A study of Jacobi-Perron boundary words for the generation of discrete planes

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Abstract

The construction of a Sturmian word, and thus of a discrete line, from a continued fraction development generalizes to higher dimension. Given any vector $v \in \mathbb{R}^3$, a list of 6-connected points approximating the line defined by v may be obtained via a generalized continued fraction algorithm. By duality, a discrete plane with normal vector v can also be generated using a related technique. We focus on such discrete planes, more precisely on the finite patterns generated at each step of the process. We show that the choice of Jacobi-Perron algorithm as a higher dimension generalization of Euclid's algorithm together with the specific substitutions deduced from it allows us to guaranty the simple connectedness of those patterns.

Keywords: discrete plane, discrete geometry, mutidimensional continued fractions, substitutions

1. Introduction

The aim of this paper is to produce a generation method for arithmetic discrete planes. This generation method is inspired by the one-dimensional case and by the generation of Sturmian words. Indeed, it is well-known that Sturmian words are infinite words over a two-letter alphabet that code discrete lines in the plane (via the Freeman chain code). For more details on Sturmian words, see e.g. the extended surveys [5, 25], and for the generation of discrete lines by using Euclid's algorithm, see also [6, 31, 30]. Moreover, these words are generated by iterating a finite number of substitutions (i.e., of morphisms of the free monoid) according to the continued fraction expansion of the slope of the discrete line they code. Most combinatorial properties of Sturmian words can be described in terms of this continued fraction expansion. If the normal vector (a, b) of the line has integer entries (which is the case in discrete geometry), we use Euclid's algorithm. Note that Euclid's algorithm allows one to compute a sequence of approximations corresponding to "truncations" in the continued fraction expansion. By using the matricial form of Euclid's algorithm, we recover the so-called Sturmian matrices (see [4, 5]), which are the incidence matrices of two-letter substitutions which belong to the Sturm monoid. We want to extend this interaction to the higher-dimensional case by working with multidimensional continued fractions. We use here as a multidimensional continued fraction algorithm the Jacobi-Perron algorithm.

Numerous generalizations of Euclid's algorithm and of the continued fraction algorithm enable similar treatment for d-tuples of real numbers, with $d \geq 3$. There is no canonical extension of the regular continued fraction algorithm, and several approaches have been proposed (see [12, 29] for a summary). We use here one of the most classical multidimensional continued fraction algorithms, namely the Jacobi-Perron algorithm. It is formulated in terms of nonnegative unimodular matrices. This algorithm generates a sequence of simultaneous rational approximations of a pair of points with the same denominator with a.e. exponential convergence (see e.g. [13]). The Jacobi-Perron d-dimensional continued fraction algorithm can thus be used to define geometric objects (lines or planes) in the discrete space $\mathbb{Z}^d$. Inspired by [24], the idea is to associate substitutions with the matricial form of this algorithm, that is, with the set of matrices in $SL_d(\mathbb{N})$ that describe this algorithm, by interpreting these nonnegative unimodular matrices as transpose matrices of

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incidence matrices of well-chosen substitutions, with this choice being here again non-canonical. We then can either generate discrete lines in the space (see e.g. [9]), or in a dual way, discrete planes, which is our aim here. We use a geometric formalism (introduced in [3]) that allows one to associate with a substitution σ a transformation $E_1^*(\sigma)$ acting on discrete planes, replacing faces of unit cubes by unions of faces. We thus get geometric Jacobi-Perron substitutions. See also [21] for a similar approach based on Brun multidimensional continued fraction algorithm.

We are not only able to substitute within the framework of generalized substitutions, but we also can desubstitute, i.e., perform the converse operation, by using the algebraic property $E_1^*(\sigma) \circ E_1^*(\sigma^{-1}) = E_1^*(\sigma^{-1}) \circ E_1^*(\sigma) = Id$ (see [3]). In [8] and [21], original algorithms for digital plane recognition and generation are given in the case of Brun's multidimensional continued fraction algorithm.

Based on results of [24, 7], the present paper first describes how the Jacobi-Perron algorithm together with these geometric Jacobi-Perron substitutions allows the generation of a sequence of patterns that are converging to a discrete plane, once we are given its normal vector in dimension $d = 3$. More precisely, if this normal vector is totally irrational, we obtain a converging sequence of patterns whose limit covers the full plane. Otherwise, we produce patterns that cover periodically the discrete plane (we use the fact that in this dimension, Jacobi-Perron algorithm detects linear dependence, see e.g. [17, 18]). See Theorem 30 and 31 for a precise statement. We are mainly interested in the topological shape of these patterns. The fact that we work in dimension $d = 3$ allows us to get a manageable description of the boundary of the generated patterns in terms of morphisms of the free group (in the flavour of [15, 16]), with the boundary being made of a finite union of segments. We then prove the main result of this paper, namely, these patterns are connected without holes (we call them polyamond patterns). Indeed, we prove that the finite sets of faces obtained by stopping the Jacobi-Perron algorithm after a finite number of steps is always a polyamond pattern (this is the discrete analogue of being simply connected). In other words, this indicates that Jacobi-Perron substitutions are "good" substitutions with respect to the topology.

This paper is organized as follows. Section 2 recalls the required background on substitutions and their associated geometric formalism, the notions of discrete plane, and boundary words. In Section 3, we introduce the Jacobi-Perron algorithm and the Jacobi-Perron substitutions. Section 4 describes the strategy we will follow based on generating patterns for discrete planes under the action of geometric Jacobi-Perron substitutions. Finally, we focus on the boundary of the generated patterns in Section 5.

2. Substitutions and geometry

2.1. Words and substitutions

Let us consider a finite set of *letters* $\mathcal{A}$, called *alphabet*. A (finite) *word* is an element of the free monoid $\mathcal{A}^*$ generated by $\mathcal{A}$. By convention the *empty word* is denoted by $\varepsilon_\mathcal{A}$ and its length is 0. The set of n-length words over $\mathcal{A}$ is denoted by $\mathcal{A}^n$. The number of occurrences in the word $w = w_1 \cdots w_n$ of a given letter $k \in \mathcal{A}$ is denoted by $|w|_k$, i.e., $|w|_k = |\{1 \leq i \leq n \mid w_i = k\}|$. Given a word $w \in \mathcal{A}^*$, a *factor* f of w is a word satisfying the following: $\exists\, x, y \in \mathcal{A}^*, w = xfy$ or, equivalently, $f = w[i \dots j]$ where $i = |x| + 1$ and $j = |w| - |y|$, with the notation $w[i \dots j]$ standing for $w_i \dots w_j$. If $x = \varepsilon_\mathcal{A}$ (resp. $y = \varepsilon_\mathcal{A}$) then f is said to be a *prefix* (resp. *suffix*) of w. Moreover, a prefix (resp. suffix) f is proper if $f \neq w$. The *mirror image* $\widetilde{w}$ of $w = w_1 w_2 \cdots w_n$ is the word $\widetilde{w} = w_n \cdots w_2 w_1$. The notion of cyclic factor recalled below will play a central role in Section 5.

Definition 1 (Conjugation)**.** Two words w and w' over the free monoid $\mathcal{A}^*$ are said to be *conjugate* if there exist u and v in $\mathcal{A}^*$ such that

$$w = uv \text{ and } w' = vu.$$

Given a word $w \in \mathcal{A}^*$, a *cyclic factor* f of w is a factor of some conjugate of w. Moreover, if f is a cyclic factor of w, then any word g such that fg is conjugate to w is called a *complement* of f in w.

We will work in the following with the d-letter alphabet $\mathcal{A}_d = \{1, 2, \dots, d\}$ as well as with the $2d$-letter alphabet $\overline{\mathcal{A}}_d = \{1, \overline{1}, 2, \overline{2}, \dots, d, \overline{d}\}$. We will also work with the free group F_d generated by $\{1, 2, \dots, d\}$.

There is a natural projection morphism π_{F_d} from $\overline{\mathcal{A}}_d^*$ onto F_d defined as

$$\pi_{F_d}(i) := i \text{ and } \pi_{F_d}(\,\bar{i}\,) := i^{-1},\ \forall i \in \{1, 2, \cdots, d\}.$$

Working with the monoid $\overline{\mathcal{A}}_d^*$ instead of working with F_d will allow us not to take reductions into account. Note furthermore that we will restrict ourselves to the case $d = 3$ in all that follows. The sets $\mathcal{A}_d^+$, $\overline{\mathcal{A}}_d^+$, F_d^+ will denote the sets of non-empty words over the respective alphabets $\mathcal{A}_d$, $\overline{\mathcal{A}}_d$, F_d.

The *abelianized* $\overrightarrow{u}$ of a word $u = u_1 \cdots u_n \in \mathcal{A}_d^*$ is defined as the following vector of $\mathbb{N}^d$

$$\overrightarrow{u} := (|u|_i)_{i\in\{1,\cdots,d\}}.$$

We thus have $\overrightarrow{u} = \sum_{1\leq j\leq |u|} e_{u_j}$, where $(e_1, \ldots, e_d)$ stands for the canonical base of $\mathbb{R}^d$. By abuse of notation, we similarly define the *abelianized* $\overrightarrow{u}$ of a word $u \in \overline{\mathcal{A}}_d^*$ as the following vector of $\mathbb{Z}^d$:

$$\overrightarrow{u} := (|u|_i - |u|_{\bar{i}})_{i\in\{1,\cdots,d\}}.$$

At last, we define the *abelianized* $\overrightarrow{u}$ of a word $u \in F_d$ as the following vector of $\mathbb{Z}^d$, $\overrightarrow{u} := (|u|_i - |u|_{i^{-1}})_{i\in\{1,\cdots,d\}}$. Let us stress the fact that all abelianized vectors belong to $\mathbb{Z}^d$.

A *substitution* σ over the alphabet $\mathcal{A}_d$ is an endomorphism of the free monoid $\mathcal{A}_d^*$. It is completely defined by its image on the letters of the alphabet. Its *abelianized matrix* (also called *incidence matrix*) $M_\sigma = (m_{i,j})_{1\leq i,j\leq d}$ is defined as the square matrix with non-negative integer entries $m_{i,j} = |\sigma(j)|_i$ for all i, j. One has

$$\text{for all } u \in \mathcal{A}_d^*,\ \overrightarrow{\sigma(u)} = M_\sigma \overrightarrow{u}.$$

According to this, we say that σ is *unimodular* if $\det(M_\sigma) = \pm 1$. Once again by abuse of terminology, if σ is a substitution over $\overline{\mathcal{A}}_d$, we define its *abelianized matrix* as the d-square matrix with integer entries whose (i, j) entry is defined by $|\sigma(j)|_i - |\sigma(j)|_{\bar{i}}$, for $(i, j) \in \{1, 2, \ldots, d\}^2$. If σ is a morphism of the free group F_d, we similarly define its *abelianized matrix* as the d-square matrix with integer entries whose (i, j) entry is defined by $|\sigma(j)|_i - |\sigma(j)|_{i^{-1}}$, for $(i, j) \in \{1, 2, \ldots, d\}^2$.

2.2. Generalized substitutions

A natural geometric interpretation of a word $w \in \mathcal{A}_d^n$ as a path $P^{(w)} := [P_0, P_1, P_2, \ldots, P_n]$ in $\mathbb{Z}^d$ is obtained by associating with each of its prefix $w[1 \ldots i]$ the integer coordinate point $P_i \in \mathbb{Z}^d$ defined as

$$P_i := \overrightarrow{w[1 \ldots i]}.$$

We say that the word w *codes the path* $P^{(w)}$, and this is known as the *Freeman chain code* [22, 23].

In order to extend the action of substitutions and free group morphisms from words to higher-dimensional geometric objects in the context of discrete geometry, we use and recall below the formalism introduced in [3, 28].

First, let $\mathfrak{F}_d$ be the vector space of mappings from $\mathbb{Z}^d \times \mathcal{A}_d$ to $\mathbb{R}$ that take everywhere zero value except for a finite set. We denote by (P, i) the element of $\mathfrak{F}_d$ which takes value 1 at (P, i) and 0 elsewhere. In the 3-dimensional case, the following geometric model can be introduced for (P, i): a pair (P, i) is simply represented by the line segment $\{P + \lambda e_i \mid 0 \leq \lambda \leq 1\} \subset \mathbb{R}^3$ endowed with the orientation provided by the vector e_i. We also interpret geometrically the sum of two segments as their union. In this context, the elements of $\mathfrak{F}_d$ that have a geometric interpretation are the finite sums of distinct (P, i) and the Freeman chain code yields the following "interpretation" map I_d:

$$\begin{aligned} I_d : \mathbb{Z}^d \times \mathcal{A}_d^+ &\longrightarrow \mathfrak{F}_d \\ (P, w_1 \cdots w_k) &\mapsto \textstyle\sum_{i=1}^k (P + \overrightarrow{w_1 \cdots w_{i-1}}, w_i). \end{aligned}$$

In particular, given a word $w \in \mathcal{A}^+$, $I_d(P, w)$ forms a path from P to $P + \overrightarrow{w}$ (see Figure 1).

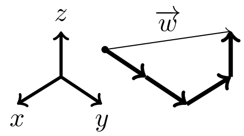


Figure 1: Let $w = 22\overline{1}3$. The geometrical interpretation of the map $I_d\left((0,1,1),w\right) = ((0,1,1),2)) + ((0,2,1),2) + \left((0,3,1),\overline{1}\right) + ((-1,3,1),3)$ is a path from $(0,1,1)$ to $(-1,3,2)$, the difference between the start point and the end point is given by the abelianized ot w which is $(-1,2,1)$.

The 1-*dimensional geometric realization* $E_1(\sigma)$ of σ is the linear mapping defined on $\mathfrak{F}_d$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{A}_d^+ & \xrightarrow{\sigma} & \mathcal{A}_d^+ \\ \Big\downarrow I_d & & \Big\downarrow I_d \\ \mathfrak{F}_d & \xrightarrow{E_1(\sigma)} & \mathfrak{F}_d \end{array}$$

More precisely:

$$E_1(\sigma)(P,i) := I_d\left(M_\sigma P, \sigma(i)\right) = \sum_{\substack{uj \text{ prefix of } \sigma(i) \\ u \in \mathcal{A}_d^*,\, j \in \mathcal{A}_d}} \left(M_\sigma P + \overrightarrow{u}, j\right). \tag{1}$$

For an illustration, see Example 2 below.

Note that E_1 can also be defined for an automorphism of the free group F_d, according to [19], with the geometric interpretation for a segment (P, i^{-1}), with $i \in \mathcal{A}_d$, being given by:

$$(P, i^{-1}) := \{P - \lambda e_i \mid 0 \leq \lambda \leq 1\} = -(P - e_i, i).$$

Hence, (P, i^{-1}) is the segment $\{P - \lambda e_i \mid 0 \leq \lambda \leq 1\}$ endowed with the orientation provided by $-e_i$. We will need and use this in Section 2.4 when handling boundary words. In order to handle cancellations (this comes from the fact that we work on the free group), it will be convenient in all that follows to also work with the monoid $\overline{\mathcal{A}}_3^*$ instead of working with the free monoid $\mathcal{A}_3^*$ (we recall that $\overline{\mathcal{A}}_3 = \{1, \overline{1}, 2, \overline{2}, 3, \overline{3}\}$). The interest of working with the free monoid $\overline{\mathcal{A}}_3^*$ instead of working with the free group F_3 will become clearer, in particular, with the combinatorial proofs of Section 5. We thus also introduce a mapping $\overline{E}_1(\sigma)$ for a substitution over $\overline{\mathcal{A}}_d$, by setting for all $i \in \{1,2,3\}$ $(P, \overline{i}) = (P, i^{-1})$. From a geometric viewpoint, there is no difference between segments of the form $(P, \overline{i})$ or (P, i^{-1}).

