodic application, we have the following. In [5] (see also [12]) topological and measure-theoretic covering numbers (i.e., the maximal measure of Rokhlin stacks having some prescribed regularity properties) are computed first for the symbolic dynamical systems associated to the rotation of argument α acting on the partition of the circle by the point β and then to exchange of three intervals; in this way, it is proved that every ergodic exchange of three intervals has simple spectrum, and a new class of exchanges of three intervals having nondiscrete spectrum is built. Results for irrational rotations of the torus $\mathbb{T}^2$ can also be obtained, by replacing intervals by Voronoï cells (see [16]).

The connections between Beatty sequences and the three distance and three gap theorems, and more precisely with the gaps in the intersection of Beatty sequences, have been investigated by Fraenkel and Holzman in [26]. We will discuss their results in Section 6.

J. Shallit introduces in [47] a *measure of automaticity* of a sequence. This measure counts the number of states in a minimal deterministic finite automaton which generates the prefix of size n of this sequence. Let us recall that a sequence has a finite measure of automaticity if and only if this sequence is a letter-to-letter projection of a fixed point of a constant length morphism of a free monoid. The author deduces in [47] a measure of automaticity of Sturmian codings of rotations from the three distance theorem, which are shown to have a high automaticity measure, even when they are fixed points of homomorphism.

Theorem 4 *Let $0 < \alpha < 1$ be an irrational number with bounded partial quotients. Let $u_n = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$, for $n \geq 1$. The automaticity of the sequence $(u_n)_{n\geq 1}$ has the same order of magnitude as $n^{1/5}$.*

Let us also note the following two applications of the three distance theorem in theoretical computer science. The first one deals with multiplicative hashing, as Fibonacci hashing, and is quoted in [34]. The second one is due to Lefèvre and gives a fast algorithm for computing a lower bound on the distance between a straight line and the points of a regular grid. This algorithm is used to find worst cases when trying to round the elementary functions correctly in floating-point arithmetic (see [36]).

Langevin studies in [35] the three distance theorem in connection with a mathematical model of the ventricular parasystole. He proves in particular the following generalization of the three distance theorem to lattices.

Theorem 5 *Let L be a lattice in $\mathbb{R}^2$. Let I be a bounded interval of $\mathbb{R}$ and let $L(I) = \{(x, y),\ x \in I\} \cap L$. For any point M of $L(I)$, let $S(M)$ denote the smallest point $M' \neq M$ of $L(I)$ such that M is smaller than M', in lexicographic order. Then there exists a basis (U, V) of the lattice L such that for any point M of $L(I)$, the difference $S(M) - M$ is either equal to U, V or $U + V$.*

Let us note that this theorem has been generalized by Fried and V. T. Sós to groups in [28].

Finally, Van Ravenstein studies in [43] the phenomenon of phyllotaxis, i.e., the regular leaf arrangement, which is given by the Fibonacci phyllotaxis for most plants (see also the work of Marzec and Kappraff in [38]). In [42] Van Ravenstein also applies the three distance theorem to evaluate some values of the discrepancy of the sequence $(n\alpha)_{n\in\mathbb{N}}$, for α irrational.

2 Codings of rotations

The aim of this section is to introduce some definitions concerning sequences defined as codings of irrational rotations on the unit circle, and more precisely measure-theoretic and topological properties of sequences with values in a finite alphabet. For $p \geq 2$, let $F = \{\beta_0 < \beta_1 < \ldots < \beta_{p-1}\}$ be a set of p consecutive points of the unit circle (identified in all that follows with $[0, 1[$ or with the unidimensional torus $\mathbb{R}/\mathbb{Z}$) and let $\beta_p = \beta_0$. Let α be an irrational number in $]0, 1[$ and let us consider the *positive orbit* of a point x of the unit circle under the rotation by angle α, i.e., the set of points $\{\{\alpha n + x\},\ n \in \mathbb{N}\}$. We denote by $\mathbb{N}$ the set of non-negative integers. The *coding* of the orbit of x under the rotation by angle α with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$ is the sequence $(u_n)_{n\in\mathbb{N}}$ defined on the finite alphabet $\Sigma = \{0, \ldots, p-1\}$ as follows:

$$u_n = k \Leftrightarrow \{x + n\alpha\} \in [\beta_k, \beta_{k+1}[, \quad \text{for} \quad 0 \leq k \leq p-1.$$

A coding of the rotation R means the coding of the orbit of a point x of the unit circle under the rotation R with respect to a finite partition of the unit circle consisting of left-closed and right-open intervals.

For instance, consider the case $F = \{0, 1-\alpha\}$, i.e., $\mathcal{P} = \{[0, 1-\alpha[, [1-\alpha, 1[\}$, where α is an irrational number in $]0, 1[$. We could also choose to code the orbit of the rotation with respect to the following partition: $\mathcal{P}' = \{]0, 1-\alpha],]1-\alpha, 1]\}$. As α is irrational, the two sequences obtained by coding with respect to $\mathcal{P}$ or to $\mathcal{P}'$ are ultimately equal. Such sequences are called *Sturmian sequences* (such a coding is called a *Sturmian coding*). Sturmian sequences have received considerable attention in the literature. We refer the reader to the impressive bibliography of [9]. A recent account on the subject can also be found in [6]. The most famous Sturmian sequence is the Fibonacci sequence ($\alpha = \tau - 1,\ x = \alpha$), where $\tau = (\sqrt{5}+1)/2$ denotes the *golden ratio*; this sequence is the fixed point of the following substitution

$$\sigma(1) = 10,\ \sigma(0) = 1.$$

Let us recall that a *substitution* defined on the finite alphabet $\mathcal{A}$ is a map from $\mathcal{A}$ to the set of words defined on $\mathcal{A}$, denoted by $\mathcal{A}^\star$, extended to $\mathcal{A}^\star$ by concatenation, or in other words, a homomorphism of the free monoid $\mathcal{A}^\star$.

The results stated for codings of rotations with respect to left-closed and right-open intervals are obviously true for left-open and right-closed partitions.

2.1 Complexity and frequencies of codings of rotations

A *factor of the infinite sequence* u is a finite block w of consecutive letters of u, say $w = u_{n+1} \cdots u_{n+d}$; d is called the *length* of w, denoted by $|w|$. Let $p(n)$ denote the *complexity function* of the sequence u with values in a finite alphabet: it counts the number of distinct factors of given length of the sequence u. For more information on the subject, we refer the reader to the survey [2].