In order to extend the definition of E_1, we first introduce maps which are analogous of the mapping I_d, that involve elements of the form (P, i^{-1}) or $(P, \overline{i})$, i.e., elements of, respectively, the vector space $\mathfrak{G}_d$ of mappings from $\mathbb{Z}^d \times F_d$ to $\mathbb{R}$, and the vector space $\overline{\mathfrak{F}}_d$ of mappings from $\mathbb{Z}^d \times \overline{\mathcal{A}}_d$ to $\mathbb{R}$. Hence, one defines

$$\begin{array}{ll} J_d : \mathbb{Z}^d \times F_d^+ & \longrightarrow \mathfrak{G}_d \\ (P, w_1 \cdots w_k) & \longmapsto \sum_{i=1}^k (P + \overrightarrow{w_1 \cdots w_{i-1}}, w_i), \end{array}$$

$$\begin{array}{ll} K_d : \mathbb{Z}^d \times \overline{\mathcal{A}}_d^+ & \longrightarrow \overline{\mathfrak{F}}_d \\ (P, w_1 \cdots w_k) & \longmapsto \sum_{i=1}^k (P + \overrightarrow{w_1 \cdots w_{i-1}}, w_i). \end{array}$$

We then set, if σ is a morphism of the free group F_d, $E_1(\sigma)(P,i) := J_d\left(M_\sigma P, \sigma(i)\right)$, and if σ is a substitution over $\overline{\mathcal{A}}_d$, $\overline{E}_1(\sigma)(P,i) := K_d\left(M_\sigma P, \sigma(i)\right)$.

We now consider $\mathfrak{F}_d^*$ the dual vector space of $\mathfrak{F}_d$ and define $E_1^*(\sigma)$ as the dual mapping of $E_1(\sigma)$. An explicit formulation of $E_1^*(\sigma)$ is given in [3], when σ is assumed furthermore to be a unimodular substitution over the alphabet $\mathcal{A}_d$:

$$E_1^*(\sigma)(P,i^*) := \sum_{j\in\mathcal{A}} \sum_{ui \text{ prefix of } \sigma(j)} \left(M_\sigma^{-1}\left(P - \overrightarrow{u}\right), j^*\right). \tag{2}$$

A mapping of the form $E_1^*(\sigma)$ is called a *generalized substitution.*

In the 3-dimensional case, a geometric model can also be given for (P, i^*): a pair (P, i^*) corresponds to the pointed face defined as the translation by $P + e_i$ of the surfel generated by $\{e_1, e_2, e_3\} \setminus \{e_i\}$ (see Figure 2 for an illustration), with the notation $((P,i^*))_{(P,i)\in\mathbb{Z}^d\times\mathcal{A}}$ standing for the dual basis of the basis $((P,i))_{(P,i)\in\mathbb{Z}^d\times\mathcal{A}_d}$ of $\mathfrak{F}_d$:

$$\begin{array}{rcl}(P,1^*) & := & P + \{\mathbf{e}_1 + \lambda\mathbf{e}_2 + \mu\mathbf{e}_3,\ (\lambda,\mu)\in[0,1]^2\} \\ (P,2^*) & := & P + \{\mathbf{e}_2 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_3,\ (\lambda,\mu)\in[0,1]^2\} \\ (P,3^*) & := & P + \{\mathbf{e}_3 + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2,\ (\lambda,\mu)\in[0,1]^2\}.\end{array}$$

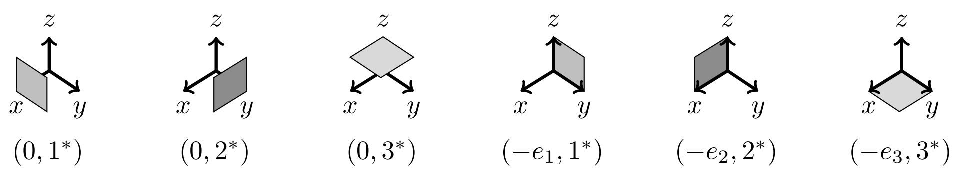


Figure 2: Geometric interpretation of faces

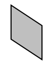
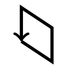
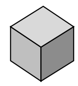
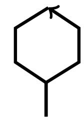
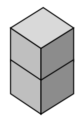
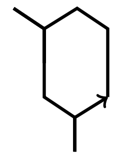
We also interpret the sum of two faces as their union.

Generalized substitutions have been introduced in [3] for unimodular substitutions. They have been extended to the case of unimodular morphisms of the free group in [19]. In the notation $E_1^*(\sigma)$, the subscript "1" stands for the codimension of the faces, while the superscript "$*$" indicates that it is the dual map of the map $E_1(\sigma)$. Extensions of $E_1(\sigma)$ to more general spaces than $\mathfrak{F}_d$ have also been provided in [28], as well as extensions of $E_1^*(\sigma)$ as k-dimensional dual mappings: the mapping $E_k^*(\sigma)$ acts on the set of $(d-k)$-dimensional pointed faces of unit cubes in $\mathbb{R}^d$ with integer vertices. These extensions satisfy a commutation relation with the natural boundary operator: the boundary of the image is the image of the boundary. We will consider the map $E_2^*(\sigma)$ in Section 2.4.

Example 2. Consider the Tribonacci substitution defined by $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$ on $\mathcal{A}_3 = \{1, 2, 3\}$. Iterations of $E_1(\sigma)$ and $E_1^*(\sigma)$ are depicted in the next figure. Let

$$\tau\colon 1 \mapsto 3,\ 2 \mapsto 1\overline{3},\ 3 \mapsto 2\overline{3},\ \overline{1} \mapsto \overline{3},\ \overline{2} \mapsto 3\overline{1},\ \overline{3} \mapsto 3\overline{2}.$$

The reason for introducing the map τ will be given in Example 17 below. The arrow indicates the orientation.

	$E_1(\sigma^k)(0,1)$	$E_1^*(\sigma^k)(0,1^*)$	$\overline{E}_1(\tau^k)(K_3(0,(23\overline{23})))$
$k=0$	 $(0,1)$	 $(0,1^*)$	 $23\overline{23}$
$k=1$	 $(0,1)+((1,0,0),2)$	 $(0,1^*)+(0,2^*)+(0,3^*)$	 $1\overline{3}2\overline{3}3\overline{1}3\overline{2}$
$k=2$	 $(0,1)+((1,0,0),2)$ $+((1,1,0),1)+((2,1,0),3)$	 $(0,1^*)+(0,2^*)+(0,3^*)$ $+((0,0,-1),1^*)+((0,0,-1),2^*)$	 $33\overline{2}1\overline{3}3\overline{2}2\overline{33}2\overline{3}3\overline{1}$

2.3. Discrete planes

According to [26], we now introduce the notion of a discrete approximation of a plane. We consider a plane in $\mathbb{R}^3$ of normal vector (a,b,c), with $(a,b,c) \in \mathbb{R}^3_+ \setminus \{0\}$. We work here in dimension 3 but the concepts introduced in this section hold in general dimension d. Note that w.l.o.g. we consider only the case $0 \le a,b \le c$ since the symmetries required to transpose any other case to one of this form are easily performed using permutations of the alphabet.

Let $(a,b,c) \in \mathbb{R}^3_+ \setminus \{0\}$. According to the yet classic terminology in discrete geometry introduced in [26, 1], the *standard arithmetic discrete plane* with normal vector (a,b,c) is defined as

$$\left\{(x,y,z) \in \mathbb{Z}^3 \;\middle|\; 0 < ax+by+cz \le a+b+c\right\}.$$

One checks (see e.g. [2]) that it is equal to the set of vertices of the union of pointed faces (P,i^*), with $(P,i) = ((x,y,z),i) \in \mathbb{Z}^3 \times \mathcal{A}_3$ satisfying

$$\begin{array}{c} -a < ax+by+cz \le 0 \text{ if } i=1, \\ -b < ax+by+cz \le 0 \text{ if } i=2, \\ -c < ax+by+cz \le 0 \text{ if } i=3. \end{array} \tag{3}$$

Hence we introduce the following notion of discrete plane defined not as a subset $\mathbb{Z}^3$ but as a union of pointed faces satisfying (3), which will be proved to be more suited to the formalism of generalized substitutions (see Theorem 8 below):

Definition 3 (Discrete planes)**.** Let $(a,b,c) \in \mathbb{R}^3_+ \setminus \{0\}$. The *discrete plane* $\mathcal{P}_{(a,b,c)}$ with normal vector (a,b,c) is defined as the union of faces (P,i^*), with $(P,i) = ((x,y,z),i) \in \mathbb{Z}^3 \times \mathcal{A}_3$ satisfying (3).

This definition is illustrated in Figure 3.

Let Π_0 be the orthogonal projection onto the plane $\mathcal{P}_0\colon x+y+z=0$. The projection mapping Π_0 is a homeomorphism from the discrete plane $\mathcal{P}_{(a,b,c)}$ onto the plane $\mathcal{P}_0$. We will use this property in order to work in a two-dimensional space (instead of $\mathbb{Z}^3$) and benefit from plane's topology results.

When working with sets of cube's faces, to suppose that they are all included in a discrete plane is a restrictive hypothesis. This is why we define a weaker condition.

Definition 4 (Pattern)**.** A *pattern* is a non-empty union of pointed faces which is homeomorphic by Π_0 to its projection.

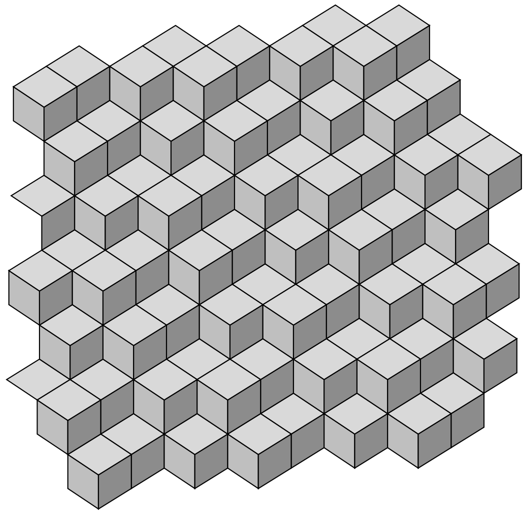
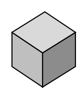


Figure 3: Pattern of a discrete plane.

A pattern is said to be *edge-connected* if any two faces F and F' of $\mathcal{W}$ can be connected by a finite path of faces sharing an edge, i.e. there exists a sequence $(F_i)_{0\leq i\leq n}$ of faces of $\mathcal{W}$ such that $F_0 = F$, $F_n = F'$, and F_i and F_{i+1} share an edge. Also, a *finite pattern* is defined as a finite union of pointed faces.

As a particular pattern, consider the upper half of the unit cube

$$\mathcal{U} := (0,1^*) + (0,2^*) + (0,3^*) =$$



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Note that one deduces from (3) that

$$\mathcal{U} \subset \mathcal{P}_{(a,b,c)} \text{ as soon as } a,b,c > 0. \tag{4}$$

This will be used in particular in Section 4.

Definition 5 (Rationality of planes)**.** If $dim_{\mathbb{Q}}(a,b,c) = 1$, then the plane is said to be *rational*, and we can assume w.l.o.g. that $a,b,c \in \mathbb{N}$ with a,b,c coprime. Otherwise, we say that the plane is irrational. Furthermore, if $dim_{\mathbb{Q}}(a,b,c) = 3$, then the plane is said to be *totally irrational*.

Definition 6 (Period patterns)**.** A non-empty pattern $\mathcal{W}$ is said to be *a period pattern* for the discrete plane $\mathcal{P}_{(a,b,c)}$ if there exist two (nonzero) vectors T_1 and T_2 in $\mathbb{Z}^3$ (called *period vectors)* such that

$$\mathcal{P}_{(a,b,c)} = \bigcup_{m,n\in\mathbb{Z}} \mathcal{W} + mT_1 + nT_2,$$

where the union is a topologically disjoint union (i.e., up to edges and vertices of pointed faces).

Remark 7. In the case where a period pattern is finite, the period vectors T_1 and T_2 must be linearly independent in order to cover the entire plane. On the other hand, in Section 4.2, we consider infinite period patterns that tile a discrete plane using only one period vector. In such case, the above definition is satisfied with $T_1 = T_2$. In particular, if $\mathcal{P}_{(a,b,c)}$ is rational, then it admits a (finite) period pattern, and two period vectors T_1 and T_2 in $\mathbb{Z}^3$ with $\dim_{\mathbb{Q}}(T_1,T_2) = 2$. More generally, the $\mathbb{Q}$-vector space generated by the period vectors of a discrete plane has dimension $3 - \dim_{\mathbb{Q}}(a,b,c)$.

The aim of Theorem 30 and 38 below will be to exhibit period patterns for discrete planes.

One key property of generalized substitutions is that they preserve discrete planes, as proved in [3, 20]. Indeed, one has the following result, by identifying a sum of faces to its union.

Theorem 8. *[3, 20] Let σ be a unimodular substitution with incidence matrix M_σ. One has*

$$E_1^*(\sigma)\mathcal{P}_{(a,b,c)} = \mathcal{P}_{(a',b',c')} \text{ with } \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} = {}^tM_\sigma \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{5}$$

Moreover, for any face $(x, j^*) \in \mathcal{P}_{(a',b',c')}$, there exist a unique $(y, i^*) \in \mathcal{P}_{(a,b,c)}$ such that $(x, j^*) \in E_1^*(\sigma)((y, i^*))$.

Remark 9. We thus deduce that the image by $E_1^*(\sigma)$ of a pattern $\mathcal{W}$ included in some discrete plane $\mathcal{P}$ is a pattern $\mathcal{W}'$ included in some discrete plane $\mathcal{P}'$.

The action of generalized substitutions on period patterns and vectors can also be completely described:

Proposition 10. *Let σ be a unimodular substitution with incidence matrix M_σ. Let $\mathcal{W}$ be a finite period pattern of $\mathcal{P}_{(a,b,c)}$ with period vectors T_1, T_2. Then, $E_1^*(\sigma)(\mathcal{W})$ is a period pattern of the plane $\mathcal{P}_{{}^tM_\sigma(a,b,c)}$ with period vectors $M_\sigma^{-1}T_1$, $M_\sigma^{-1}T_2$.*

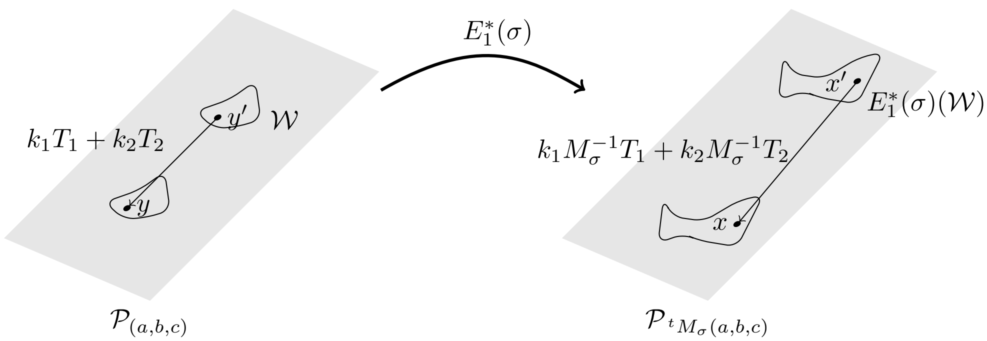


Figure 4: Illustration of the proof of Proposition 10. Given a face $(x, i^*) \in \mathcal{P}_{{}^tM_\sigma(a,b,c)}$ there exists a unique face $(x', i^*) \in E_1^*(\mathcal{W})$ such that $x = k_1M_\sigma{}^{-1}T_1 + k_2M_\sigma{}^{-1}T_2$.

Proof. Let $\mathcal{W}$ be a finite period pattern of $\mathcal{P}_{(a,b,c)}$ with period vectors T_1, T_2. Let $(x, j^*) \in \mathcal{P}_{{}^tM_\sigma(a,b,c)}$. Let us prove that there exist a unique face $(x', j^*) \in E_1^*(\sigma)(\mathcal{W})$ and $k_1, k_2 \in \mathbb{Z}$ such that $x = x' + k_1M_\sigma T_1 + k_2M_\sigma t_2$.

Existence: By Theorem 8 there exists a unique $(y, i^*) \in \mathcal{P}_{(a,b,c)}$ such that $(x, j^*) \in E_1^*(\sigma)((y, i^*))$. Since $\mathcal{W}$ is a period pattern of $\mathcal{P}_{(a,b,c)}$, there exist $k_1, k_2 \in \mathbb{Z}^3$ and a point $y' \in \mathbb{Z}^3$ such that $(y', i^*) \in \mathcal{W}$ and $y = y' + k_1T_1 + k_2T_2$ (see Figure 4).

By (2), there exist a prefix ui of $\sigma(j)$ such that

$$x = M_\sigma^{-1}(y - \overrightarrow{u}) = M_\sigma^{-1}(y' + k_1T_1 + k_2T_2 - \overrightarrow{u}).$$

Let $x' = x - k_1M_\sigma^{-1}T_1 - k_2M_\sigma^{-1}T_2$. From the above equality we deduce that $x' = M_\sigma^{-1}(y' - \overrightarrow{u})$ and so $(x', j^*) \in E_1^*(\sigma)((y', i^*)) \subset E_1^*(\sigma)(\mathcal{W})$ (E_1^* is one-to-one).

Unicity: Suppose there exist $(x', j^*), (x'', j^*) \in E_1^*(\sigma)(\mathcal{W})$ such that

$$x = x' + k_1'M_\sigma^{-1}T_1 + k_2'M_\sigma^{-1}T_2 = x'' + k_1''M_\sigma^{-1}T_1 + k_2''M_\sigma^{-1}T_2.$$

Let $(y, i^*) \in \mathcal{P}_{(a,b,c)}$ such that $(x, j^*) \in E_1^*(\sigma)((y, i^*))$. Let $y' = y - k_1'T_1 - k_2'T_2$, and $y'' = y - k_1''T_1 - k_2''T_2$. One has

$$E_1^*(\sigma)((y, i^*)) = k_1'M_\sigma^{-1}T_1 + k_2'M_\sigma^{-1}T_2 + E_1^*(\sigma)((y', i^*)),$$

with a similar equality also holding for y''. Hence $(x', j^*) \in E_1^*(\sigma)((y', i^*))$ and $(x'', j^*) \in E_1^*(\sigma)((y'', i^*))$. This implies that $(y', i^*), (y'', i^*)$ both belong to $\mathcal{W}$. Since $y' + k_1'T_1 + k_2'T_2 = y'' + k_1''T_1 + k_2''T_2$, we deduce that $y' = y$", and thus, $x' = x''$, $k_i' = k_i''$, for $i = 1, 2$.

□

2.4. Boundary words

We would like to understand the action of $E_1^*(\sigma)$ on the topology of particular patterns of discrete planes, for a family of substitutions, namely the Jacobi-Perron substitutions that will be introduced in Section 3. For that purpose we work with the topological boundary of the projection by Π_0 of these patterns, which is made of a finite union of segments.

Definition 11 (Polyamond pattern)**.** A finite pattern is said to be a *polyamond pattern* if the topological boundary of its projection by Π_0 is a Jordan curve, i.e., a simple closed curve.

Polyamonds aim at generalizing polyominoes: a finite pattern is a polyamond pattern if it is edge-connected and without holes. For an illustration, see Figure 5.

Definition 12 (Boundary)**.** Let $\mathcal{W}$ be a finite pattern. The segments whose projection belongs to the boundary of $\Pi_0(\mathcal{W})$ are called *boundary segments* of $\mathcal{W}$. Their union is called the *boundary* of $\mathcal{W}$

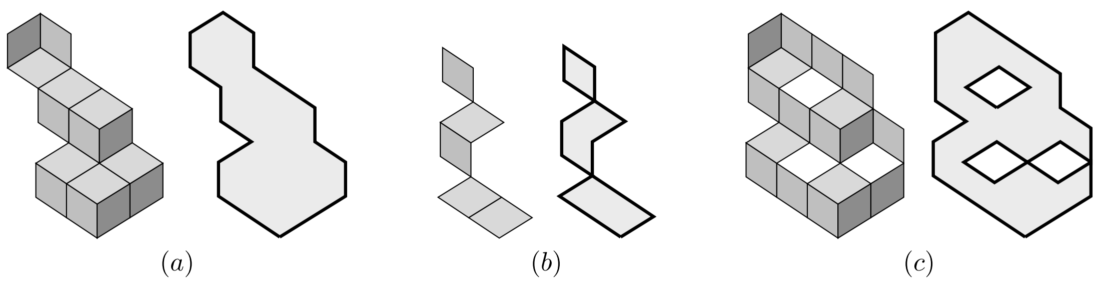


Figure 5: Three finite patterns and their projections by Π_0 showing that (a) is a polyamond while (b) and (c) are not.

The formalism of generalized substitutions also produces the boundary segments of images of finite patterns. Indeed, in [28], a mapping, denoted by $E_2^*(\sigma)$, which acts on segments and satisfies a commutation relation with the natural boundary operator (on faces) is introduced: if $\mathcal{W}$ is a finite pattern, $E_2^*(\sigma)(\mathcal{W})$ produces the boundary segments of $E_1^*(\sigma)(\mathcal{W})$. This mapping is called the *boundary map* of $E_1^*(\sigma)$. Contrary to the mapping $E_1^*(\sigma)$ which involves sums of faces, the boundary map produces sums and differences of segments, that is, segments with negative weights (which is a natural consequence of the commutation with the boundary operator). When working in dimension $d = 3$, there is a very convenient way to describe the mapping $E_2^*(\sigma)$ in terms of $E_1(\widetilde{\sigma^{-1}})$, when σ is an invertible substitution, that is, when it extends to an automorphism of the free group. Indeed, it is proved in [19] that $E_2^*(\sigma)$ and $E_1(\widetilde{\sigma^{-1}})$ are isomorphic mappings. For a precise statement, see Theorem 15 below. We thus can deduce the boundary of $E_1^*(\mathcal{W})$, for a finite pattern $\mathcal{W}$, from the boundary of $\mathcal{W}$ by application of a simple combinatorial rule, namely a free group automorphism, through the notion of boundary word that we introduce below. Note that the boundary morphism is closely related to the endomorphisms of the free group of rank 3 considered in [15, 16].

Let us recall that $\mathfrak{G}_3$ and $\overline{\mathfrak{F}}_3$ stand respectively for the vector space of mappings from $\mathbb{Z}^3 \times F_3$ to $\mathbb{R}$, and from $\mathbb{Z}^3 \times \overline{\mathcal{A}}_3$ to $\mathbb{R}$. In order to make the notion of boundary word more precise, we first define a notion of *reduction* for elements of $\mathfrak{G}_3$ and $\overline{\mathfrak{F}}_3$, obtained by replacing by 0 each two-term sum of the form $(P, i^{-1}) + (P - e_i, i)$, or $(P, \bar{i}) + (P - e_i, i)$. An element of $\mathfrak{G}_3$ and $\overline{\mathfrak{F}}_3$ is said to be *reduced* if no further reduction can be applied. A sum of segments admits a unique reduced form and is called *geometric* if all its coefficients are in $\{\pm 1, 0\}$. We thus define the two following geometric representation maps

$$\mathcal{G}_3 : \mathfrak{G}_3 \longrightarrow \mathbb{R}^3 \text{ and } \overline{\mathcal{G}}_3 : \overline{\mathfrak{F}}_3 \longrightarrow \mathbb{R}^3$$

that send geometric sums of segments onto the union of segments that occur in their reduced forms.

Example 13. As shows Figure 6, the geometric sum

$$(e_1,2)+(e_1+e_2,\overline{1})+(e_2,3)+(e_2+e_3,\overline{2})+(e_3,1)+(e_1+e_3,\overline{2})+(e_1+e_3-e_2,2)+(e_1+e_3,\overline{3}) = K_3(e_1,2\overline{1}3\overline{2}1\overline{2}2\overline{3})$$

admits as reduced sum

$$(e_1,2)+(e_1+e_2,\overline{1})+(e_2,3)+(e_2+e_3,\overline{2})+(e_3,1)+(e_1+e_3,\overline{3}) = K_3(e_1,2\overline{1}3\overline{2}1\overline{3}).$$

The union of segments

$$(e_1,2)\cup(e_1+e_2,\overline{1})\cup(e_2,3)\cup(e_2+e_3,\overline{2})\cup(e_3,1)\cup(e_1+e_3,\overline{3}) = \overline{\mathcal{G}}_3\circ K_3(e_1,2\overline{1}3\overline{2}1\overline{3}) = \overline{\mathcal{G}}_3\circ K_3(e_1,2\overline{1}3\overline{2}1\overline{2}2\overline{3})$$

is equal to the boundary of $\mathcal{U}=(0,1^*)+(0,2^*)+(0,3^*)$.

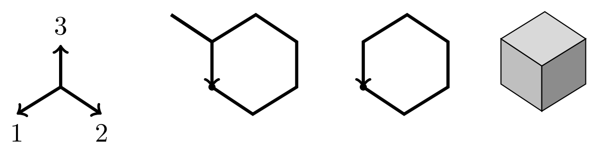


Figure 6: The unions of segments $K_3(e_1,2\overline{1}3\overline{2}1\overline{2}2\overline{3})$ (left) and $K_3(e_1,2\overline{1}3\overline{2}1\overline{3})$ (middle). Their image by $\overline{\mathcal{G}}_3$ is equal to the boundary of $\mathcal{U}$ (right).

With this dictionary at hand between words and sums/unions of segments, we can code in a natural way the boundary segments of a polyamond pattern in terms of words of $\overline{\mathcal{A}}_3^*$, in the spirit of the Freeman code: we code any finite connected union of segments of the form (x,a) for $a\in\overline{\mathcal{A}}_3$ as a finite word over $\overline{\mathcal{A}}_3$ by labeling them in a consecutive manner. For example, the word $2\overline{1}3\overline{2}1\overline{3}$ codes the boundary of $\mathcal{U}$. Furthermore, we also want to code the boundary of finite patterns that are not necessarily polyamonds. Note that the boundary word of a pattern whose boundary is a closed curve (which is not necessarily simple) is far from being defined in a canonical way. Indeed, there might be at least three reasons for an ambiguity to occur. First, the choice of the starting point for the description of the segments on the boundary is arbitrary. Second, in the case for instance where the boundary is not a simple closed curve (if it contains a loop), an ambiguity occurs after a crosspoint for the choice of the path to follow. Third, one might involve some non-reduced terms such as illustrated in Figure 6. This yields the following definition by stressing the fact that a finite pattern admits infinitely many boundary words. For illustrations, see Figure 7 and 8.

Definition 14 (Boundary word)**.** A word w over $\overline{\mathcal{A}}_3$ is said to be a *boundary word* for the finite pattern $\mathcal{W}$ if there exists $P\in\mathbb{Z}^3$ such that $\overline{\mathcal{G}}_3\circ K_3(P,w)$ is equal to the boundary of $\mathcal{W}$.

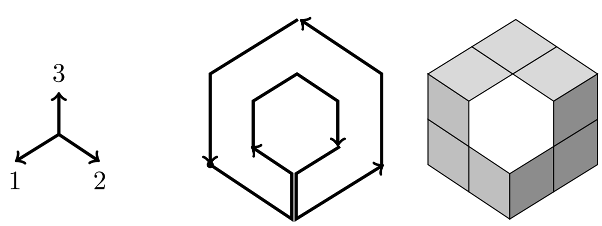


Figure 7: The word $223\overline{2}\ 3\overline{1}2\overline{3}\ 1\overline{3}\overline{1}\overline{1}\ 33\overline{2}\overline{2}\ 11\overline{3}\overline{3}$ is a boundary word for $\mathcal{W}$ (right).

We now can recall the theorem from [19] giving an expression of the boundary map of $E_1^*(\sigma)$ in terms of $\widetilde{\sigma^{-1}}$.

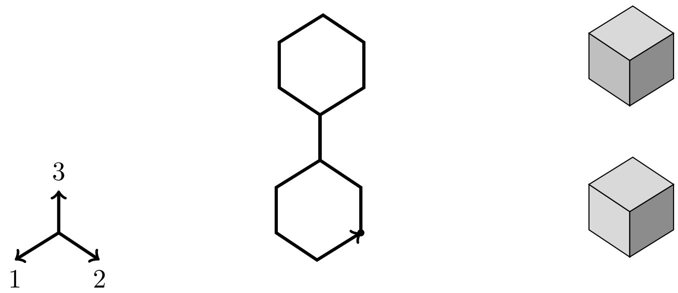


Figure 8: The word $3\bar{2}3\bar{1}3\bar{2}1\bar{3}2\bar{3}1\bar{3}2\bar{1}$ is boundary word for $\mathcal{U} \cup (-e_1 - e_2 + e_3 + \mathcal{U})$.

Theorem 15. *[19] Let σ be an invertible substitution over the alphabet $\mathcal{A}_3$. Let $\tau = \widetilde{\sigma^{-1}}$ considered as an automorphism of the free group F_3. Let $\mathcal{W}$ be a finite pattern and let w be a boundary word of $\mathcal{W}$ over F_3. The boundary of the pattern $E_1^*(\sigma)(\mathcal{W})$ is equal to*

$$\mathcal{G}_3(E_1(\tau)(w)).$$

Remark 16. This theorem can also be stated over $\overline{\mathcal{A}}_3$, which will be the form we will work with here. Let σ be an invertible substitution over the alphabet $\mathcal{A}_3$. Let $\tau = \widetilde{\sigma^{-1}}$ considered as a morphism of the alphabet $\overline{\mathcal{A}}_3$. Let $w \in \overline{\mathcal{A}}_3^*$ be a boundary word of the finite pattern $\mathcal{W}$. Then, the boundary of the pattern $E_1^*(\sigma)(\mathcal{W})$ is also equal to $\overline{\mathcal{G}}_3(E_1(\tau)(w))$, and the word $\tau(w)$ is a boundary word of $E_1^*(\sigma)(\mathcal{W})$.

Example 17. We continue Example 2, with the Tribonacci substitution defined by $\sigma(1) = 12$, $\sigma(2) = 13$, $\sigma(3) = 1$ on $\mathcal{A}_3 = \{1, 2, 3\}$. By seeing σ as an automorphism of the free group, we have:

$$\widetilde{\sigma^{-1}}(1) = 3, \quad \widetilde{\sigma^{-1}}(2) = 13^{-1}, \quad \widetilde{\sigma^{-1}}(3) = 23^{-1}.$$

Consequently we introduce τ the substitution over $\overline{\mathcal{A}}_3^*$ defined by:

$$\tau\colon 1 \mapsto 3,\ 2 \mapsto 1\bar{3},\ 3 \mapsto 2\bar{3},\ \bar{1} \mapsto \bar{3},\ \bar{2} \mapsto 3\bar{1},\ \bar{3} \mapsto 3\bar{2}.$$

Iterations of τ are depicted in the figure of Example 2.

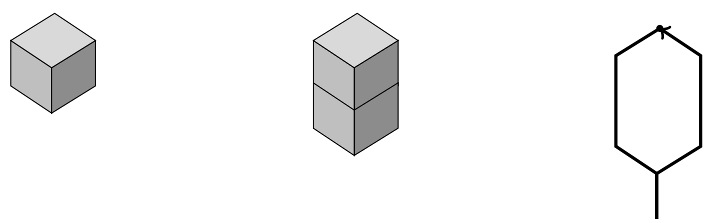


Figure 9: The pattern $\mathcal{U}$ (left) and its image $E_1^*(\sigma)(\mathcal{U})$ (middle). The word $1\bar{3}\bar{3}2\bar{3}3\bar{1}33\bar{2} = \tau(2\bar{1}3\bar{2}1\bar{3})$ is a boundary word for the pattern $E_1^*(\sigma)(\mathcal{U})$ (right).

We have seen that the word $2\bar{1}3\bar{2}1\bar{3}$ is a boundary word for the pattern $\mathcal{U}$, and we check that $1\bar{3}\bar{3}2\bar{3}3\bar{1}33\bar{2} = \tau(2\bar{1}3\bar{2}1\bar{3})$ is a boundary word for the pattern $E_1^*(\sigma)(\mathcal{U})$ (see Figure 9 for an illustration).