With the above notation, consider a coding u of the orbit of a point x under the rotation by angle α with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$. Let $I_k = [\beta_k, \beta_{k+1}[$ and let R denote the rotation by angle α. A finite word $w_1 \cdots w_n$ defined on the alphabet $\Sigma = \{0, 1, \ldots, p-1\}$ is a factor of the sequence u if and only if there exists an integer k such that

$$\{x + k\alpha\} \in I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}}).$$

As α is irrational, the sequence $(\{x + n\alpha\})_{n\in\mathbb{N}}$ is dense in the unit circle, which implies that $w_1 w_2 \dots w_n$ is a factor of u if and only if $I(w_1, \dots, w_n) \neq \emptyset$. In particular, the set of factors of a coding does not depend on the initial point x of this coding. Furthermore, the connected components of these sets are bounded by the points

$$\{k(1-\alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n-1,\ 0 \leq i \leq p-1.$$

Let us recall that the *frequency* $f(B)$ of a factor B of a sequence is the limit, if it exists, of the number of occurrences of this block in the first k terms of the sequence divided by k. Thus the frequency of the factor $w_1 \dots w_n$ exists and is equal to the density of the set

$$\{k \mid \{x + k\alpha\} \in I(w_1, \dots, w_n)\},$$

which is equal to the length of $I(w_1, \dots, w_n)$, by uniform distribution of the sequence $(\{x + n\alpha\})_{n\in\mathbb{N}}$. These sets consist of finite unions of intervals. More precisely, if for every k, $\beta_{k+1} - \beta_k \leq \sup(\alpha, 1-\alpha)$, then these sets are connected; if there exists K such that $\beta_{K+1} - \beta_K > \sup(\alpha, 1-\alpha)$, then the sets are connected except for $w_1 \dots w_n$ of the form a_K^n (see [1]) (the notation a_K^n denotes the word of length n obtained by successive concatenations of the letter a_K). Let us note that there exists at most one integer K satifying $\beta_{K+1} - \beta_K > \sup(\alpha, 1-\alpha)$. We thus have the following lemma which links the three distance theorem and related results to the frequencies of codings of rotations.

Lemma 1 *Let u be a coding of a rotation by irrational angle α on the unit circle with respect to the partition*

$$\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \dots, [\beta_{p-1}, \beta_p[\},$$

such that the lengths of the intervals of the partition are less than or equal to $\sup(\alpha, 1-\alpha)$. Then the frequencies of factors of length n of the sequence u are equal to the lengths of the intervals bounded by the points

$$\{k(1-\alpha) + \beta_i\}, \textit{ for } 0 \leq k \leq n-1,\ 0 \leq i \leq p-1.$$

In particular, if the partition is equal to $\{[0, 1-\alpha[, [1-\alpha, 1[\}$, i.e., if u is a Sturmian sequence, the intervals $I(w_1, \dots, w_n)$ are exactly the $(n+1)$ intervals bounded by the points

$$0, \{(1-\alpha)\}, \dots, \{(n(1-\alpha)\}.$$

Therefore there are exactly $n+1$ factors of length n and the complexity of a Sturmian sequence satisfies $p(n) = n+1$, for every n. Furthermore, the lengths of these intervals are equal to the frequencies of factors of length n.

In fact, this complexity function characterizes Sturmian sequences. Indeed, any sequence of complexity $p(n) = n + 1$, for every n, is a Sturmian sequence, i.e., there exists α irrational in $]0, 1[$ and x such that this sequence is the coding of the orbit of x under the rotation by angle α with respect either to the partition $\{[0, 1-\alpha[, [1-\alpha, 1[\}$ or $\{]0, 1-\alpha],]1-\alpha, 1]\}$ (see [40]) (the coding of the orbit of α is called the *characteristic sequence* of α). Note that a sequence whose complexity satisfies $p(n) \leq n$, for some n, is ultimately periodic (see [19] and [39]). Sturmian sequences thus have the minimal complexity among sequences not ultimately periodic. Sturmian sequences are also characterized by the following properties.

- Sturmian sequences are exactly the non-ultimately periodic balanced sequences over a two-letter alphabet. A sequence is balanced if the difference between the number of occurrences of a letter in any two factors of the same length is bounded by one in absolute value.
- Sturmian sequences are codings of trajectories of irrational initial slope in a square billiard obtained by coding horizontal sides by the letter 0 and vertical sides by the letter 1.

In the general case of a coding of an irrational rotation, the complexity has the form $p(n) = an + b$, for n large enough (see Theorem 10 below and [1], for the whole proof). The converse is not true: every sequence of ultimately affine complexity is not necessarily obtained as a coding of rotation. Didier gives in [23] a characterization of codings of rotations. See also [46], where Rote studies the case of sequences of complexity $p(n) = 2n$, for every n. However, if the complexity of a sequence u has the form $p(n) = n + k$, for n large enough, then u is the image of a Sturmian sequence by a morphism, up to a prefix of finite length (see for instance [22] or [1]).

2.2 The graph of words

The aim of this section is to introduce the Rauzy graph of words of a sequence, in order to obtain results concerning the frequencies of factors of this sequence. This follows an idea of Dekking who expressed the block frequencies for the Fibonacci sequence, by using the graph of words (see [20] and also [8]). Note that Boshernitzan also introduces in [8] a graph for interval exchange maps (homeomorphic to the Rauzy graph of words) in order to prove Theorem 3, which can be seen as a result on frequencies.

Let us note that precise knowledge of the frequencies of a sequence with values in a finite alphabet $\mathcal{A}$ allows a precise description of the measure associated with the dynamical system $(\overline{\mathcal{O}(u)}, T)$: T denotes here the one-sided shift which associates to a sequence $(u_n)_{n\in\mathbb{N}}$ the sequence $(u_{n+1})_{n\in\mathbb{N}}$ and $\overline{\mathcal{O}(u)}$ is the orbit closure under the shift T of the sequence u in $\mathcal{A}^{\mathbb{N}}$, equipped with the product of the discrete topologies (it is easily seen that $\overline{\mathcal{O}(u)}$ is the set of sequences of factors belonging to the set of factors of u). Indeed, we define a probability measure μ on the Borel sets of $\overline{\mathcal{O}(u)}$ as follows: the measure μ is the unique T-invariant measure defined by assigning to each cylinder $[w]$ corresponding to the sequences of $\overline{\mathcal{O}(u)}$ of prefix w, the frequency of w, for any finite block w with letters from $\mathcal{A}$. Let us note that a dynamical system obtained via a coding of irrational rotation is *uniquely ergodic*, i.e., there exists a unique T-invariant probability measure on this dynamical system, which is thus determined by the block frequencies.

The Rauzy graph Γ_n of words of length n of a sequence with values in a finite alphabet is an oriented graph (see, for instance, [41]), which is a subgraph of the de Bruijn graph of words. Its vertices are the factors of length n of the sequence and the edges are defined as follows: there is an edge from U to V if V follows U in the sequence, i.e., more precisely, if there exists a word W and two letters x and y such that $U = xW$, $V = Wy$ and xWy is a factor of the sequence (such an edge is labelled by xWy). Thus there are $p(n+1)$ edges and $p(n)$ vertices, where $p(n)$ denotes the complexity function. A sequence is said to be *recurrent* if every factor appears at least twice, or equivalently if every factor appears an infinite number of times in this sequence. For instance, codings of rotations are recurrent. Note that the Rauzy graphs of words of a sequence are always connected; furthermore, they are strongly connected if and only if this sequence is recurrent.