We now want to state a sufficient condition (see Proposition 21 below) in terms of boundary words for a pattern of the form $E_1^*(\sigma)(\mathcal{W})$ to be a polyamond. Recall that π_{F_3} is the canonical projection morphism from $\overline{\mathcal{A}_3}^*$ to the free group F_3 defined as

$$\pi_{F_3}(i) := i \text{ and } \pi_{F_3}(\bar{i}) := i^{-1},\ \forall i \in \{1, 2, 3\}.$$

Definition 18 (Closed path, eight-curve, simple closed path)**.** A word $w \in \overline{\mathcal{A}}_3^+$ is said to be a *closed path* (or a *loop*) if its abelianized $\overrightarrow{w}$ satisfies $\overrightarrow{w} = 0$ and $\pi_{F_3}(w) \neq \varepsilon_{F_3}$. Moreover a word $w \in \overline{\mathcal{A}}_3^+$ is said to be an *eight-curve* if it is a closed path and if it admits a cyclic factor u (i.e., a factor of some conjugate of w) such that both u and its complement in w are closed paths. A closed path w that is not an eight-curve is said to be a *simple closed path.*

These definitions are illustrated in Figure 10. We will use the following proposition in order to study the topological action of $E_1^*(\sigma)$ (for the Jacobi-Perron substitutions introduced in Section 3.2) on some polyamond patterns.

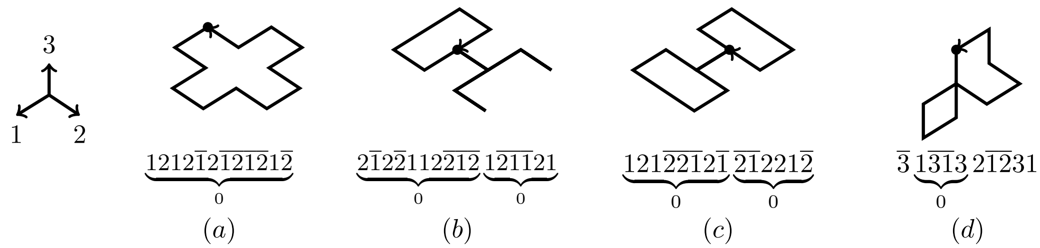


Figure 10: Examples of closed paths. Path (a) is a simple closed path. Path (b) is also a simple closed path since $\pi_{F_d}(2\bar{1}2\bar{2}112\bar{2}\bar{1}\bar{2}) = \varepsilon_{F_d}$. Paths (c) and (d) are eight-curves.

Lemma 19. *Let $\mathcal{W}$ be a finite pattern, and let $w \in \overline{\mathcal{A}}_3^*$ be the boundary word of $\mathcal{W}$. If w is a simple closed path, then $\mathcal{W}$ is a polyamond pattern.*

Proof. We proceed by contradiction. Suppose that the pattern $\mathcal{W}$ is not a polyamond pattern. Then its topological boundary $\partial\mathcal{W}$ is not a Jordan curve. Since $\mathcal{W}$ admits w as a boundary word, it must be finite and so, there are two cases to consider: either $\partial\mathcal{W}$ is disconnected, or there exists a continuous map of a circle to $\partial\mathcal{W}$ which is not injective.

In the first case, the boundary word w must be of the same form than in Figure 10(c), and so, it is an eight-curve. In the latter case, the boundary word is of the same form than in Figure 10(d) and, again, it is an eight-curve. □

Remark 20. The reciprocal of Lemma 19 is false. As shows Figure 11, an eight-curve may be a valid boundary word for a polyamond pattern.

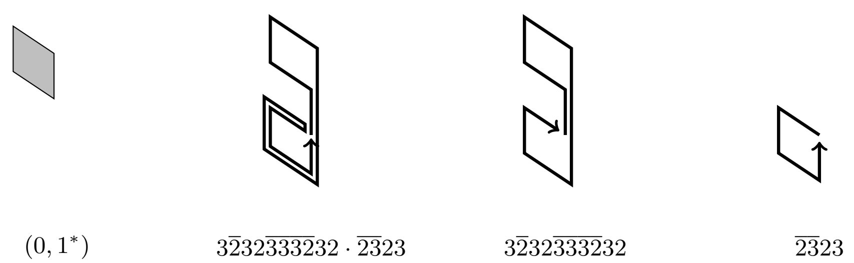


Figure 11: The word $w = 3\bar{2}32\bar{3}\bar{3}\bar{3}\bar{2}32 \cdot \bar{2}\bar{3}23$ is a boundary word for the polyamond pattern $(0, 1^*)$. Indeed w is a conjugate of $w' = \bar{3}\bar{3}\bar{2}32\bar{2}\bar{3}233 \cdot \bar{2}32\bar{3}$ and $\pi_{F_d}(w') = 2^{-1}323^{-1}$. But w is also an eight-curve since $w = uv$ with $u = 3\bar{2}32\bar{3}\bar{3}\bar{3}\bar{2}32$, $v = \bar{2}\bar{3}23$ and both word u, v are simple closed paths.

Proposition 21. *Let $\mathcal{W}$ be a polyamond pattern included in some discrete plane, and let w be a boundary word for $\mathcal{W}$. Let σ be a substitution over $\mathcal{A}_3$. Let $\tau = \widetilde{\sigma^{-1}}$ considered as a substitution over $\overline{\mathcal{A}_3}$. If $\tau(w)$ is a simple closed path, then $E_1^*(\sigma)(\mathcal{W})$ is a polyamond pattern, and $\tau(w)$ is a boundary word for $E_1^*(\sigma)(\mathcal{W})$.*

Proof. By Remark 9, $E_1^*(\sigma)(\mathcal{W})$ is also included in a discrete plane. We conclude thanks to Remark 16 and Lemma 19. $\square$

3. Jacobi-Perron substitutions

We now introduce the Jacobi-Perron algorithm. Let us note that in contrast with the Sturmian/discrete line/continued fraction case, the choice of the Jacobi-Perron algorithm is arbitrary, in the sense that there exist quite close algorithms, such as Brun algorithm [14], that are also used in the framework of discrete geometry, and in particular for the recognition of discrete planes (see for Brun algorithm [8, 21]). We consider here Jacobi-Perron algorithm mainly because of the fact that the patterns that are generated thanks to this algorithm cover any given discrete plane in totality, and not only a part of it (see Section 4 for more precise statements).

3.1. The Jacobi-Perron algorithm

The Jacobi-Perron algorithm can be described as follows. Starting from a triple (a, b, c) of real numbers satisfying $0 \leq a, b \leq c$, the algorithm consists in subtracting a from b and c as much as possible, and to cyclically permute the resulting coordinates by sending a at the end. It follows that a and b must be smaller than c, but a and b are not necessarily ordered. For more details, see e.g. [29, 13, 2].

The *projective* Jacobi-Perron algorithm is defined on $(0, 1] \times [0, 1]$ by the following transformation:

$$(\alpha, \beta) \mapsto \left(\frac{\beta}{\alpha} - \lfloor \frac{\beta}{\alpha} \rfloor, \frac{1}{\alpha} - \lfloor \frac{1}{\alpha} \rfloor \right).$$

The *linear* Jacobi-Perron algorithm is defined on the cone $\{(a, b, c) \in \mathbb{R}^3 | 0 \leq a, b \leq c,\ a \neq 0\}$ by the following transformation:

$$(a, b, c) \mapsto (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a).$$

We deduce the projective version of the Jacobi-Perron algorithm from its linear version by setting $\alpha := a/c$ and $\beta := b/c$. We will work in the present paper with the linear version of Jacobi-Perron agorithm.

In order to handle parameters for which $a = 0$, we introduce here an extended version of Jacobi-Perron algorithm: if $a = 0$, $b \neq 0$ and $c \neq 0$, then we apply Euclid's algorithm. We thus define on the cone $\{(a, b, c) \in \mathbb{R}^3 | 0 \leq a, b \leq c\}$ the transformation F:

$$F(a, b, c) = \begin{cases} (b - \lfloor b/a \rfloor a, c - \lfloor c/a \rfloor a, a) & \text{if } a \neq 0, \\ (0, c - \lfloor c/b \rfloor b, b) & \text{if } b \neq 0,\ a = 0, \\ (0, 0, c) & \text{if } a = b = 0. \end{cases}$$

Let $(a, b, c) \in \mathbb{R}^3$ with $0 \leq a, b \leq c$ and set

$$(a_n, b_n, c_n) := F^n(a, b, c) \text{ for all } n \in \mathbb{N}.$$

Note that if we start with coprime integer parameters a, b, c, then the extended Jacobi-Perron algorithm terminates with $(0, 0, 1)$. Indeed, one checks that the sequence $(c_n)_n$ is non-increasing. If there exists n such that $c_n = c_{n+1}$, then this implies either that $a_n = b_n = 0$, which yields directly the desired conclusion, or else that $a_n = c_n$, $b_{n+1} = 0$, and $a_{n+2} = 0, b_{n+2} = 0$. Furthermore, a_{n+1} and c_{n+1} are coprime, as well as b_{n+2} and c_{n+2}. We then deduce the conclusion from the fact that we then apply Euclid's algorithm on b_{n+2} and c_{n+2}. Furthermore, one checks that the extended Jacobi-Perron algorithm detects rational dependencies (see e.g. [17, 18], and note that this behaviour is specific to the dimension $d = 3$):

Theorem 22. *[17, 18] The extended Jacobi-Perron algorithm leads to a coefficient $a_n = 0$ for some $n \geq 1$ if and only if $\dim_{\mathbb{Q}}(a, b, c) < 3$. The Jacobi-Perron algorithm leads to coefficients $a_n = 0$ and $b_n = 0$ for some $n \geq 1$ if and only if $\dim_{\mathbb{Q}}(a, b, c) = 1$.*

The iteration of the Jacobi-Perron algorithm yields a sequence of digits $(B_n, C_n, E_n)_{n\geq 1}$, called a *sequence of Jacobi-Perron digits*, and defined for all n as follows

$$\begin{aligned}
&B_n := \lfloor b_n/a_n \rfloor,\ C_n := \lfloor c_n/a_n \rfloor,\ E_n := 0 \text{ if } a_n \neq 0,\\
&B_n := 0,\ C_n := 0,\quad E_n := \lfloor c_n/b_n \rfloor \text{ if } a_n = 0,\ b_n \neq 0,\\
&B_n := 0,\ C_n := 0,\quad E_n := 0 \text{ otherwise.}
\end{aligned}$$

We have furthermore the following admissibility conditions holding for every $n \in \mathbb{N}$ for which $a_n \neq 0$ (see e.g. [29]):

$$0 \leq B_n \leq C_n,\ C_n \geq 1;\ \text{furthermore if } B_n = C_n \text{ and } a_{n+1} \neq 0, \text{ then } B_{n+1} \neq 0, \tag{6}$$

and for every $n \in \mathbb{N}$ for which $a_n = 0$:

$$E_n = 0 \text{ implies } E_{n+1} = 0. \tag{7}$$

Let us recall here a short proof for (6). Assume that $B_n = C_n$ and $a_{n+1} \neq 0$. One has $a_{n+1} = b_n - B_n a_n \leq b_{n+1} = c_n - C_n a_n$, since $b_n \leq c_n$ and $B_n = C_n$. Hence $B_{n+1} = \lfloor b_{n+1}/a_{n+1} \rfloor \geq 1$.

Conversely, evey sequence of digits satisfying (6) and (7) corresponds to the expansion of a unique triple (a, b, c) with $0 \leq a, b \leq c$.

In matricial terms, one has

$$\begin{aligned}
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} &= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & B_n \\ 0 & 1 & C_n \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} \text{ if } a_n \neq 0,\\
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & E_n \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} \text{ if } a_n = 0,\ b_n \neq 0,\\
\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} \text{ if } a_n = b_n = 0.
\end{aligned}$$

For $B, C, E \in \mathbb{N}$, we introduce the transpose of the matrices describing Jacobi–Perron algorithm

$$M_{B,C} := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & B & C \end{pmatrix},\ M_E := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & E \end{pmatrix} \text{ if } E \neq 0,\ \text{ and } M_0 := \mathrm{Id} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The reason why we introduce transpose matrices comes from Theorem 8. These matrices which describe the action of Jacobi-Perron algorithm are unimodular matrices with entries in $\mathbb{N}$. The Jacobi-Perron algorithm is thus said to be a unimodular nonnegative continued fraction algorithm (for more details, see [12]). The iteration of the Jacobi-Perron algorithm yields a sequence of (unimodular) matrices $(M_{<n>})_{n\geq 1}$, called a *sequence of Jacobi-Perron matrices*, and defined for all n as follows

$$\text{if } a_n \neq 0,\ M_{<n>} := {}^t M_{B_n, C_n},\quad \text{if } a_n = 0,\ M_{<n>} := {}^t M_{E_n} = M_{E_n}.$$

Hence for every $n \geq 1$:

$$\begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = M_{<n>} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = M_{<1>} \cdots M_{<n>} \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix}.$$

3.2. Jacobi-Perron substitutions

We now interpret the nonnegative unimodular matrices $M_{B,C}$ as incidence matrices of well-chosen substitutions. Let us stress the fact that this choice is highly non-canonical.

Proper Jacobi-Perron substitutions. For $B, C \in \mathbb{N}$, we denote by $\sigma_{B,C}$ the substitution over the three-letter alphabet $\{1,2,3\}$ defined by:

$$\sigma_{B,C} : \ 1 \mapsto 3, \ 2 \mapsto 13^B, \ 3 \mapsto 23^C.$$

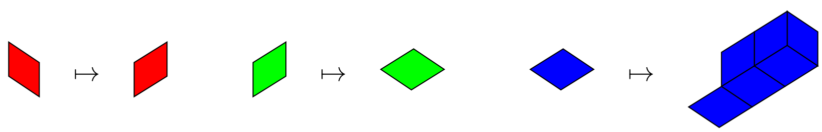
The incidence matrix of the substitution $\sigma_{B,C}$ is $M_{B,C}$. Substitutions $\sigma_{B,C}$ have been discussed in particular in [24, 2]. Let us compute $E_1^*(\sigma_{B,C})$. The inverse of the matrix $M_{B,C}$ is given by:

$$M_{B,C}^{-1} = \begin{pmatrix} -B & -C & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

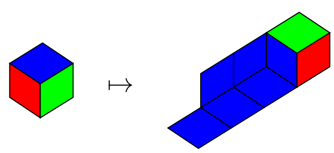
One gets

$$\begin{array}{rcl} E_1^*(\sigma_{B,C})((x,y,z),1^*) & = & ((-Bx - Cy + z, x, y), 2^*) \\ E_1^*(\sigma_{B,C})((x,y,z),2^*) & = & ((-Bx - Cy + z, x, y), 3^*) \\ E_1^*(\sigma_{B,C})((x,y,z),3^*) & = & ((-Bx - Cy + z, x, y), 1^*) + \Sigma_{i=1}^{B}((-Bx - Cy + z + i, x-1, y), 2^*) + \\ & & \Sigma_{i=1}^{C}((-Bx - Cy + z + i, x, y-1), 3^*). \end{array}$$

This is illustrated as follows for $B = 2$ and $C = 3$ and $(x,y,z) = 0$:



and



The substitution $\sigma_{B,C}$ has been chosen such as an automorphism of the free group. Its inverse equals

$$\sigma_{B,C}^{-1} : \ 1 \mapsto 21^{-B}, \ 2 \mapsto 31^{-C}, \ 3 \mapsto 1.$$

Let $\tau_{B,C}$ be the substitution of the free monoid $\overline{A}_3^*$ defined for $B, C \in \mathbb{N}$ by

$$\tau_{B,C} : 1 \to \overline{1}^B 2, \ 2 \to \overline{1}^C 3, \ 3 \to 1, \ \overline{1} \to \overline{2} 1^B, \ \overline{2} \to \overline{3} 1^C, \ \overline{3} \to \overline{1}$$

One checks that the abelianized matrix of $\tau_{B,C}$ is equal to $M_{B,C}^{-1}$ and that $\pi_{F_3} \circ \tau_{B,C} = \widetilde{\sigma_{B,C}}^{-1}$.

Euclid's substitutions. For E a positive integer, we denote by γ_E the substitution over the alphabet

$$\gamma_E : \ 1 \mapsto 1, \ 2 \mapsto 3, \ 3 \mapsto 23^E,$$

and we set $\gamma_0 := \mathrm{Id} : \ 1 \mapsto 1, \ 2 \mapsto 2, \ 3 \mapsto 3$.

If $E \neq 0$, the incidence matrix of γ_E is equal to

$$M_E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & E \end{pmatrix}.$$

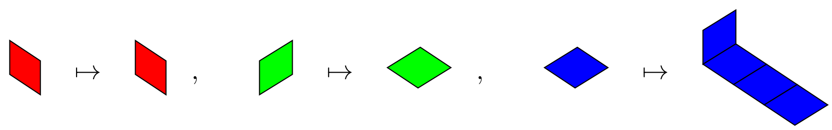
Let us compute $E_1^*(\gamma_E)$ for $E \neq 0$. The inverse of the matrix M_E is given by:

$$M_E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -E & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

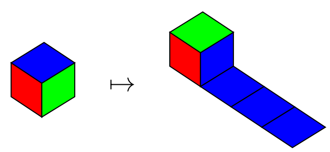
One gets

$$\begin{array}{rcl} E_1^*(\gamma_E)((x,y,z),1^*) & = & ((x,-Ey+z,y),1^*) \\ E_1^*(\gamma_E)((x,y,z),2^*) & = & ((x,-Ey+z,y),3^*) \\ E_1^*(\gamma_E)((x,y,z),3^*) & = & ((x,-Ey+z,y),2^*) + \displaystyle\sum_{i=0}^{E-1}((x,-Ey+z+E-i,y-1),3^*). \end{array}$$

This is illustrated as follows for $E = 3$ and $(x,y,z) = 0$:



and



Let $(B_n, C_n, E_n)_{n\geq 1}$ be the sequence of Jacobi-Perron digits produced by the Jacobi-Perron algorithm applied to the triple (a,b,c). For all $n \in \mathbb{N}$, we set

$$\sigma_{<n>} := \sigma_{B_n,C_n} \text{ if } a_n \neq 0, \text{ and } \sigma_{<n>} := \gamma_{E_n} \text{ otherwise,}$$

and

$$\mathbf{M}_n = M_{<1>}M_{<2>}\cdots M_{<n>}.$$

Consequently with this notation, we write $\tau_{<n>}$ for the corresponding substitution of the free monoid $\overline{\mathcal{A}}^*$, i.e.,

$$\pi_{F_3} \circ \tau_{<n>} = \widetilde{\sigma_{<n>}}^{-1}.$$

Definition 23 (Jacobi-Perron substitutions and geometric substitutions)**.** The sequence $(\tau_{<n>})_{n\geq 1}$ is called the sequence of *Jacobi-Perron substitutions* generated by (a,b,c), and the sequence $(E_1^*(\sigma_{<n>}))_{n\geq 1}$ is called the sequence of *geometric Jacobi-Perron substitutions* generated by (a,b,c). Substitutions $\tau_{<n>}$ for which $C_n \neq 0$ are called proper Jacobi-Perron substitutions.

We will make a strong use the following remarks.

Remark 24. Let $u, v \in \overline{A}_3^*$. If for some $B, C \geq 0$, $\tau_{B,C}(v) = u$ then

$$M_{B,C}\overrightarrow{v} = \overrightarrow{u} \text{ and } \overrightarrow{v} = M_{B,C}^{-1}\overrightarrow{u}.$$

Remark 25. Let $\mathcal{W}$ be a finite pattern of a discrete plane with normal vector (a,b,c) that satisfies $0 \leq a, b \leq c$. Then, for any n, $E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{W})$ is also a pattern of a discrete plane with normal vector (a', b', c') with a', b', c' satisfying $0 \leq a', b' \leq c'$. This comes from Theorem 8.

4. Generating patterns for discrete planes

The aim of this section is to exhibit patterns generating the whole discrete plane $\mathcal{P}_{(a,b,c)}$ when applying the geometric Jacobi-Perron algorithm to the triple (a, b, c) and using the associated sequence of geometric Jacobi-Perron substitutions.

Let (a, b, c) with $0 \le a, b \le c$ and let $(a_n, b_n, c_n) = F^n(a, b, c)$ for all n. We deduce from Theorem 8 that

$$E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{P}_{(a_n,b_n,c_n)}) = \mathcal{P}_{(a,b,c)},$$

by using the notation of Definition 23. We distinguish two cases according to $\dim_{\mathbb{Q}}(a, b, c)$, assumed to be equal to 3 in Section 4.1, and to 1 or 2 in Section 4.2. These two cases are illustrated in Figure 12.

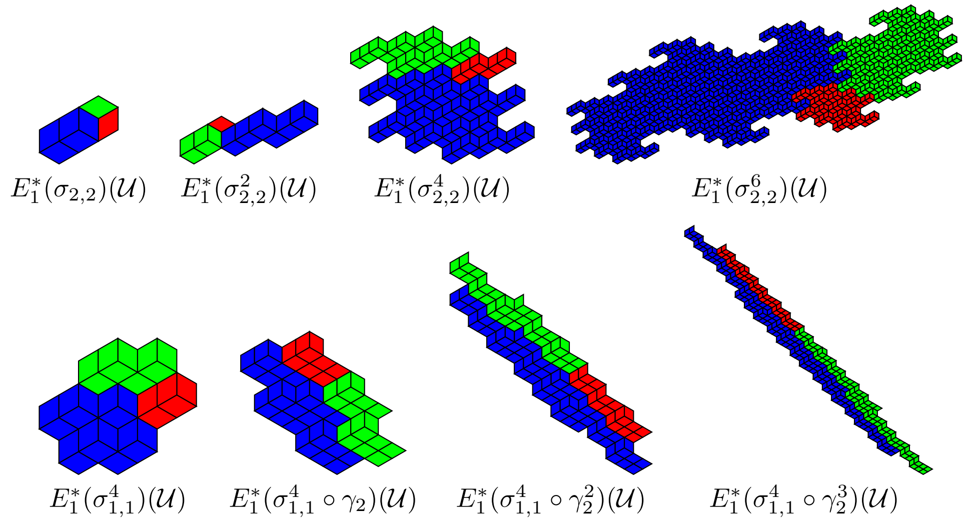


Figure 12: Example of patterns produced during the plane generation process. The image of face $(0, 1^*)$ is shown in red, while the image of $(0, 2^*)$ image is in green, and the image of $(0, 3^*)$ is in blue.

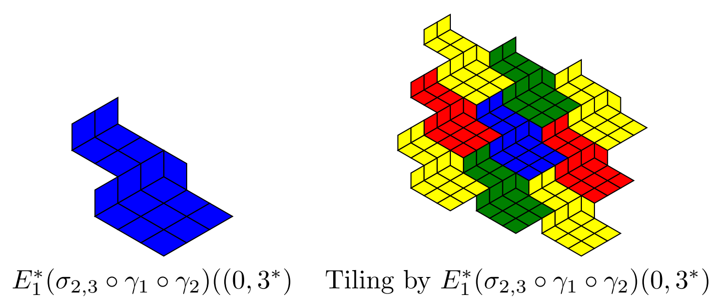


Figure 13: Example of a tiling by a pattern obtained as the image of the face $((0, 0, 0), 3^*)$ in the rational case.

4.1. Totally irrational case

We first assume $\dim_{\mathbb{Q}}(a, b, c) = 3$. Note that $\sigma_{<n>} = \sigma_{B_n,C_n}$ for all n (according to Theorem 22). In particular, $a, b, c > 0$. We thus can use the fact that $\mathcal{U} = (0, 1^*) + (0, 2^*) + (0, 3^*)$ is included in $\mathcal{P}_{(a,b,c)}$ (see (4)), and deduce from Theorem 8 that

$$E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{U}) \subset \mathcal{P}_{(a,b,c)}.$$

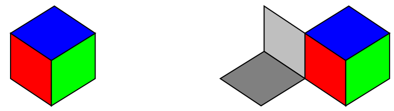


Figure 14: The generating patterns $\mathcal{U}$ (left) and $\mathcal{V}$ (right).

The question is whether the patterns $E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{U})$ generate the whole plane $\mathcal{P}_{(a,b,c)}$, that is, whether

$$\lim_{n\to\infty} E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{U}) = \lim_{n\to\infty} E_1^*(\sigma_{(B_1,C_1)}) \circ \cdots \circ E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U}) = \mathcal{P}_{(a,b,c)}.$$

The notion of limit refers here to the metric inspired by the rigid version of the so-called local metric often used for tilings (see e.g. [27]): the projections by Π_0 of $E_1^*(\sigma_{<1>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{U})$ and of $\mathcal{P}_{(a,b,c)}$ coincide on balls with increasing radius with n.

Note that this question is a substitutive counterpart of the Finiteness Property in beta-numeration, which states that every $x \in \mathbb{Z}[1/\beta] \cap [0, 1)$ has a finite β-expansion. For more details, see [10].

This question has been answered in the Jacobi-Perron case in [24, 7]: one can generate any plane $\mathcal{P}_{(a,b,c)}$ with the assumption $\dim_{\mathbb{Q}}(a, b, c) = 3$ by starting either with the pattern $\mathcal{U}$ or with the pattern $\mathcal{V} = \mathcal{U} \cup (-e_2, 1^*) \cup (e_1 - e_2 - e_3, 3^*)$ (see Figure 14). (Note that the original aim of [24] was to prove the uniform recurrence of discrete planes, see also [11] for an arithmetic proof of this property.) More precisely, given a sequence of Jacobi-Perron digits $(B_n, C_n, E_n)_{n\geq 1}$, consider the following condition:

$$\exists n \geq 1 \text{ such that } \forall k,\ B_{n+3k} = C_{n+3k},\ C_{n+3k+1} - B_{n+3k+1} \geq 1,\ B_{n+3k+2} = 0. \quad (8)$$

Theorem 26. *[24, 7] Let (a, b, c) with $0 \leq a, b \leq c$ and $\dim_{\mathbb{Q}}(a, b, c) = 3$. Let $(B_n, C_n)_{n\geq 1}$ be the sequence of Jacobi-Perron digits produced by the triple (a, b, c). If (8) holds, then*

$$\lim_{n\to\infty} E_1^*(\sigma_{(B_1,C_1)}) \circ \cdots \circ E_1^*(\sigma_{(B_n,C_n)})(\mathcal{V}) = \mathcal{P}_{(a,b,c)},$$

otherwise, if (8) does not hold, then

$$\lim_{n\to\infty} E_1^*(\sigma_{(B_1,C_1)}) \circ \cdots \circ E_1^*(\sigma_{(B_n,C_n)})(\mathcal{U}) = \mathcal{P}_{(a,b,c)}.$$

An illustration of the limit when starting with $\mathcal{U}$ without the assumption (8) is depicted in the top part of Figure 12, and under the assumptions (8), in the left part of Figure 15 (this corresponds to the sequence $(B_n, C_n)_{n\geq 1}$ which is purely periodic with period $[(1, 1), (1, 2), (0, 1)]$). The limit is strictly included in the plane $\mathcal{P}_{(a,b,c)}$.

4.2. Rational and non-totally irrational case

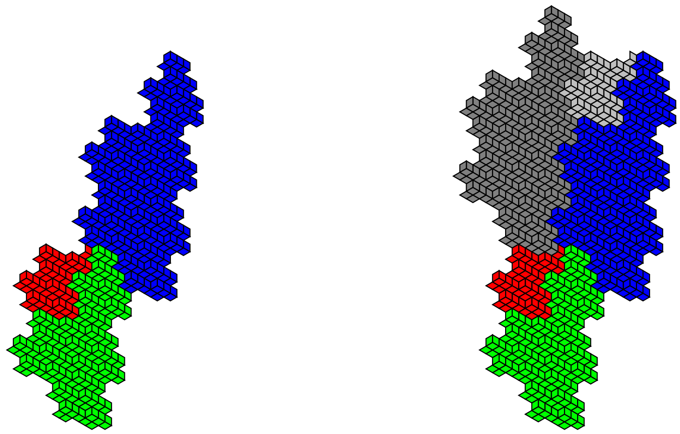
We now assume $\dim_{\mathbb{Q}}(a, b, c) = 1$ or 2. According to Theorem 22, the sequence $(B_n, C_n, E_n)_{n\geq 1}$ of Jacobi-Perron digits of (a, b, c) satisfies $E_n \neq 0$ for all n large enough if and only if $\dim_{\mathbb{Q}}(a, b, c) = 2$.

We want now to exhibit period patterns. We first prove a basic statement concerning the patterns generated by the substitutions $E_1^*(\gamma_n)$.

Lemma 27. *Let $(a, b, c) \in \mathbb{R}_+^3$ with $1 \leq \dim(a, b, c) \leq 2$, and let $(B_n, C_n, E_n)_{n\geq 1}$ be its sequence of Jacobi-Perron digits. Let $m, n \in \mathbb{N}$. The union of faces*

$$E_1^*(\gamma_{E_m}) \circ \cdots \circ E_1^*(\gamma_{E_{m+n}})(0, 3^*)$$

is a polyamond pattern of a discrete plane.



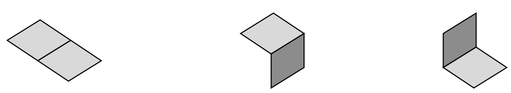
$E_1^*\left((\sigma_{1,1}\circ\sigma_{1,2}\circ\sigma_{0,1})^3\right)(\mathcal{U})$ $\qquad$ $E_1^*\left((\sigma_{1,1}\circ\sigma_{1,2}\circ\sigma_{0,1})^3\right)(\mathcal{V})$

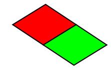
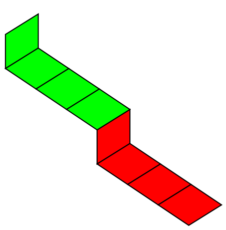
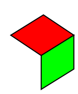
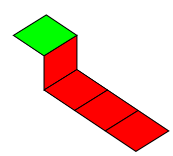
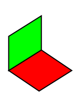
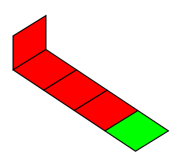
Figure 15: Patterns generated by the purely periodic sequence $(B_n, C_n)_{n\geq 1}$ with period $[(1,1),(1,2),(0,1)]$. The color of each pattern is given by the color of its preimage in Figure 14. On the left, the three patterns meet at $(0,0,0)$, indicating that the north west part of the discrete plane is not generated when starting from $\mathcal{U}$.

Proof. Let $m, n \in \mathbb{N}$. We can assume $E_{m+n} \neq 0$ without loss of generality. We first note that $E_1^*(\gamma_{E_n}) \circ \cdots \circ E_1^*(\gamma_{E_{n+m}})(0, 3^*)$ is covered by patterns of the following form, for $P \in \mathbb{Z}^3$:

$$(P,3)^* \cup (P+e_2, 3^*),\ (P,2)^* \cup (P,3^*),\ (P,2)^* \cup (P+e_2-e_3, 3^*),$$

i.e.,



Indeed, the image of each of these three types of patterns is itself covered by them. We illustrate it with $E = 3$. Consider the image of $(P,3)^* \cup (P+e_2, 3^*) =$  by $E_1^*(\gamma_E)$. It is of the form  . The image of $(P,3)^* \cup (P, 2^*) =$  is of the form  . The image of $(P, 2^*) \cup (P+e_2-e_3, 3^*) =$  by $E_1^*(\gamma_E)$ is of the form  .