If B is a factor, then a letter x such that Bx (respectively xB) is also a factor is called *right extension* (respectively *left extension*). Let U be a vertex of the graph. Denote by U^+ the number of edges of Γ_n with origin U and U^- the number of edges of Γ_n which end vertex U. In other words, U^+ (respectively U^-) counts the number of right (respectively left) extensions of U. Note that

$$p(n+1) - p(n) = \sum_{U\in V(\Gamma_n)} (U^+ - 1) = \sum_{U\in V(\Gamma_n)} (U^- - 1),$$

where $V(\Gamma_n)$ is the vertex set of Γ_n.

In this section we restrict ourselves to sequences with values in a finite alphabet, for which the frequencies exist. Note that the function which associates to an edge labelled by xWy the frequency of the factor xWy is a *flow*. Indeed, it satisfies Kirchhoff's current law: the total current flowing into each vertex is equal to the total current leaving the vertex. This common value is equal to the frequency of the word corresponding to this vertex. Let us see how to deduce, from the topology of a graph of words, information on the number of frequencies for factors of given length. We will use the following obvious result.

Lemma 2 *Let U and V be two vertices joined by an edge such that $U^+ = 1$ and $V^- = 1$. Then the two factors U and V have the same frequency.*

A *branch* of the graph Γ_n is a maximal directed path of consecutive vertices $(U_1, \ldots, U_m)$ (possibly $m = 1$), satisfying

$$U_i^+ = 1, \quad \text{for} \quad i < m, \quad U_i^- = 1, \quad \text{for} \quad i > 1.$$

Therefore, the vertices of a branch have the same frequency and the number of frequencies of factors of given length is bounded by the number of branches of the corresponding graph, as expressed below (for a proof of this result due to Boshernitzan, see [8]).

Theorem 6 *For a recurrent sequence of complexity function $(p(n))$, the frequencies of factors of given length, say n, take at most $3(p(n+1) - p(n))$ values.*

Remark In fact, on can prove the following: the frequencies of factors of length n take at most $p(n+1) - p(n) + r_n + l_n$ values, where r_n (respectively l_n) denotes the number of factors having more than one right (respectively left) extension.

We deduce from this theorem that if $p(n+1) - p(n)$ is uniformly bounded with n, the frequencies of factors of given length take a finite number of values. Indeed, using a theorem of Cassaigne quoted below (see [10]), we can easily state the following corollary.

Theorem 7 *If the complexity $p(n)$ of a sequence with values in a finite alphabet is sub-affine then $p(n+1) - p(n)$ is bounded.*

Corollary 1 *If a sequence over a finite alphabet has a sub-affine complexity, then the frequencies of its factors of given length take a finite number of values.*

Examples of sequences with sub-affine complexity function include the fixed point of a uniform substitution (i.e., of a substitution such that the images of the letters have the same length), the coding of a rotation or the coding of the orbit of a point under an interval exchange map with respect to the intervals of the interval exchange map.

2.3 Frequencies of factors of Sturmian sequences

In particular, in the Sturmian case ($p(n) = n + 1$, for every integer n), Theorem 6 implies the following result (see [3]).

Theorem 8 *The frequencies of factors of given length of a Sturmian sequence take at most three values.*

Consider a Sturmian sequence of angle α. We have seen in Section 2.1 that the frequency of a factor $w_1 \ldots w_n$ of u is equal to the length of the interval

$$I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}}),$$

and that these sets $I(w_1, \ldots, w_n)$ are exactly the intervals bounded by the points

$$0, \{1 - \alpha\}, \ldots, \{n(1 - \alpha)\}.$$

We deduce from Theorem 8 that the lengths of the intervals $I(w_1, \ldots, w_n)$, and thus the lengths of the intervals obtained by placing the points $0, \{1 - \alpha\}, \ldots, \{n(1 - \alpha)\}$ on the unit circle, take at most three values. Hence Theorem 8 is equivalent to the three distance theorem and provides a combinatorial proof of this result.

Remarks In fact this point of view, and more precisely the study of the evolution of the graphs of words with respect to the length n of the factors, allows us to give a proof of the most complete version of the three distance theorem as given in [53] (for more details, the reader is referred to [3]).

3 The three distance theorem

The three distance theorem was initially conjectured by Steinhaus, first proved V. T. Sós (see [53] and also [52]), and then by Świerczkowski [56], Surányi [55], Slater [51], Halton [31]. More recent proofs have also been given by Van Ravenstein [44] and Langevin [35]. A survey of the different approaches used by these authors is to be found in [44, 51, 35]. In the literature this theorem is called *the Steinhaus theorem*, *the three length, three gap* or *the three step theorem*. In order to avoid any ambiguity, we will always call it the three distance theorem, reserving the name *three gap* for the theorem introduced in the next section.

Three distance theorem *Let $0 < \alpha < 1$ be an irrational number and n a positive integer. The points $\{i\alpha\}$, for $0 \leq i \leq n$, partition the unit circle into $n + 1$ intervals, the lengths of which take at most three values, one being the sum of the other two.*

More precisely, let $(\frac{p_k}{q_k})_{k\in\mathbb{N}}$ *and* $(c_k)_{k\in\mathbb{N}}$ *be the sequences of convergents and partial quotients associated to* α *in its continued fraction expansion (if* $\alpha = [0, c_1, c_2, \ldots]$*, then* $\frac{p_n}{q_n} = [0, c_1, \ldots, c_n]$*). Let* $\eta_k = (-1)^k(q_k\alpha - p_k)$*. Let* n *be a positive integer. There exists a unique expression for* n *of the form*

$$n = mq_k + q_{k-1} + r,$$

with $1 \leq m \leq c_{k+1}$ *and* $0 \leq r < q_k$*. Then, the circle is divided by the points* $0, \{\alpha\}, \{2\alpha\}, \ldots, \{n\alpha\}$ *into* $n+1$ *intervals which satisfy:*

- $n+1-q_k$ *of them have length* η_k *(which is the largest of the three lengths),*
- $r+1$ *have length* $\eta_{k-1} - m\eta_k$*,*
- $q_k - (r+1)$ *have length* $\eta_{k-1} - (m-1)\eta_k$*.*

Remarks

- One can reformulate this result in terms of *n-Farey points*. Let us recall that an n-Farey point is a rational element $\frac{p}{q}$ of $[0,1]$ such that $p \geq 0$, $1 \leq q \leq n$ and p, q are coprime (see [32] for instance). Note that the two successive n-Farey points, say $\frac{p^{(1)}}{q^{(1)}}$ and $\frac{p^{(2)}}{q^{(2)}}$, satisfying $\frac{p^{(1)}}{q^{(1)}} < \alpha < \frac{p^{(2)}}{q^{(2)}}$ are $\frac{p_k}{q_k}$ and $\frac{mp_k+p_{k-1}}{mq_k+q_{k-1}}$, with the above notation. The three distance theorem states that the lengths of the intervals belong to the set

$$\{p^{(2)} - \alpha q^{(2)},\ \alpha q^{(1)} - p^{(1)},\ \alpha(q^{(1)} - q^{(2)}) + p^{(2)} - p^{(1)}\}.$$

- As α is irrational, the three lengths are distinct. The third length in the above theorem, which is the largest since it is the sum of the two others, appears if and only if

$$n \neq q^{(1)} + q^{(2)} - 1 = (m+1)q_k + q_{k-1} - 1.$$

  Thus there are infinitely many integers n for which there are only two lengths. The other two lengths do always appear.