Furthermore, since $E_{m+n} \neq 0$, $c_{n+m} \neq 0$. We deduce from (3) that $(0, 3^*)$ belongs to $\mathcal{P}_{(a_{n+m}, b_{n+m}, c_{n+m})}$. Hence, by Theorem 8, $E_1^*(\gamma_{E_n}) \circ \cdots \circ E_1^*(\gamma_{E_{n+m}})(0, 3^*)$ is a subset of a discrete plane, namely $\mathcal{P}_{(a_n, b_n, c_n)}$. This concludes the proof. $\square$

We can now state an analogue of Theorem 26 in the case of $\dim(a,b,c) = 1, 2$. For an ilustration, see Figure 12 and 13. We recall that for all $n \geq 1$,

$$\mathbf{M}_n = M_{<1>} M_{<2>} \cdots M_{<n>}.$$

Proposition 28. *Let* $(a,b,c) \in \mathbb{R}^3_+$ *and let* $(B_n, C_n, E_n)_{n\geq 1}$ *be its sequence of Jacobi-Perron digits. Let* n_0 *be the smallest positive integer for which* $a_{n_0} = 0$.

Assume $\dim(a,b,c) = 2$. *The infinite union of faces*

$$\mathcal{T} = \lim_{n\to+\infty} E_1^*(\sigma_{<1>}) \circ E_1^*(\sigma_{<2>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(0,3^*)$$

is a period pattern for $\mathcal{P}_{(a,b,c)}$ *with period vector* $\overrightarrow{v_0} = {}^t\mathbf{M}_{n_0}^{-1}(1,0,0)$, *i.e.,* $\mathcal{P}_{(a,b,c)} = \sum_{k\in\mathbb{Z}}(k\overrightarrow{v_0} + \mathcal{T})$.

Assume $\dim(a,b,c) = 1$. *Let* $m_0 \geq n_0$ *be the smallest integer for which* $c_{m_0} = 0$. *The pattern*

$$\mathcal{W} = E_1^*(\sigma_{<1>}) \circ E_1^*(\sigma_{<2>}) \circ \cdots \circ E_1^*(\sigma_{<m_0>})(0,3^*)$$

is a period pattern for $P_{(a,b,c)}$ *with period vectors* ${}^t\mathbf{M}_{m_0}^{-1}(1,0,0)$ *and* ${}^t\mathbf{M}_{m_0}^{-1}(0,1,0)$.

Proof. We first assume $\dim(a,b,c) = 2$, i.e., the sequence $(E_i)_{i\geq n_0}$ only takes positive values for $n \geq n_0$. For all $n \geq n_0$, one has

$$E_1^*(\gamma_{E_{n_0}}) \circ \cdots \circ E_1^*(\gamma_{E_n})(0,3^*) \subset \mathcal{P}_{(0,b_{n_0},c_{n_0})},$$

by Theorem 8, since $c_n \neq 0$ for all n. Furthermore, by Lemma 27, $E_1^*(\gamma_{E_{n_0}}) \circ \cdots \circ E_1^*(\gamma_{E_n})(0,3^*)$ is a polyamond pattern. Furthermore, the orthogonal projection of $\lim_{n\to+\infty} E_1^*(\gamma_{E_{n_0}}) \circ \cdots \circ E_1^*(\gamma_{E_n})(0,3^*)$ onto the line $x=0$, $bx+cz=0$ covers this line (it is onto). We then conclude thanks to Proposition 10 by noticing that $\begin{pmatrix}1\\0\\0\end{pmatrix}$ is a period vector for $\lim_{n\to+\infty} E_1^*(\gamma_{E_{n_0}}) \circ \cdots \circ E_1^*(\gamma_{E_n})(0,3^*)$ in $\mathcal{P}_{(0,b_{n_0},c_{n_0})}$.

Assume now that $\dim(a,b,c) = 1$. We again conclude by using Proposition 10 and by noticing first that $b_{m_0} = 0$ and then, that $(0,3^*)$ is a period pattern with period vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$ for $\mathcal{P}_{(0,0,c_{m_0})}$. □

4.3. Generating patterns

According to Theorem 26 and Proposition 28, we introduce the following definition.

Definition 29. We call *generating pattern* of the discrete plane $\mathcal{P}_{(a,b,c)}$ the pattern

- $\mathcal{U}$ if $\mathcal{P}_{(a,b,c)}$ is irrational and (8) does not hold;
- $\mathcal{V}$ if $\mathcal{P}_{(a,b,c)}$ is irrational and (8) does hold;
- $(0,3^*)$ if $\mathcal{P}_{(a,b,c)}$ is rational.

Moreover, with the generating pattern $\mathcal{W}$ we associate the translation vectors Trans($\mathcal{W}$):

$$\mathrm{Trans}(\mathcal{W}) = \begin{cases} \{-e_1+e_3, -e_2+e_3\} & \text{if } \mathcal{W} = \mathcal{U} \\ \{-e_1+e_3, -e_1+2e_2\} & \text{if } \mathcal{W} = \mathcal{V} \\ \{e_1, e_2\} & \text{if } \mathcal{W} = (0,3^*) \end{cases}$$

Theorem 26 and Proposition 28 can be summarized as follows.

Theorem 30. *Let* $(a,b,c) \in \mathbb{R}^3_+$. *Let* $\mathcal{W}$ *be the generating pattern of* $\mathcal{P}_{(a,b,c)}$. *The pattern*

$$E_1^*(\sigma_{<1>}) \circ E_1^*(\sigma_{<2>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(0,3^*)$$

is a period pattern for $\mathcal{P}_{\mathbf{M}_n(0,0,1)}$ *with period vectors* ${}^t\mathbf{M}_n^{-1}\mathrm{Trans}(0,3^*)$. *Moreover, let*

$$\mathcal{T} = \lim_{n\to+\infty} E_1^*(\sigma_{<1>}) \circ E_1^*(\sigma_{<2>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{W}).$$

- *If* $\dim_{\mathbb{Q}}(a,b,c) = 3$, *then* $\mathcal{T} = \mathcal{P}_{(a,b,c)}$.

- *if* $\dim_{\mathbb{Q}}(a,b,c) = 2$*, then there exist* $n_0 \geq 1$ *such that* $\overrightarrow{v_0} = {}^t\mathbf{M}_{n_0}^{-1}(1,0,0)$ *is such that*

$$\sum_{k \in \mathbb{Z}} (k\overrightarrow{v_0} + \mathcal{T}) = \mathcal{P}_{(a,b,c)}.$$

- *If* $\dim_{\mathbb{Q}}(a,b,c) = 1$*, then there exists* $m_0 \geq 1$ *such that* $\mathcal{P}_{\mathbf{M}_{n_0}(0,0,1)} = \mathcal{P}_{(a,b,c)}$.

The aim of next section will be to study the topological properties of the patterns

$$E_1^*(\sigma_{<1>}) \circ E_1^*(\sigma_{<2>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{W}),$$

and to prove the main result of this paper, namely Theorem 31.

5. Boundary words for the Jacobi-Perron algorithm

We now have gathered all the required material to state and prove the main result of this paper:

Theorem 31. *Let* (a,b,c) *with* $0 \leq a,b \leq c$ *and* $a \neq 0$*. Let* $(E_1^*(\sigma_{<n>}))_{n\geq 1}$ *be the sequence of* geometric Jacobi-Perron substitutions *generated by* (a,b,c)*, according to Definition 23. Let* $\mathcal{W}$ *be the generating pattern of* $\mathcal{P}_{(a,b,c)}$ *(see Definition 29). Then,*

$$E_1^*(\sigma_{<1>}) \circ E_1^*(\sigma_{<2>}) \circ \cdots \circ E_1^*(\sigma_{<n>})(\mathcal{W})$$

is a polyamond pattern for all $n \geq 1$.

In other words, if $\dim_{\mathbb{Q}}(a,b,c) = 3$, one generates arbitrarily large polyamond patterns that will eventually cover the full plane. If $\dim_{\mathbb{Q}}(a,b,c) = 1$, which is the usual situation in discrete geometry, one produces polyamond period patterns. In the intermediate case $\dim_{\mathbb{Q}}(a,b,c) = 2$, one covers periodically the plane by infinite edge-connected stripes of faces that have no holes.

Theorem 31 will be a direct corollary of the more technical result below which deals with the substitutions $\tau_{<n>}$ defined over the alphabet $\overline{A}_3$ (see Definition 23), which requires in particular the introduction of the two following sets. Let

$$S := \left\{ \pm \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \pm \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}, \tag{9}$$

and

$$T := \left\{ \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{Z}^3 \,\middle|\, \forall a,b,c \in \mathbb{N} \text{ with } 0 \leq a,b \leq c, \ (\alpha,\beta,\gamma) \notin \mathcal{P}_{(a,b,c)} \right\}. \tag{10}$$

Proposition 32. *Let* (a,b,c) *with* $0 \leq a,b \leq c$ *and* $a \neq 0$*. Let* $(\tau_{<n>})_{n\geq 1}$ *be the sequence of* Jacobi-Perron substitutions *generated by* (a,b,c)*. Let* $\mathcal{W}$ *be a polyamond pattern and* w *be a boundary word for* $\mathcal{W}$*. We assume furthermore that* w *contains no cyclic factor*

(i) *of the form* $2f$ *with* $\overrightarrow{2f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$*, nor of the form* $\overline{2}f$ *with* $\overrightarrow{f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$,

(ii) *nor no cyclic factor whose abelianization belongs to* $S \cup T$.

Then, for all $n \geq 1$ *for which* $C_n \neq 0$*, i.e., for which* $a_n \neq 0$,

$$\tau_{<1>} \circ \tau_{<2>} \circ \cdots \circ \tau_{<n>}(w)$$

is a simple closed path.

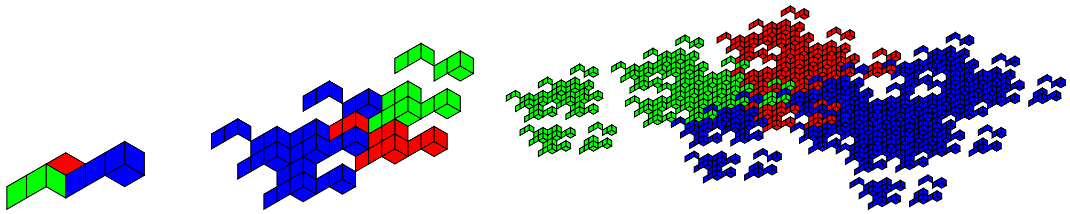


Figure 16: Jacobi-Perron's substitutions with $B_n > C_n$ produce patterns that are not connected. Note that Jacobi-Perron's algorithm only produces pairs of the form (B, C) with $B \leq C$.

5.1. Proof of Theorem 31 and Proposition 32

Remark 33. Our proofs will not make use of the admissibility condition (6) on the Jacobi-Perron digits, but will only use the fact that $0 \leq B_n \leq C_n$, $C_n \geq 1$, for all n. See Figure 16 for examples of patterns for which this latter condition does not hold.

Proof of Proposition 32. The proof is by induction and makes use of Lemmas 34, 35 and 36 below. In particular, Lemma 34 relies on the following idea. Let $w \in \overline{\mathcal{A}}_3^*$ with $\overrightarrow{w} = 0$. When applying $\tau_{B,C}$ to w, one gets a closed path: $\overrightarrow{\tau_{B,C}(w)} = 0$. The question is to know whether it is simple closed path or an eight-curve. We will prove that $\tau_{B,C}(w)$ is an eight-curve for some (B, C) (see Definition 18) either if w is itself an eight-curve, or else if w contains a factor whose abelianization belongs $S \cup T$.

We now state Lemmas 34, 35 and 36. Lemma 34 and Lemma 35 will be proved in Appendix A and Appendix B, respectively. The proof of Lemma 36 which is straightforward will be given in the present section.

Lemma 34. *Let $w \in \overline{\mathcal{A}}_3^*$ be a word such that $\tau_{B,C}(w)$ is an eight-curve, with $0 \leq B \leq C$ and $C \geq 1$. Then, one of the following holds:*

(i) w is an eight-curve,

(ii) w contains a cyclic factor whose abelianization belongs to $S \cup T$.

Lemma 35 (resp. Lemma 36) handles the case of the set S (resp. T). Note that the technical assumptions (i) of Proposition 32 are required here for handling the case of the set S.

Lemma 35. *Let $\mathcal{W}$ be a polyamond pattern and w be a boundary word for $\mathcal{W}$. We assume furthermore that w contains no cyclic factor*

(i) of the form $2f$ with $\overrightarrow{2f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, nor of the form $\overline{2}f$ with $\overrightarrow{f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$,

(ii) nor no cyclic factor whose abelianization belongs to S.

For any positive integer n and for any product Γ of n proper Jacobi-Perron substitutions, $\Gamma(w)$ contains no cyclic factor whose abelianization belongs to S.

Lemma 36. *Let $\mathcal{W}$ be a polyamond pattern and w be a boundary word for $\mathcal{W}$ such that the exists $P \in \mathbb{Z}^3$ such that $K_3(0, w)$ is included in some discrete plane with normal vector (a, b, c) with $0 \leq a, b \leq c$ and $c \neq 0$. For any positive integer n and for any product Γ of n proper Jacobi-Perron substitutions, $\Gamma(w)$ contains no cyclic factor whose abelianization belongs to T.*

Proof of Lemma 36. Let n be a positive integer and let Γ be a product of n proper Jacobi-Perron substitutions. According to Theorem 8, $\Gamma(w)$ is a boundary word for a pattern that belongs to some discrete plane that we denote by $\mathcal{P}$. According to Remark 25, its normal vector (a', b', c') satisfies $0 \leq a', b' \leq c'$. Hence any cyclic factor f of $\Gamma(w)$ satisfies $\overrightarrow{f} \in \mathcal{P} - \mathcal{P}$, and thus does not belong to T. ∎

End of proof of Proposition 32. We can now come back to the proof of Proposition 32. Let $\mathcal{W}$ be a polyamond pattern and w be a boundary word for $\mathcal{W}$ that satisfies Conditions (*i*) and (*ii*). Note that $\overrightarrow{w} = 0$. The proof works by contradiction. We assume that there exist $n \geq 1$ and a product Γ of n proper Jacobi-Perron substitutions such that $\Gamma(w)$ is an eight-curve. Let n be the smallest such integer. We write Γ as $\Gamma = \tau_{B,C} \circ \Gamma'$, where Γ' is a product of $n-1$ proper Jacobi-Perron substitutions. Note that $\overrightarrow{\Gamma'(w)} = 0$ and that w is a simple closed path by the minimality assumption made on n. By applying Lemma 34 (and thus Condition (ii)) to

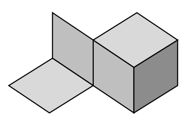
$$\Gamma(w) = \tau_{B,C} \circ \Gamma'(w),$$

one deduces that $\Gamma'(w)$ contains a cyclic factor whose abelianization belongs to $S \cup T$. The contradiction comes from Lemma 35 and 36. ∎

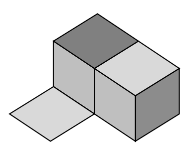
Proof of Theorem 31. Let (a, b, c) with $0 \leq a, b \leq c$ and $a \neq 0$.

We first assume that $\dim_{\mathbb{Q}}(a, b, c) = 3$ and that (8) holds. We check that that the word $2\bar{1}3\bar{2}1\bar{3}$ which is a boundary word for the polyamond pattern $\mathcal{U}$ (see Example 13) contains no cyclic factor of the form $2f$ with $\overrightarrow{2f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, nor of the form $\bar{2}f$ with $\overrightarrow{f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, nor no cyclic factor whose abelianization belongs to S. Since it belongs to $\mathcal{P}_{(a,b,c)}$ (as soon as $a, b, c > 0$), it contains no cyclic factor in T. It remains to apply Proposition 32 to $\mathcal{U}$, and then Proposition 21.