- The structure and the transformation rules for the partitioning in two-length intervals are studied in details in [44]. Furthermore, in [45] van Ravenstein, Winley and Tognetti prove the following: for α having as sequence of partial quotients the constant sequence $aaaa\cdots$, label by large and small the lengths of intervals of the partition $\{i\alpha\}$, for $0 \leq i \leq q_n + q_{n-1} - 1$, where q_n is the denominator of a reduced convergent of α (there are only two lengths in this case); this binary finite sequence of lengths is a prefix after a permutation of the characteristic sequence of α (i.e., the Sturmian coding of the orbit of α). For a precise study of the limit points of these finite binary sequences (coresponding to the two-length case), see [48].

- In the two-length case, it is easily seen that the largest length is less than or equal to twice the second one. In [14] (see also [15, 16]) Chevallier extends this result to the two-dimensional torus $\mathbb{T}^2$, by studying the notion of best approximation.
- The point $\{(n+1)\alpha\}$ belongs to an interval of largest length in the partition of the unit circle by the points $\{i\alpha\}$, for $0 \leq i \leq n$.
- The three distance theorem is a geometric illustration of the properties of good approximation of the n-Farey points. Indeed, the two intervals containing 0 are of distinct lengths and their lengths are the two smallest. We thus have

$$\alpha q^{(1)} - p^{(1)} = \inf\{k\alpha, \quad \text{for } 0 \leq k \leq n\}$$

and

$$p^{(2)} - \alpha q^{(2)} = 1 - \sup\{k\alpha, \quad \text{for } 0 \leq k \leq n\}.$$

- For a deeper study of the rational case, the reader is referred for instance to [51].

4 The three gap theorem

The following theorem, called the *three gap theorem*, is in some sense the dual of the three distance theorem. This theorem was first proved by Slater (see [49] and see also [50, 51]), in the early fifties, whereas the first proofs of the three distance theorem date back to the late fifties. For other proofs of the three-gap theorem, see also [25], and more recently, [58] and [35].

The formulation of the three gap theorem quoted below is due to Slater. Following the notation of [51], let k_i be the sequence of integers k satisfying $k\alpha < \beta$. Then any difference $k_{i+1} - k_i$ is called a *gap*. Moreover, the *frequency* of a gap is defined as its frequency in the sequence of the successive gaps $(k_{i+1} - k_i)_{i \in \mathbb{N}}$.

Three gap theorem *Let α be an irrational number in $]0,1[$ and let $\beta \in]0,1/2[$. The gaps between the successive integers j such that $\{\alpha j\} < \beta$ take at most three values, one being the sum of the other two.*

More precisely, let $(\frac{p_k}{q_k})_{k \in \mathbb{N}}$ and $(c_k)_{k \in \mathbb{N}}$ be the sequences of the convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. There exists a unique *expression for β of the form*

$$\beta = m\eta_k + \eta_{k+1} + \psi,$$

with $k \geq 0$, $0 < \psi \leq \eta_k$, and if $k = 0$, $1 \leq m \leq c_1 - 1$, otherwise, $1 \leq m \leq c_{k+1}$. Then, the gaps between two successive j such that $\{j\alpha\} \in [0, \beta[$ satisfy the following:

- *the gap q_k has frequency $(m-1)\eta_k + \eta_{k+1} + \psi$,*
- *the gap $q_{k+1} - mq_k$ has frequency ψ,*
- *the gap $q_{k+1} - (m-1)q_k$ has frequency $\eta_k - \psi$.*

Remarks

- Suppose that α is an irrational number. By density of the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$, this theorem still holds when considering the gaps between the successive integers k such that $\{\alpha k\} \in I$, where I denotes any interval of the unit circle of length β.
- Furthermore, the third gap, which is the largest, can have frequency 0, when $\eta_k = \psi$, with the above notation. This means that this gap does not appear at all, as a consequence of the uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$ in the circle.
- The other two gaps do always appear (infinitely often, in fact, because of their positive frequencies) and are shown to be equal to the smallest positive integers l_1 and l_2 such that $\{l_1\alpha\} < \beta$ and $\{l_2\alpha\} > 1 - \beta$ (see [51]).
- The study of the rational case proves the equivalence between the three distance and the three gap theorems, as observed by Slater [51] in the case of an open interval and by Langevin, for any interval, in [35].

4.1 Connectedness index

Let $u = (u_n)_{n\in\mathbb{N}}$ be a coding of a rotation by irrational angle $0 < \alpha < 1$ with respect to the partition

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}.$$

We have seen in Section 2.1 that the sets $I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}})$, where $I_k = [\beta_k, \beta_{k+1}[$, for $0 \leq j \leq p-1$, are connected except for $w_1 \ldots w_n$ of the form a_K^n, where K denotes the index of the interval of $\mathcal{P}$ (if such an interval exists) of length greater than $\sup(\alpha, 1-\alpha)$.

Let us suppose that there exists an interval of $\mathcal{P}$ of length L greater than $1 - \alpha$ and index K, say. We deduce from the three gap theorem that the set of integers n such that a_K^n is a factor of the sequence u is bounded. More precisely, let us define $n^{(1)}$ as the largest integer n such that a_K^n is a factor of the sequence u. We will call the integer $n^{(1)}$ the *index of connectedness* of the sequence u. (If every interval of $\mathcal{P}$ has length smaller than or equal to $\sup(\alpha, 1-\alpha)$ then the connectedness index of u is equal to 1.) The three gap theorem enables us to give an exact expression for the connectedness index. Indeed $n^{(1)}+1$ is the largest gap between the consecutive values of k for which $0 < \{k\alpha\} < 1-L$. We thus have the following.

Theorem 9 *Let $u = (u_n)_{n\in\mathbb{N}}$ be a coding of the rotation by irrational angle α. Suppose that there exists an interval of $\mathcal{P}$ of length $L > \sup(\alpha, 1-\alpha)$. Let $(\frac{p_k}{q_k})_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ be the sequences of convergents and partial quotients associated to α in its continued fraction expansion. Let $\eta_k = (-1)^k(q_k\alpha - p_k)$. Write*

$$1 - L = m\eta_k + \eta_{k+1} + \psi,$$

with $k \geq 1$, $0 < \psi \leq \eta_k$ and $1 \leq m \leq c_{k+1}$. The connectedness index $n^{(1)}$ of the sequence u satisfies

$$n^{(1)} = q_{k+1} - (m-1)q_k - 1, \ \text{if } \psi \neq \eta_k,$$

$$n^{(1)} = q_{k+1} - mq_k - 1, \ \text{if } \psi = \eta_k \text{ and } m < c_{k+1},$$

$$n^{(1)} = q_k - 1, \ \text{if } \psi = \eta_k \text{ and } m = c_{k+1}.$$

4.2 Applications

Precise knowledge of the connectedness index is useful, as shown by the following. Indeed Lemma 1 can be rephrased as follows.

Lemma 3 *Let u be a coding of an irrational rotation on the unit circle with respect to the partition $\{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}$. The frequencies of factors of u of length $n \geq n^{(1)}$, where $n^{(1)}$ denotes the connectedness index, are equal to the lengths of the intervals bounded by the points*

$$\{k(1-\alpha) + \beta_i\}, \text{ for } 0 \leq k \leq n-1, \ 0 \leq i \leq p-1.$$

The complexity of a coding on p letters of an irrational rotation ultimately has the form $p(n) = an + b$, where $a \leq p$, and depends on the algebraic relations between the angle and the lengths of the intervals of the coding. More precisely, we have the following theorem proved in [1].

Theorem 10 *Let $u = (u_n)_{n\in\mathbb{N}}$ be a coding of the irrational rotation R of irrational angle α with respect to the partition*

$$\mathcal{P} = \{[\beta_0, \beta_1[, [\beta_1, \beta_2[, \ldots, [\beta_{p-1}, \beta_p[\}.$$

Let $(k_n)_{n\in\mathbb{N}}$ be the sequence defined by

$$k_0 = p = card(F),$$

$$k_n = card\{\beta_i \in F; \forall k \in [1 \ldots n], R^{-k}(\beta_i) \notin F\}.$$

Let a be the limit of this sequence, $n^{(2)}$ the smallest index such that $k_n = a$, and let

$$b = \sum_{i=0}^{n^{(2)}-1} (k_i - a).$$

Let $n^{(1)}$ denote the connectedness index of u.

If $n \geq \max(n^{(1)}, n^{(2)})$, then the complexity of the sequence u satisfies

$$p(n) = an + b.$$

Remarks

- Note that if $1, \alpha, \beta_1, \ldots, \beta_p$ are rationally independent, then $n^{(2)} = 0$, $b = 0$ and $a = p$.
- Theorem 10 answers the question of the existence of sequences of ultimately affine complexity (for more details, the reader is referred to [1], see also the result of Cassaigne in [11]).

4.3 Beatty sequences

The connections between the three gap theorem and the Beatty sequences have been investigated by Fraenkel and Holzman in [26]. Let us recall that a Beatty sequence is a sequence $u(\alpha, \rho) = (u_n)_{n\in\mathbb{N}}$ of the form $u_n = \lfloor \alpha n + \rho \rfloor$, where α and ρ are real numbers such that $\alpha \geq 1$. The number α is called the *modulus* and ρ is called the *residue* or *intercept*. For an impressive bibliography on the subject, we refer the reader to [27] and [54]. Fraenkel and Holzman have noticed in [26] that the three gap theorem answers the question of the gaps in the intersection of a Beatty sequence and an arithmetical sequence $(an + c)_{n\in\mathbb{N}}$, for a a positive integer and c an integer. This result has been obtained independently by Wolff and Pitman in [58]. By intersection of the two

Beatty sequences $s = (s_n)_{n\in\mathbb{N}}$ and $t = (t_n)_{n\in\mathbb{N}}$, we mean the strictly increasing sequence u defined as:

$$\{u_n,\ n \in \mathbb{N}\} = \{u,\ \exists k, l \in \mathbb{N} \text{ such that } u = s_k = t_l\}.$$

Hence a gap in the intersection denotes the difference between two distinct elements of the intersection.

Note that Beatty sequences and Sturmian sequences are related: let u be a Beatty sequence of modulus α and residue ρ; the characteristic sequence $(v_n)_{n\in\mathbb{N}}$ of u defined as

$$v_n = 1 \text{ if and only if there exists } m \text{ such that } n = \lfloor \alpha m + \rho \rfloor$$

is the Sturmian sequence obtained as the coding of the orbit of $-\rho/\alpha$ under the rotation by angle $1/\alpha$, with respect to the partition $\{]0, 1 - 1/\alpha],]1 - 1/\alpha, 1]\}$. Indeed, if $n = \lfloor \alpha m + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = m + 1 = 1 + \lceil n/\alpha - \rho/\alpha \rceil$, and if $\lfloor \alpha m + \rho \rfloor < n < \lfloor \alpha(m+1) + \rho \rfloor$, then $\lceil 1/\alpha(n+1) - \rho/\alpha \rceil = \lceil n/\alpha - \rho/\alpha \rceil$.

5 The recurrence function

Let us deduce now from the three distance and three gap theorems a simple proof of the following result originally due to Morse and Hedlund concerning the recurrence function of a Sturmian sequence (see [40]).

Recall that a sequence u is called *minimal* or *uniformly recurrent* if every factor of u appears infinitely often and with bounded gaps or, equivalently, if for any integer n, there exists an integer m such that every factor of u of length m contains every factor of u of length n. Note that it is equivalent (see [30]) to the *minimality* of the dynamical system $(\overline{\mathcal{O}}(u), T)$, i.e., the orbit of every element of $\overline{\mathcal{O}}(u)$ is dense, or equivalently every sequence in the orbit closure of u has the same set of factors as u.

The recurrence function φ of a minimal sequence u is defined by:

$$\varphi(n) = \min\{m \in \mathbb{N} \text{ such that } \forall B \in L_m, \forall A \in L_n,\ \text{A is a factor of B}\}$$

where L_n denotes the set of factors of u of length n, i.e., $\varphi(n)$ is the size of the smallest window that contains all factors of length n whatever its position on the sequence.

Theorem 11 *Let u be a Sturmian sequence with angle α. Let $(q_k)_{k\in\mathbb{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α. The recurrence function φ of this sequence satisfies for any non zero integer n:*

$$\varphi(n) = n - 1 + q_k + q_{k-1},\ \textit{where}\ \ q_{k-1} \le n < q_k.$$

Proof of Theorem 11 Let $u \in \{0,1\}^{\mathbb{N}}$ be a Sturmian sequence. There exist a real number x and an irrational number α in $]0,1[$ such that $u_n = 0 \Leftrightarrow \{x+n\alpha\} \in I_0$, with $I_0 = [0,\alpha[$ or $I_0 =]0,\alpha]$ (see Section 2.1). Let $I_1 = [\alpha,1[$ (respectively $]\alpha,1]$) if $I_0 = [0,\alpha[$ (respectively $I_0 =]0,\alpha]$). Let us denote by R the rotation of the circle by angle α. Assume we are given a positive integer n. We have seen in Section 2.1 that the word $w_1 w_2 \ldots w_n$ defined on $\{0,1\}$ appears in u if and only if

$$I(w_1,\ldots,w_n) = \bigcap_{j=0}^{n-1} R^{-j}(I_{w_{j+1}}) \neq \emptyset.$$

We deduce from this that every Sturmian sequence of angle α has the same factors as u and thus belongs to the orbit closure of u. Conversely, each sequence of the orbit closure of u is a Sturmian sequence of angle α. Hence the closed orbit of any Sturmian sequence is equal to the set of all Sturmian sequences of the same angle. This implies the minimality of any Sturmian sequence and that Sturmian sequences of the same angle have the same recurrence function; hence we can suppose here that $x = 0$.