We now assume that $\dim_{\mathbb{Q}}(a, b, c) = 3$ but that (8) that does not hold. The word $2\bar{1}3\bar{2}1\overline{23}12\bar{1}$ is a boundary word for $\mathcal{V}$ with

$\mathcal{V} =$ 

It contains a cyclic factor that belongs to S, namely $\overline{23}$. Nevertheless, one deduces from (3) that if $\mathcal{V}$ is included in some discrete plane whose normal vector (a, b, c) satisfies $0 < a, b \leq c$, then

$\mathcal{V} \cup (-e_2, 3^*) =$ 

is also included in this very same discrete plane (we use (3) together with that fact that $c \geq b$): this is the only way for the segment $(e_3, 1)$ to be shared with a face in a discrete plane that contains $\mathcal{V}$ when $b \leq c$. Furthermore, $\mathcal{V} \cup (-e_2, 3^*)$ is also a polyamond pattern. The word $2\bar{1}3\overline{222}1\bar{3}12\bar{1}$ is a boundary word for $\mathcal{V} \cup (-e_2, 3^*)$. One checks that it satisfies Conditions (*i*) and (*ii*) of Proposition 32. In particular, since $\mathcal{V} \cup (-e_2, 3^*)$ belongs to $\mathcal{P}_{(a,b,c)}$ $(a, b, c > 0)$, $2\bar{1}3\overline{222}1\bar{3}12\bar{1}$ contains no cyclic factor in T. It remains to apply Proposition 32 to $\mathcal{V} \cup (-e_2, 3^*)$ and which is a boundary word for $\mathcal{V} \cup (-e_2, 3^*)$, and then again Proposition 21.

Assume now that $\dim_{\mathbb{Q}}(a, b, c) = 1$ or 2. One checks that for $m, n \in \mathbb{N}$ and for $(B, C) \in \mathbb{N}^2$ with $0 \leq B \leq C$ and $C \geq 1$, if w is a boundary word for a polyamond pattern of the form

$$\mathcal{W} := E_1^*(\gamma_{E_n}) \circ \cdots \circ E_1^*(\gamma_{E_{n+m}})(0, 3^*),$$

with w itself of the form

$$w = 2^{k_1}\bar{3}2^{k_2}\bar{3} \cdots 2^{k_\ell}\overline{12}^{k_\ell}3 \cdots \bar{2}^{k_2}2\bar{2}^{k_1}1,$$

then w contains no cyclic factor of the form $2f$ with $\overrightarrow{2f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, nor of the form $\overline{2}f$ with $\overrightarrow{f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$, nor no cyclic factor whose abelianization belongs to S. See the figures of the proof of Lemma 27 for an illustration. We thus conclude by applying Theorem 32 and Proposition 21. ■

Remark 37. Before proving respectively Lemma 34 in Appendix A and Lemma 35 in Appendix B, let us show that

$$\left\{ \pm \begin{pmatrix}\alpha\\\beta\\\gamma+2\end{pmatrix} \middle| \alpha,\beta,\gamma \in \mathbb{N}, \alpha+\beta+\gamma \geq 1 \right\} \cup \left\{ \pm \begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} \middle| \alpha,\beta,\gamma \in \mathbb{N}, \alpha,\beta,\gamma \geq 1 \right\} \subset T. \tag{11}$$

Let $0 \leq a, b \leq c$. For any $(x, y, z) \in \mathcal{P}_{(a,b,c)}$, one has by definition $0 < ax + by + cz \leq a + b + c$. If $(\alpha, \beta, \gamma + 2) \in \mathbb{N}^3$ satisfies $\alpha + \beta + \gamma \geq 1$, then for any $(x, y, z) \in \mathcal{P}_{(a,b,c)}$

$$a + b + c \leq 2c + \min(a, b) < a(x + \alpha) + b(y + \beta) + c(z + \gamma + 2).$$

Similarly, one proves that $a(x - \alpha) + b(y - \beta) + c(z - \gamma - 2) \leq 0$. Hence, $\pm(\alpha, \beta, \gamma + 2) \notin \mathcal{P}_{(a,b,c)} - \mathcal{P}_{(a,b,c)}$. Moreover, if $(\alpha, \beta, \gamma) \in \mathbb{N}^3$ satisfies $\alpha, \beta, \gamma \geq 1$, then for any $(x, y, z) \in \mathcal{P}_{(a,b,c)}$, one has $a + b + c < a(x+\alpha)+b(y+\beta)+c(z+\gamma+2)$ and $a(x-\alpha)+b(y-\beta)+c(z-\gamma+2) \leq 0$. Hence, $\pm(\alpha, \beta, \gamma) \notin \mathcal{P}_{(a,b,c)} - \mathcal{P}_{(a,b,c)}$.

In particular, note that

$$\left\{ \pm \begin{pmatrix}0\\1\\\gamma\end{pmatrix}, \pm \begin{pmatrix}1\\0\\\gamma\end{pmatrix} \middle| \gamma \in \mathbb{N}, \gamma \geq 1 \right\} \subset S \cup T. \tag{12}$$

We will use these remarks below when proving Lemmas 34 and 35.

By combining Theorem 26, Theorem 30 and Corollary 31, we obtain the following:

Theorem 38. *Let $(a, b, c) \in \mathbb{R}^3_+$. Let $\mathcal{W}$ be the generating pattern of $\mathcal{P}_{(a,b,c)}$ (according to Definition 29). Then,*

$$E_1^*(\tau^{(1)}) \circ E_1^*(\tau^{(2)}) \circ \cdots \circ E_1^*(\tau^{(n)})(\mathcal{W})$$

is a polyamond pattern for all $n \geq 1$ which satisfies

- *if* $\dim(a, b, c) = 3$
$$\lim_{n\to+\infty} E_1^*(\tau^{(1)}) \circ E_1^*(\tau^{(2)}) \circ \cdots \circ E_1^*(\tau^{(n)})(\mathcal{W}) = \mathcal{P}_{(a,b,c)}$$

- *if* $\dim(a, b, c) = 2$, *then*
$$\lim_{n\to\infty} E_1^*(\sigma_{(B_1,C_1)}) \circ E_1^*(\sigma_{(B_2,C_2)}) \circ \cdots \circ E_1^*(\sigma_{(B_{n_0-1},C_{n_0-1})}) \circ E_1^*(\gamma_{E_{n_0}}) \circ \cdots \circ E_1^*(\gamma_{E_n})(0, 3^*)$$
is a polymaond period pattern for $\mathcal{P}_{(a,b,c)}$ with period vector ${}^tM^{-1}_{B_1,C_1} \cdots {}^tM^{-1}_{B_{n_0-1},C_{n_0-1}} \begin{pmatrix}1\\-1\\0\end{pmatrix}$.

- *if* $\dim(a, b, c) = 1$, *then*
$$E_1^*(\sigma_{(B_1,C_1)}) \circ E_1^*(\sigma_{(B_2,C_2)}) \circ \cdots \circ E_1^*(\sigma_{(B_{n_0-1},C_{n_0-1})}) \circ E_1^*(\gamma_{E_{n_0}}) \circ \cdots \circ E_1^*(\gamma_{E_{m_0}})(0, 3^*)$$
is a polyamond period vector for $\mathcal{P}_{(a,b,c)}$ with period vectors ${}^tM^{-1}_{B_1,C_1} \cdots {}^tM^{-1}_{B_{n_0-1},C_{n_0-1}} \, {}^tM_{E_{n_0}} \cdots {}^tM_{E_{m_0}} \begin{pmatrix}1\\0\\0\end{pmatrix}$
and ${}^tM^{-1}_{B_1,C_1} \cdots {}^tM^{-1}_{B_{n_0-1},C_{n_0-1}} \, {}^tM^{-1}_{E_{n_0}} \cdots {}^tM^{-1}_{E_{m_0}} \begin{pmatrix}0\\1\\0\end{pmatrix}$.

Appendix A. Proof of Lemma 34

Let $w \in \overline{A}_3^*$ such that $\tau_{B,C}(w)$ is an eight-curve, with $0 \leq B \leq C$ and $C \geq 1$. Hence, $\tau_{B,C}(w)$ contains a cyclic factor u that is a closed path, with its complement in w being also a closed path. In particular, $\overrightarrow{w} = \overrightarrow{u} = 0$ and $\pi_{F_3}(u) \neq \varepsilon$. We recall that

$$\tau_{B,C} : 1 \to \overline{1}^B 2,\ 2 \to \overline{1}^C 3,\ 3 \to 1,\ \overline{1} \to \overline{2}1^B,\ \overline{2} \to \overline{3}1^C,\ \overline{3} \to \overline{1}$$

and that the abelianized matrix of $\tau_{B,C}$ is

$$M_{B,C}^{-1} = \begin{pmatrix} -B & -C & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

There are four possible cases according to the way u can be "desubstituted" by $\tau_{B,C}$.

1. We first assume that there exists a cyclic factor v of w such that $\tau_{B,C}(v) = u$. According to Remark 24, this implies $\overrightarrow{v} = M_{B,C}0 = 0$. Hence, v is a also closed path since $\pi_{F_3}(u) \neq \varepsilon$ implies $\pi_{F_3}(v) \neq \varepsilon$. The same holds for its complement in w. We conclude that w is an eight-curve.

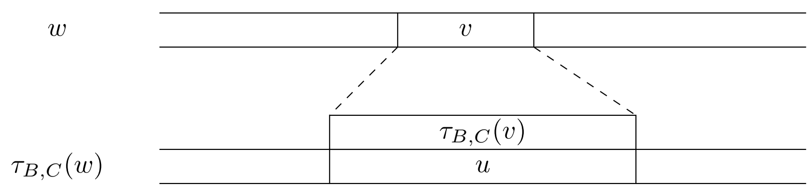


Figure A.17: Illustration of Case 1.

2. We now assume that there exist a letter $a \in \overline{A}_3^*$ and a cyclic factor v of w such that av is a cyclic factor of w that satisfies $u = s\tau_{B,C}(v)$ where s is a proper suffix of $\tau_{B,C}(a)$. We consider two subcases according to the value of a.

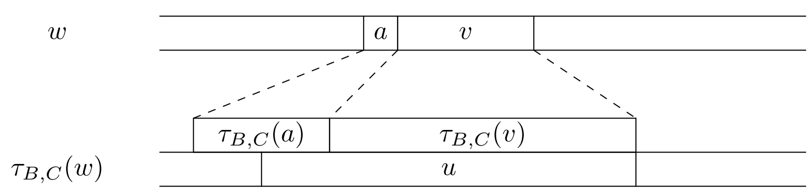


Figure A.18: Illustration of Case 2.

(1) We assume $a \in \{\overline{1}, \overline{2}\}$. Hence, there exists $i \geq 1$ such that $u = 1^i \tau_{B,C}(v)$, so we deduce from Remark 24 and from $\overrightarrow{u} = 0$ that

$$\overrightarrow{v} = M_{B,C}\,(-i\,\overrightarrow{e_1}) = \begin{pmatrix} 0 \\ 0 \\ -i \end{pmatrix}.$$

One deduces that $\overrightarrow{av} \in S \cup T$, according to (12), which implies one of the desired conclusions, namely w contains a cyclic factor whose abelianization belongs to $S \cup T$.

(2) We assume $a \in \{1, 2\}$. In this case, there exists $i \geq 0$ such that $u = \overline{1}^i a' \tau_{B,C}(v)$ where a' is either the letter 2 or 3 depending on the value of a. In order to consider both cases at once, we set $D := \begin{cases} B \text{ if } a' = 2 \\ C \text{ if } a' = 3 \end{cases}$. Hence, $1^{i-D}u = \tau_{B,C}(av)$. One gets

$$\overrightarrow{av} = M_{B,C} \begin{pmatrix} i - D \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ i - D \end{pmatrix}.$$

Since $0 \leq i < D$, we have that $i - D \leq -1$. We deduce that $\overrightarrow{v} \in S \cup T$ by (12).

3. We assume that there exist a letter b and a cyclic factor of the form vb of w such that $u = \tau_{B,C}(v)p$ where p is a proper prefix of $\tau_{B,C}(b)$.

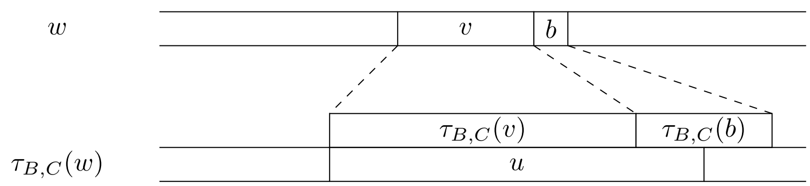


Figure A.19: Illustration of Case 3.

Considering the complement of u in the word $\tau_{B,C}(w)$, which is also a closed path, we come back to Case 2.

4. We assume that there exist two letters a and b and a cyclic factor of w of the form avb such that $s\tau_{B,C}(v)p = u$ where p (resp. s) is a proper prefix (resp. suffix) of $\tau_{B,C}(b)$ (resp. $\tau_{B,C}(a)$). There are now four cases to consider.

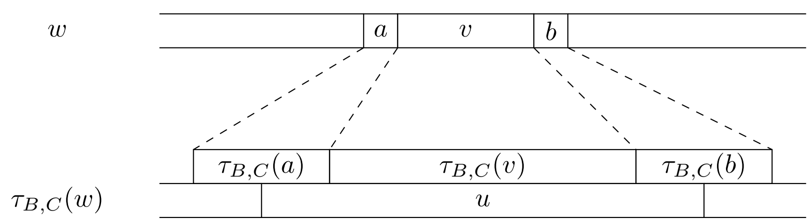


Figure A.20: Illustration of Case 4.

(1) We assume $a \in \{\overline{1}, \overline{2}\}$ and $b \in \{1, 2\}$. Thus, there exist $i \geq 0$ and $j \geq 1$ such that $u = 1^i \tau_{B,C}(v) \overline{1}^j$. Hence,

$$\overrightarrow{v} = M_{B,C} \left((j - i)\, e_1 \right) = \begin{pmatrix} 0 \\ 0 \\ j - i \end{pmatrix}.$$

First of all, note that $j = i$ implies that $\tau_{B,C}(v)$ is also a closed path. Hence, this was already considered in Case 1. Otherwise, if $j - i \geq 1$, then $\overrightarrow{vb} \in S \cup T$, while if $j - i \leq 1$, then $\overrightarrow{av} \in S \cup T$.

(2) We assume $a \in \{1, 2\}$ and $b \in \{\overline{1}, \overline{2}\}$. Considering again as in Case 3 the complement of u in $\tau_{B,C}(w)$, we come back to Case 4 (1).

(3) We assume $a \in \{1,2\}$ and $b \in \{1,2\}$. There exist $i \geq 0$ and $j \geq 1$ such that $u = \overline{1}^i a' \tau_{B,C}(v) \overline{1}^j$ where a' is equal either to the letter 2 or to 3 depending on the value of a. We set $D := \begin{cases} B \text{ if } a' = 2 \\ C \text{ if } a' = 3 \end{cases}$. One has $\overline{1}^{D-i} u = \tau_{B,C}(av)\overline{1}^j$. One has

$$\overrightarrow{av} = M_{B,C} \begin{pmatrix} i+j-D \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ (i+j)-D \end{pmatrix}.$$

We conclude by noticing that

- if $(i+j) - D \geq 1$ then $\overrightarrow{avb} \in S \cup T$, by (12),
- if $(i+j) - D \leq -1$ then $\overrightarrow{v} \in S$, again by (12),
- if $(i+j) - D = 0$ then av is a closed path, as well as its complement in w, hence w is an eight-curve. Indeed, one has $\pi_{F_3}(av) \neq \varepsilon_{F_3}$ since $\overline{1}^{D-i} u = \tau_{B,C}(av)\overline{1}^j$ and $\pi_{F_3}(u) \neq \varepsilon_{F_3}$. The same holds for its complement.

(4) We assume $a \in \{\overline{1}, \overline{2}\}$ and $b \in \{\overline{1}, \overline{2}\}$. There exist $i \geq 1$ and $j \geq 0$ such that $u = 1^i \tau_{B,C}(v) b' 1^j$ where b' is either the letter $\overline{2}$ or $\overline{3}$ depending on the value of b. Up to a sign change, this case is identical to the previous one. ∎

Appendix B. Proof of Lemma 35

The proof of Lemma 35 is a direct consequence of the two following lemmas.