Theorem 11 can easily been deduced from the following two lemmata. We omit the proof of Lemma 5 which is straightforward.

Lemma 4 *Let δ_n be the smallest length of the nonempty intervals $I(w_1,\ldots,w_n)$, where $w_1,\ldots,w_n$ belong to $\{0,1\}$. Let l_n be the greatest gap between the succesive integers k such that $\{k\alpha\} \in [0,\delta_n[$. We have*

$$\varphi(n) = n - 1 + l_n.$$

Lemma 5 *Let $(q_k)_{k\in\mathbb{N}}$ denote the sequence of denominators of the convergents of the continued fraction expansion of α. Let k be an integer such that $q_{k-1} \leq n < q_k$. Then we have*

$$\delta_n = \eta_{k-1} \text{ and } l_n = q_k + q_{k-1}.$$

Proof of Lemma 4 A set of points is said to visit an interval if one of these points belongs to this interval. By definition of l_n, every set of l_n consecutive points of the sequence $(\{k\alpha\})_{k\in\mathbb{N}}$ visits every interval of length δ_n (see Remark 4). Therefore they visit every nonempty interval of the form $I(w_1,\ldots,w_n)$, by definition of δ_n. Let B be a factor of u of length $n-1+l_n$; there exists an integer K such that B corresponds to the $n-1+l_n$ consecutive points $\{K\alpha\},\ldots,\{(K+n-1+l_n-1)\alpha\}$. The set of the l_n first points of this sequence of points visits every interval of the form $I(w_1,\ldots,w_n)$, thus B contains every factor of u of length n. This implies that $\varphi(n) \leq n - 1 + l_n$.

By definition of l_n and by density of $(\{k\alpha\})_{k\in\mathbb{N}}$, there exists a sequence of $l_n - 1$ points of the sequence $(\{k\alpha\})_{k\in\mathbb{N}}$ which do not visit an interval of the form $I(w_1, \ldots, w_n)$ of length δ_n; therefore, there exists a factor of u of length $l_n - 2 + n$ which does not contain the factor $w_1 \ldots w_n$. This shows that $\varphi(n) \geq n - 1 + l_n$. The lemma is thus proved.

Remark Note that in the case of the Fibonacci sequence ($\alpha = \frac{\sqrt{5}-1}{2}$), the recurrence function satisfies, for $F_{k-1} < n \leq F_k$,

$$\varphi(n) = n - 1 + F_{k+1},$$

where $(F_n)_{n\in\mathbb{N}}$ denotes the Fibonacci sequence $F_{n+1} = F_n + F_{n-1}$, with $F_0 = 1$ and $F_1 = 2$.

This result is extended in [13] to the fixed point of the substitution σ introduced by Rauzy which generalizes the Fibonacci substitution and defined by $\sigma(0) = 01$, $\sigma(1) = 02$, $\sigma(2) = 0$.

Theorem 12 *Let T_n denote the so-called Tribonacci sequence defined as follows: $T_{k+3} = T_{k+2} + T_{k+1} + T_k$, with $T_0 = 0$, $T_1 = 0$, $T_1 = 1$. The recurrence function φ of the fixed point beginning with 0 of the Rauzy substitution satisfies for any positive integer n:*

$$\varphi(n) = n - 1 + T_{k+6}, \text{ where } \sum_{0}^{k+1} T_i < n \leq \sum_{0}^{k+2} T_i.$$

6 Higher dimensional generalisations

6.1 Two-dimensional generalisations and Beatty sequences

Let us consider now some two-dimensional versions of the three distance and three gap theorems. Such generalisations were introduced by Fraenkel and Holzman in [26] in order to give an upper bound for the number of gaps in the intersection of two Beatty sequences. They first reduce this problem to a two-dimensional version of the three distance theorem, conjectured by Simpson and Holzman and proved by Geelen and Simpson (see [29]). Then they deduce from this theorem a bound for the number of gaps in the intersection of two Beatty sequences, when at least one of the moduli is rational.

Let us first give the two-dimensional version of the three gap theorem introduced by Fraenkel and Holzman. We will use the same notation as in [26]: for any pair of real numbers (x, y), $\{(x, y)\}$ means the equivalence class of (x,y) mod $\mathbb{Z}^2$, i.e., $\{(x,y)\}$ belongs to the torus $\mathbb{T}^2$.

Theorem 13 *Let α_1, α_2, β_1, β_2, μ_1 and μ_2 be real numbers in* $[0,1[$. *The gaps between the successive values of the integers n such that the following points of the torus $\mathbb{T}^2$*

$$\{(n\alpha_1, n\alpha_2)\}$$

belong to the rectangle

$$\mathcal{R} = \{\{(x,y)\}; \mu_1 - \beta_1 < x \leq \mu_1,\ \mu_2 - \beta_2 < y \leq \mu_2\}$$

take a finite number of values which depend only on α_1, α_2, β_1 and β_2.

Furthermore, if at least one of the two angles α_1 and α_2 is rational, then the number of gaps is bounded by $q+3$, where q is the minimum of the denominators of α_1 and α_2 in lowest terms (the denominator of an irrational number is considered as $+\infty$).

Let us state now the two-dimensional version of the three distance theorem proved in [29] by Geelen and Simpson.

Theorem 14 *Assume we are given two real numbers α_1, α_2 and two positive integers n_1, n_2. The set of points*

$$\{i\alpha_1 + j\alpha_2 + \rho,\ 0 \leq i \leq n_1 - 1,\ 0 \leq j \leq n_2 - 1\}$$

partitions the unit circle into intervals having at most $\min\{n_1, n_2\} + 3$ *lengths.*

Note that the bound $\min\{n_1, n_2\} + 3$ is not the best possible when n_1 or $n_2 = 1$. Indeed, in this case, the statement reduces to the three distance theorem. For a discussion on the achievability of the bound, the reader is referred to [29].

Fraenkel and Holzman have proved in [26] that Theorems 13 and 14 together answer the question of the intersection of two Beatty sequences, when at least one modulus is rational. We define a *gap* in the intersection of two Beatty sequences to be a difference between two successive elements of the intersection, and an *index-gap* to be the difference between the two corresponding indices in the same Beatty sequence.

Theorem 15 *Let $(\lfloor n\alpha_1 + \rho_1 \rfloor)_{n\in\mathbb{N}}$ and $(\lfloor n\alpha_2 + \rho_2 \rfloor)_{n\in\mathbb{N}}$ be two Beatty sequences, with at least one of the two moduli α_1 and α_2 rational. Let q denote the minimum of the denominators of α_1 and α_2 in lowest terms (the denominator of an irrational number is considered as $+\infty$). The number of gaps and index-gaps in the intersection is bounded by $q+3$, if $q \geq 2$, and bounded by 3 otherwise.*

Fraenkel and Holzman show furthermore that this bound is achievable and that the number of gaps can be made arbitrarily large, when at least one of the moduli is rational.

6.2 Combinatorial applications

Now let us review some applications of Theorems 13 and 14. For instance we can deduce the following result for the intersection of two Sturmian sequences.

Theorem 16 *Let $s = (s_n)_{n\in\mathbb{N}}$ and $t = (t_n)_{n\in\mathbb{N}}$ be two Sturmian sequences. The number of gaps between the successive integers n such that $s_n = t_n$ is finite.*

Proof Let $s = (s_n)_{n\in\mathbb{N}}$ and $t = (t_n)_{n\in\mathbb{N}}$ be two Sturmian sequences of angles α and β, with corresponding partitions $\{I_0, I_1\}$ and $\{J_0, J_1\}$. The gaps between the integers n such that the points $\{(n\alpha, n\beta)\}$ in $\mathbb{T}^2$ belong to the rectangle $I_0 \times J_0$ (respectively $I_1 \times J_1$) take a finite number of values, hence so do the gaps between the successive integers n such that the points $\{(n\alpha, n\beta)\}$ in $\mathbb{T}^2$ belong to the set $I_0 \times J_0 \cup I_1 \times J_1$.

We also deduce from Theorem 14 and Lemma 3 the following.

Theorem 17 *Let u be a coding of the irrational rotation by angle $0 < \alpha < 1$ with respect to a partition into d intervals of length $1/d$. The frequencies of factors of u of length $n \geq \sup\{n^{(1)}, d\}$ take at most $d + 3$ values, where $n^{(1)}$ denotes the connectedness index.*

Proof This result is a direct application of Lemma 3 and Theorem 14. Indeed, the intervals $I(w_1, \ldots, w_n)$ (corresponding to the factors $w_1 \ldots w_n$ of length n) are bounded by the points $\{i(1-\alpha) + j/d,\ 0 \leq i \leq n-1,\ 0 \leq j \leq d-1\}$.

Vuillon has introduced in [57] two-dimensional generalisations of Sturmian sequences obtained by considering the approximation of a plane of irrational normal by square faces oriented along the three coordinates planes. Theorem 14 can also be applied to give an upper bound for the number of frequencies of blocks of a given size for such double sequences (see [4]).

We will give in Section 7 a direct combinatorial proof of Theorem 14 in the particular case $\min\{n_1, n_2\} = 2$, and give an interpretation in terms of frequencies of binary codings: the frequencies of the factors of given length of a coding of an irrational rotation with respect to a partition in two intervals take ultimately at most 5 values.

6.3 The $3d$ distance theorem

Let us consider another generalisation of the three distance theorem, known as the $3d$ *distance theorem*. This result, conjectured by Graham (see [17] and [34]), was first proved by Chung and Graham in [18] and secondly by Liang who gave a very nice proof in [37]. Geelen and Simpson remark in [29] that their proof uses ideas from Liang's proof.

The $3d$ distance theorem *Assume we are given $0 < \alpha < 1$ irrational, $\gamma_1, \ldots, \gamma_d$ real numbers and $n_1, \ldots, n_d$ positive integers. The points $\{n\alpha + \gamma_i\}$, for $0 \le n < n_i$ and $1 \le i \le d$, partition the unit circle into at most $n_1 + \cdots + n_d$ intervals, having at most $3d$ different lengths.*

We will give a combinatorial proof of this result in Section 8 and express the coresponding result for frequencies of codings of rotations, i.e., that the frequencies of the factors of given length of a coding of a rotation by the unit circle under a partition in d intervals take ultimately at most $3d$ values.

6.4 Other generalisations

Slater has studied in [50] the following generalisation of the three gap theorem, which should be compared with Theorem 13: there is a bounded number of gaps between the successive values of the integers n such that $\{n(\eta_1, \ldots, \eta_d)\} \in C$, where C is a closed convex region on the d-dimensional torus and where $1, \eta_1 \ldots, \eta_d$ are rationally independent. However, Fraenkel and Holzman prove Theorem 13 even in the case where α_1, α_2 and 1 are rationally independent.

Chevallier studies in [16] a d-generalization of the three distance theorem to $\mathbb{T}^d$, where intervals are replaced by Voronoï cells: the number of Voronoï cells (up to isometries) is shown to be connected to the number of sides of a Voronoï cell. The notion of continued fraction expansion is generalized by properties of best approximation.

Finally, note the unsolved problems quoted in [29] concerning further generalisations of the three distance theorem. For instance, an upper bound for the number of distinct lengths in the partition of the unit circle by the points $k_1\alpha_1 + k_2\alpha_2 + \cdots + k_d\alpha_d$, for $k_i \le n_i - 1$ and $1 \le i \le d$ is conjectured to be of the form $c_d + \prod_{i=1}^{d-1} n_i$, where c_d is a constant independent of $n_1, \cdots, n_d$.

7 Frequencies of factors for binary codings of rotations

We will prove in this section the following result, which corresponds to the case $\min\{n_1, n_2\} = 2$ in Theorem 14. The idea of using a reflection of the unit circle can also be found in the original proof in [29].

Theorem 18 *Let α be an irrational number in $]0,1[$, $\beta \neq 0$ a real number and n a non zero integer. The set of points $\{0\}, \{\beta\}, \{\alpha\}$, $\{\beta+\alpha\}, \ldots, \{n\alpha\}$, $\{\beta+n\alpha\}$ divides the circle into a finite number of intervals, whose lengths take at most five values.*

7.1 A combinatorial proof

We will prove Theorem 18 by introducing a coding of the rotation by angle α with respect to the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}$, $\{\beta+\alpha\}, \cdots, \{n\alpha\}, \{\beta+n\alpha\}$.

Let α be an irrational number, β a non-zero real number and n an integer. Let $I_1, \cdots, I_p$ denote the intervals of the unit circle bounded by the points $\{0\}, \{\beta\}, \{\alpha\}, \{\beta+\alpha\}, \cdots, \{n\alpha\}, \{\beta+n\alpha\}$. Let $u = (u_n)_{n\in\mathbb{N}}$ be the sequence defined on the alphabet $\Sigma = \{a_1, \ldots, a_p\}$ as the coding of the orbit of 0 under the rotation R of angle α under the partition $\{I_1, \cdots, I_p\}$:

$$u_n = a_k \Leftrightarrow \{n\alpha\} \in I_k.$$

The frequency of the letter a_k in the sequence u is equal to the length of the interval I_k, by uniform distribution of the sequence $(\{n\alpha\})_{n\in\mathbb{N}}$. We must now prove that the frequencies of the letters of u take at most five values Let us consider the graph Γ_1 of words of u of length 1. There is one edge from a_k to $a_{k'}$ if $I_{k'}$ is the image of I_k by the rotation R or if $I_{k'}$ contains $\{-\alpha\}$ or $\{-\alpha+\beta\}$. Therefore the graph Γ_1 contains p vertices (one for each letter) and $p+2$ edges: indeed, every vertex has only one leaving edge, except the ones associated with the intervals containing $\{-\alpha\}$ or $\{n-\alpha+\beta\}$, which have two leaving edges (if both of these points belong to the same interval I_k, then a_k has three leaving edges and all the other intervals have only one edge). In other words, we have $p(1) = p$ and $p(2) = p+2$. As in the proof of Theorem 6, this implies that there are at most 6 branches in Γ_1: indeed, each branch starts with a vertex with more than one entering edge (this provides at most two branches) or just after a vertex with at least two leaving edges (at most four branches are of this kind). We deduce from this that the frequencies of the letters in u take at most 6 values. Let us prove that at least two branches of Γ_1 have the same frequency, which will complete the proof.