Lemma 39. *Let $w \in \overline{\mathcal{A}}_3^*$ and let $\tau_{B,C}$ be a proper Jacobi-Perron substitution such that $\tau_{B,C}(w)$ contains a cyclic factor whose abelianization belongs to S. We also assume $\overrightarrow{w} = 0$. Then, one of the following holds:*

(i) w contains a cyclic factor whose abelianization belongs to $S \cup T$,

(ii) w contains a cyclic factor of the form $2f$ with $\overrightarrow{2f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, or of the form $\overline{2}f$ with $\overrightarrow{f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Lemma 40. *Let $w \in \overline{\mathcal{A}}_3^*$. Let $\tau_{B,C}$ be a proper Jacobi-Perron substitution. We assume that $\tau_{B,C}(w)$ contains a cyclic factor of the form $2f$ with $\overrightarrow{2f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, or of the form $\overline{2}f$ with $\overrightarrow{f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. Then, w contains a cyclic factor whose abelianization belongs to $S \cup T$.*

Proof of Lemma 35. We assume that w satifies the assumption of the lemma. The proof works by induction on the number of proper Jacobi-Perron substitutions $\tau_{B,C}$ applied to the word w, with the induction property being the following: for any product Γ of n proper Jacobi-Perron substitutions, $\Gamma(w)$ contains no cyclic factor whose abelianization belongs to S. The induction property holds for $n = 0$ by assumption. We then use Lemma 36, 39, 40 to prove that if the induction property holds for all k smaller than n for some $n \geq 1$, then it holds for $n + 1$. ∎

It remains now to prove Lemmas 39 and 40.

Proof of Lemma 39. Let $w \in \overline{\mathcal{A}}_3^*$ with $\overrightarrow{w} = 0$. Since $\overrightarrow{w} = 0$ (and thus $\overrightarrow{\tau_{B,C}(w)} = 0$) we can handle w.o.l.o.g. the case where $\tau_{B,C}(w)$ contains a cyclic factor u such that $\overrightarrow{u} = \begin{pmatrix} \delta \\ 1-\delta \\ 1 \end{pmatrix}$ for some $\delta \in \{0, 1\}$. There are again four cases to consider according to the preimage of u with respect to $\tau_{B,C}$.

1. We first assume that there exists a cyclic factor v of w such that $\tau_{B,C}(v) = u$. Then,

$$\overrightarrow{v} = M_{B,C}\overrightarrow{u} = \begin{pmatrix} 1-\delta \\ 1 \\ \delta + (1-\delta)B + C \end{pmatrix},$$

so if $\delta = 1$, $\overrightarrow{v} \in S \cup T$ by (12), and otherwise, by (11), $\overrightarrow{v} \in T$.

2. We now assume that there exist a letter a and a cyclic factor of w of the form av which satisfies $u = s\tau_{B,C}(v)$, where s is a proper suffix of $\tau_{B,C}(a)$. We consider two further subcases according to the value of a.

   (1) We first assume $a \in \{1, 2\}$. There exists $i \geq 0$ such that $u = \overline{1}^i a' \tau_{B,C}(v)$ where a' is either the letter 2 or 3 depending on the value of a. We set $D := \begin{cases} B \text{ if } a' = 2 \\ C \text{ if } a' = 3 \end{cases}$. One has $\overline{1}^{D-i} u = \tau_{B,C}(av)$, with $0 \leq i \leq D-1$. One gets

$$\overrightarrow{av} = M_{B,C}\begin{pmatrix} \delta + i - D \\ 1-\delta \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\delta \\ 1 \\ \delta + i - D + (1-\delta)B + C \end{pmatrix}.$$

   • Assume $\delta = 0$. One has $\overrightarrow{av} = \begin{pmatrix} 1 \\ 1 \\ i - D + B + C \end{pmatrix}$ and $i - D + B + C \geq 0$. Moreover, $i - D + B + C = 0$ if and only if $i = B = 0$ and $D = C$. If $i - D + B + C \geq 1$, then $\overrightarrow{v} \in S \cup T$ by (11). Otherwise, if $i = B = 0$ and $D = C$, then $a = 2$, and w contains a cyclic factor of the form $2v$ with $\overrightarrow{av} = \overrightarrow{2v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

   • Assume $\delta = 1$. One has $\overrightarrow{av} = \begin{pmatrix} 0 \\ 1 \\ 1 + i - D + C \end{pmatrix} \in S \cup T$, since $D \leq C$.

   (2) We assume $a \in \{\overline{1}, \overline{2}\}$. There exists $i \geq 1$ such that $u = 1^i \tau_{B,C}(v)$, so

$$\overrightarrow{v} = M_{B,C}\begin{pmatrix} \delta - i \\ 1-\delta \\ 1 \end{pmatrix} = \begin{pmatrix} 1-\delta \\ 1 \\ \delta - i + (1-\delta)B + C \end{pmatrix}.$$

   • Assume $\delta = 0$. One has $\overrightarrow{v} = \begin{pmatrix} 1 \\ 1 \\ -i + B + C \end{pmatrix}$ and $-i + B + C \geq 0$. Moreover, $i - B + C = 0$ if and only if $B = 0$ and $i = C$. If $i - D + B + C \geq 1$, then $\overrightarrow{v} \in S \cup T$ by (11). Otherwise, if $B = 0$ and $i = C$, then $a = \overline{2}$, and w contains a cyclic factor of the form $\overline{2}v$ with $\overrightarrow{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

   • Assume $\delta = 1$. One has $\overrightarrow{v} = \begin{pmatrix} 0 \\ 1 \\ 1 - i + C \end{pmatrix} \in S$ by (12).

3. We now assume that there exist a letter b and a cyclic factor of w of the form vb that satisfies $u = \tau_{B,C}(v)p$ where p is a proper prefix of $\tau_{B,C}(b)$. Note that we cannot work anymore with the complement of u in $\tau_{B,C}(w)$, since $\overrightarrow{u}$ of the form $\begin{pmatrix} \delta \\ 1-\delta \\ 1 \end{pmatrix}$, does not imply that its complement is so. We again consider two subcases.

(1) We assume $b \in \{1,2\}$. There exists $i \geq 1$ such that $u = \tau_{B,C}(v)\overline{1}^i$, so

$$\overrightarrow{v} = M_{B,C}\begin{pmatrix}\delta+i\\1-\delta\\1\end{pmatrix} = \begin{pmatrix}1-\delta\\1\\\delta+i+(1-\delta)B+C\end{pmatrix}.$$

One checks in both cases ($\delta = 0$ and $\delta = 1$) that $\overrightarrow{v} \in S \cup T$.

(2) We assume $b \in \{\overline{1},\overline{2}\}$. There exists $i \geq 0$ such that $u = \tau_{B,C}(v)b'1^j$ where b' is either the letter $\overline{2}$ or $\overline{3}$ depending on the value of b. We set $D := \begin{cases} B \text{ if } b' = \overline{2} \\ C \text{ if } b' = \overline{3}\end{cases}$. One has $u1^{D-j} = \tau_{B,C}(vb)$, with $j \leq D-1$. One gets

$$\overrightarrow{vb} = \ = M_{B,C}\begin{pmatrix}\delta+D-j\\1-\delta\\1\end{pmatrix} = \begin{pmatrix}1-\delta\\1\\\delta+D-j+(1-\delta)B+C\end{pmatrix}.$$

One checks again that in both cases ($\delta = 0$ and $\delta = 1$) that $\overrightarrow{vb} \in S \cup T$.

4. We assume that there exist two letters a and b and a cyclic factor v of w such that avb is a cyclic factor of w that satisfies $u = s\tau_{B,C}(v)p$, where p (resp. s) is a proper prefix (resp. suffix) of $\tau_{B,C}(b)$ (resp. $\tau_{B,C}(a)$). There are four cases to consider according to the values of a and b.

(1) We assume $a \in \{\overline{1},\overline{2}\}$ and $b \in \{1,2\}$. There exist $i \geq 1$ and $j \geq 1$ such that $u = 1^i\tau_{B,C}(v)\overline{1}^j$. Hence,

$$\overrightarrow{v} = M_{B,C}\begin{pmatrix}\delta+j-i\\1-\delta\\1\end{pmatrix} = \begin{pmatrix}1-\delta\\1\\\delta+(j-i)+(1-\delta)B+C\end{pmatrix}.$$

If $\delta = 0$, $\overrightarrow{v} = \begin{pmatrix}1\\1\\(j-i)+B+C\end{pmatrix}$. If $\delta = 1$, $\overrightarrow{v} = \begin{pmatrix}0\\1\\1+(j-i)+C\end{pmatrix}$.

In both cases, $\overrightarrow{v} \in S \cup T$, since $i \leq C$ and $j \geq 1$.

(2) We assume $a \in \{1,2\}$ and $b \in \{\overline{1},\overline{2}\}$. Let u' be the complement of u in $\tau_{B,C}(w)$ and v' the cyclic factor of w defined as the complement of avb in w. We have that $\overrightarrow{u'} = -\overrightarrow{u}$ (since $\overrightarrow{w} = 0$) and there exist $i,j \geq 0$ such that $u' = 1^j\tau_{B,C}(v')\overline{1}^i$ with $0 \leq i,j \leq C$. Hence,

$$\overrightarrow{v'} = M_{B,C}\left(-\begin{pmatrix}\delta\\1-\delta\\1\end{pmatrix} + \begin{pmatrix}i-j\\0\\0\end{pmatrix}\right) = \begin{pmatrix}-(1-\delta)\\-1\\-\delta-(j-i)-(1-\delta)B-C\end{pmatrix}.$$

• Assume $\delta = 0$. One has $-\overrightarrow{v}' = \begin{pmatrix}1\\1\\(j-i)+B+C\end{pmatrix}$, with $j-i+B+C \geq 0$. If $j-i+B+C \geq 1$, then $\overrightarrow{v}' \in T$, by (11). Moreover, $j-i+B+C = 0$ if and only if $i = C$, $j = 0 = B$. In this latter case, $a = 2$ (this comes from $i = C$ and $C \neq B$), $b = \overline{2}$ (this comes from $j = 0 = B$ and p is a proper prefix of $\tau_{B,C}(b)$), and thus, $u = 3\tau_{B,C}(v)\overline{3}$. We thus consider the factor $\tau_{B,C}(v)$ that satisfies $\overrightarrow{\tau_{B,C}(v)} = \overrightarrow{u}$ and we come back to Case 1.

• Assume $\delta = 1$. One has $-\overrightarrow{v}' = \begin{pmatrix}0\\1\\1+(j-i)+C\end{pmatrix}$, with $1+j-i+C \geq 1$, and thus, $\overrightarrow{v}' \in S \cup T$.

(3) We now assume $a \in \{1,2\}$ and $b \in \{1,2\}$. There exist $i \geq 0$ and $j \geq 1$ such that $u = \overline{1}^i a' \tau_{B,C}(v)\overline{1}^j$ where a' is either the letter 2 or 3 depending on the value of a. We set $D := \begin{cases} B \text{ if } a' = 2 \\ C \text{ if } a' = 3 \end{cases}$. One has $\overline{1}^{D-i} u = \tau_{B,C}(av)\overline{1}^j$, with $i \leq D-1$ and $j \geq 1$. One gets

$$\overrightarrow{av} = M_{B,C} \left(\begin{pmatrix} \delta \\ 1-\delta \\ 1 \end{pmatrix} + \begin{pmatrix} -D+i+j \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1-\delta \\ 1 \\ \delta - D + i + j + (1-\delta)B + C \end{pmatrix}.$$

If $\delta = 0$, $\overrightarrow{av} = \begin{pmatrix} 1 \\ 1 \\ -D+i+j+B+C \end{pmatrix}$. Since $j \geq 1$, we get $\overrightarrow{av} \in T$ by (11).

If $\delta = 1$, $\overrightarrow{av} = \begin{pmatrix} 0 \\ 1 \\ 1-D+i+j+C \end{pmatrix}$. Since $D \leq C$ and $j \geq 1$, we again deduce that $\overrightarrow{av} \in S \cup T$.

(4) We assume $a \in \{\overline{1}, \overline{2}\}$ and $b \in \{\overline{1}, \overline{2}\}$. There exist $i \geq 1$ and $j \geq 0$ such that $u = 1^i \tau_{B,C}(v) b' 1^j$ where b' is either the letter $\overline{2}$ or $\overline{3}$ depending on the value of b. We set $D := \begin{cases} B \text{ if } b' = \overline{2} \\ C \text{ if } b' = \overline{3} \end{cases}$. One has $u 1^{D-j} = 1^i \tau_{B,C}(vb)$, with $i \geq 1$ and $j \leq D-1$. One gets

$$\overrightarrow{vb} = M_{B,C} \left(\begin{pmatrix} \delta \\ 1-\delta \\ 1 \end{pmatrix} + \begin{pmatrix} D-i-j \\ 0 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 1-\delta \\ 1 \\ \delta + D - i - j + (1-\delta)B + C \end{pmatrix}.$$

If $\delta = 0$, $\overrightarrow{av} = \begin{pmatrix} 1 \\ 1 \\ D-i-j+B+C \end{pmatrix}$. One has $D-i-j+B+C \geq 1$ since $j \leq D-1$, hence $\overrightarrow{av} \in T$.

If $\delta = 1$, $\overrightarrow{av} = \begin{pmatrix} 0 \\ 1 \\ 1+D-i-j+C \end{pmatrix}$. One has again $1+D-i-j+C \geq 1$ since $j \leq D-1$, hence $\overrightarrow{av} \in S \cup T$.

■

Proof of Lemma 40. We first assume that $\tau_{B,C}(w)$ contains a cyclic factor of the form $2f$, with $\overrightarrow{2f} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

Let us "desubstitute" $2f$. There exist a letter b and a cyclic factor of w of the form $1vb$ which satisfies $\overline{1}^B 2f = \tau_{B,C}(1v)p$, where p is a prefix of $\tau_{B,C}(b)$ (that may be empty). We consider two cases according to the value of b.

(1) We assume $b \in \{1,2\}$. There exists $i \geq 0$ such that $f = \tau_{B,C}(v)\overline{1}^i$. Since $\overline{1}^B 2f = \tau_{B,C}(1v)\overline{1}^i$, we deduce

$$\overrightarrow{1v} = M_{B,C} \begin{pmatrix} 1+i-B \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1+i \end{pmatrix} \in S \cup T.$$

(2) We assume $b \in \{\overline{1}, \overline{2}\}$. There exist $i \geq 0$ and b' such that $f = \tau_{B,C}(v) b' 1^i$. We set $D := \begin{cases} B \text{ if } b' = \overline{2} \\ C \text{ if } b' = \overline{3} \end{cases}$.

One has $i \leq D$. Since $\overline{1}^B 2f 1^{D-i} = \tau_{B,C}(1vb)$, we deduce

$$\overrightarrow{1vb} = M_{B,C} \begin{pmatrix} 1+D-i-B \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1+D-i \end{pmatrix} \in S \cup T.$$

We now assume that $\tau_{B,C}(w)$ contains a cyclic factor of the form $\overline{2}f$, with $\overrightarrow{f} = \begin{pmatrix}1\\1\\0\end{pmatrix}$. We again "desubstitute" $\overline{2}f$. We thus assume that there exist a letter b and a cyclic factor of w of the form $\overline{1}vb$ which satisfy $\overline{2}f = \tau_{B,C}(\overline{1}v)p$, where p is a prefix of $\tau_{B,C}(b)$. Hence, one has $f = 2^B\tau_{B,C}(v)p$. There are again two cases to consider.

(1) We assume $b \in \{1, 2\}$. There exists $i \geq 0$ such that $f = 1^B\tau_{B,C}(v)\overline{1}^i$. We deduce

$$\overrightarrow{v} = M_{B,C}\begin{pmatrix}1-B+i\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1+i\end{pmatrix} \in S \cup T.$$

(2) We assume $b \in \{\overline{1}, \overline{2}\}$. There exist $i \geq 0$ and b' such that $f = 1^B\tau_{B,C}(v)b'1^i$, where b' depends on the value of b. We set $D := \begin{cases} B \text{ if } b' = \overline{2} \\ C \text{ if } b' = \overline{3} \end{cases}$. One has $i \leq D$. Since $f1^{D-i} = 1^B\tau_{B,C}(vb)$, we deduce

$$\overrightarrow{vb} = M_{B,C}\begin{pmatrix}1-B+D-i\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1+D-i\end{pmatrix} \in S \cup T.$$

■

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