Let s denote the reflection of the circle defined by $s : x \to \{\beta + n\alpha - x\}$. This reflection leaves invariant the endpoints of the intervals $I_1, \cdots, I_p$ and thus induces a permutation σ of the interiors of the intervals I_k, which can also be seen as a permutation of Σ. The length of I_k is equal to the length of $I_{\sigma(k)} = s(I_k)$. The frequency of the letter a_k is thus equal to the frequency of the letter $\sigma(a_k)$. Note that if $a_i a_j$ is a factor of u, then $\sigma(a_j)\sigma(a_i)$ is also a factor. We deduce from this that if there is an edge in Γ_1 from a_i to a_j, then there is also an edge from $\sigma(a_j)$ to $\sigma(a_i)$, or in other words, that Γ_1 is invariant by the following action of σ: the image of the vertex associated with the letter a is equal to the vertex associated with $\sigma(a)$ and the image of the edge $a \to b$ is the edge $\sigma(b) \to \sigma(a)$, i.e., each letter is replaced by its image and the direction of every edge is changed. Furthermore, the image of a branch is a branch. Let us prove that at most four branches of the graph Γ_1 are invariant by σ. Let $B = U_1 \to U_2 \to \ldots \to U_q$ be an invariant branch of the graph. We have $B = \sigma(B) = \sigma(U_q) \to \ldots \to \sigma(U_1)$. We thus get $\sigma(U_k) = U_{q+1-k}$.

- Suppose that there exists i such that $U_i = \sigma(U_i)$. Therefore the interval I_i must contain a fixed point for s. Since there are only two such fixed points, at most two branches can satisfy this property.
- Let us suppose that $U_i \neq \sigma(U_i)$ for each $1 \leq i \leq q$. We thus have q even and $\sigma(U_{q/2}) = U_{q/2+1}$. Let I (respectively I') be the closure of the interval associated with $U_{q/2}$ (respectively $U_{q/2+1}$). We thus get $s(I) = I'$. Furthermore, I' is the image of I by the rotation R, because of the edge $U_{q/2} \to U_{q/2} + 1$. This implies that I' contains a fixed point of the symmetry $s \circ R^{-1}$, which has at most two fixed points. Hence, at most two branches are of this kind.

We have proved that at most four basic paths can be their own image by σ. Therefore, there exist among the six branches at least two different branches, say A and B, such that $B = \sigma(A)$. Thus A and B have the same frequency, which implies that there are at most five possible frequencies for the letters of u.

7.2 Application to binary codings

A more natural coding of the rotation R would have been with respect to the partition $[0, \beta[, [\beta, 1[$. The points $\{0\}, \{\beta\}, \{\alpha\}$, $\{\beta + \alpha\}, \ldots, \{n\alpha\}$, $\{\beta + n\alpha\}$ are the endpoints of the sets $I(w_1, \ldots, w_n)$, following the notation of Section 2. But these sets might not be connected. Thus the frequencies of factors of length n are the sums of the lengths of the connected components of the sets $I(w_1, \ldots, w_n)$. Despite this disandvantage, this coding allows us to deduce the following result from Lemma 3.

Theorem 19 *Let u be a coding of an irrational rotation with respect to the partition into two intervals $\{[0, \beta[, [\beta, 1[\}$, where $0 < \beta < 1$. Let $n^{(1)}$ denote the connectedness index of u. The frequencies of factors of given length $n \geq n^{(1)}$ of u take at most 5 values. Furthermore, the set of factors of u is stable by mirror image, i.e., if the word $a_1 \cdots a_n$ is a factor of the sequence u, then $a_n \cdots a_1$ is also a factor and furthermore, both words have the same frequency.*

Proof It remains to prove the part of this theorem concerning the stability by mirror image. Assume we are given a fixed positive integer n. Let s_n be the reflection of the circle defined by $s_n : x \to \{\beta - (n-1)\alpha - x\}$. We have $s_n(R^{-k}(I_j)) = R^{(-n+1+k)}(I_j)$, for $j = 0, 1$, following the previous notation. The image of $I(w_1, \ldots, w_n)$ by s_n is $I(w_n, \cdots, w_1)$; they thus have the same length, which gives the result.

Remark A study of the topology of the graph of words for a binary coding of an irrational rotation of complexity satisfying ultimately $p(n+1) - p(n) = 2$ can be found in [24] or in [46].

8 The $3d$ distance theorem

Following the idea of the above proof, let us give a combinatorial proof of the $3d$-*distance theorem.*

The $3d$ distance theorem *Assume we are given $0 < \alpha < 1$ irrational, $\gamma_1, \ldots, \gamma_d$ real numbers and $n_1, \ldots, n_d$ positive integers. The points $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$, partition the unit circle into at most $n_1 + \cdots + n_d$ intervals, having at most $3d$ different lengths.*

Proof Let us consider a coding of the rotation by angle α under the left-closed and right-open partition of the unit circle bounded by all the points of the form $\{n\alpha + \gamma_i\}$, for $0 \leq n < n_i$ and $1 \leq i \leq d$; let $\beta_0, \ldots, \beta_{p-1}$ denote these consecutive points. The letter associated with the interval $I_k = [\beta_k, \beta_{k+1}[$ has a unique right extension, except when I_k contains points of the form $\{\beta_i - \alpha\}$. Suppose there are $q \geq 2$ pints of this form; the associated letter has $q+1$ right extensions. Since there are at most d points of this type, we obtain $p(2) - p(1) \leq d$. We deduce from Theorem 6 that there are at most $3d$ different frequencies for the letters of the coding, i.e., there are at most $3d$ different lengths for the intervals I_k.

Remark The start and finish intervals as introduced by Liang in his proof in [37] correspond exactly to the beginning of the branches in the graph of words. Indeed, Liang shows that any interval is associated either with a start point $\{\gamma_i\}$ (i.e., with one extension of a factor having more than one right extension) or with a finish point $\{(n_i - 1)\alpha + \gamma_i\}$ (i.e., with a factor having more than one left extension). Counting the finish and start points defined in [37] (there are $3d$ such points) is equivalent to counting the number of branches in the graph of words.

As in the remark of the previous section, we can consider a coding of the rotation by irrational angle $1 - \alpha$ under the partition $\{[\gamma_1, \gamma_2[, \ldots, [\gamma_d, \gamma_1[\}$. For such a coding, the $3d$ distance theorem can be rephrased as follows.

Theorem 20 *The frequencies of the factors of given length $n \geq n^{(1)}$ of a coding of a rotation by irrational angle under a partition in d intervals take at most $3d$ values, where $n^{(1)}$ denotes the connectedness index.*

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