DISCRETE GEOMETRY AND SYMBOLIC DYNAMICS

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Avec mes pensées les plus amicales pour Christer Kiselman.

1. Introduction

The aim of this survey is to illustrate various connections that exist between word combinatorics and arithmetic discrete geometry through the discussion of some discretizations of elementary Euclidean objects (lines, planes, surfaces). We focus on the role played by dynamical systems (toral rotations mainly) that can be associated in a natural way with these discrete structures. We show how classical techniques in symbolic dynamics applied to some codings of such discretizations allow one to obtain results concerning the enumeration of configurations and their statistical properties. Note that we have no claim to exhaustivity: the examples that we detail here have been chosen for their simplicity.

Let us first illustrate this interaction with Figure 1.1 below where a piece of an arithmetic discrete plane in $\mathbb{R}^3$ is depicted, as well as its orthogonal projection onto the antidiagonal plane $\Delta\colon x_1+x_2+x_3=0$ in $\mathbb{R}^3$, which can be considered as a piece of a tiling of the plane by three kinds of lozenges, and lastly, its coding as a two-dimensional word over a three-letter alphabet.

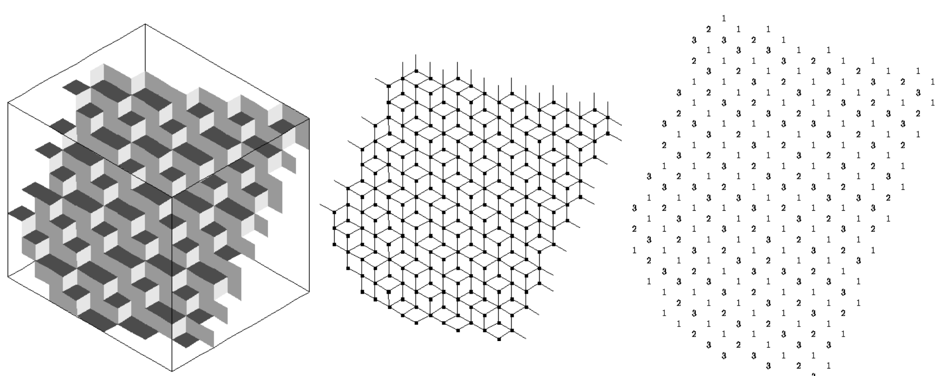

Figure 1.1. Left: arithmetic discrete plane. Middle: tiling of the plane. Right: two-dimensional word.

This paper is organized as follows. We first start with the most simple situation, namely discrete lines and Sturmian words (see Section 2). Section 3 is devoted to the higher-dimensional case, i.e., to the study of arithmetic discrete planes. We generalize this study performed mainly in the so-called naive case, first, to a broader class of arithmetic discrete planes in Section 4, and second, to functional stepped surfaces in Section 5. Section 6 is concerned with the generation of arithmetic discrete planes by so-called generalized substitutions. Special focus is given to the duality between arithmetic discrete planes and discrete lines in the three-dimensional space. One key tool will be the so-called Rauzy fractal associated with the cubic Pisot number of minimal polynomial $X^3 - X^2 - X - 1 = 0$.

2. Sturmian words and discrete lines

This section is concerned with the connections between arithmetic discrete lines and Sturmian words. A substantial literature has been devoted to the study of discrete lines, as illustrated for instance in the surveys [KR04, BCK07]. Let us start by recalling the definition of an arithmetic discrete line, introduced by Reveillès in [Rev91].

Definition 1. Let $\mathbf{v} \in \mathbb{R}^2$, and $\mu, \omega \in \mathbb{R}$. The (lower) arithmetic *discrete line* $\mathfrak{D}(\mathbf{v}, \mu, \omega)$ is defined as

$$\mathfrak{D}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^2;\ 0 \leq \langle \mathbf{v}, \mathbf{v} \rangle + \mu < \omega\}.$$

Parameter μ is called the *translation parameter* of $\mathfrak{D}(\mathbf{v}, \mu, \omega)$, and ω is called the *width* of $\mathfrak{D}(\mathbf{v}, \mu, \omega)$.

Two natural cases are more particularly studied: if $\omega = ||\mathbf{v}||_\infty$, then $\mathfrak{D}(\mathbf{v}, \mu, \omega)$ is said *naive*, and if $\omega = ||\mathbf{v}||_1$, then $\mathfrak{D}(\mathbf{v}, \mu, \omega)$ is said *standard*. One checks that a naive (resp. standard) arithmetic discrete line is made of

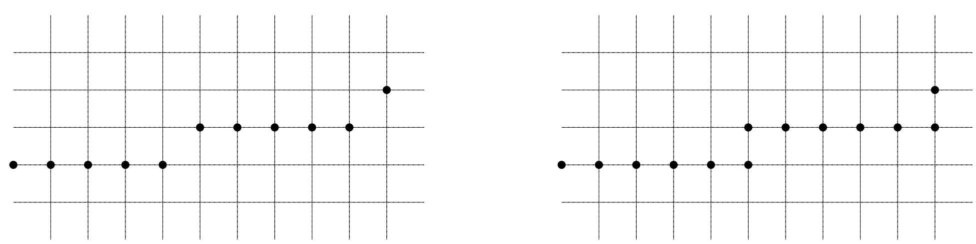

Figure 2.1. Left: a naive discrete line. Right: a standard discrete line.

horizontal and diagonal (resp. horizontal and vertical) steps. One can code such a standard line by using the *Freeman code* [Fre70] over the two-letter alphabet $\{0, 1\}$ as follows: one codes horizontal steps by a 0, and diagonal ones by a 1. One gets a *so-called* Stumian word $(u_n)_{n \in \mathbb{N}} \in \{0, 1\}^{\mathbb{N}}$. More precisely, Sturmian words satisfy:

Definition 2 (Morse-Hedlund [MH40]). Let $R_\alpha \colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z},\ x \mapsto x + \alpha \bmod 1$ be the rotation of angle α of the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Let $u = (u_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$. The infinite word u is a Sturmian word if there exist $\alpha \in (0,1)$, $\alpha \notin \mathbb{Q}$, $x \in \mathbb{R}$ such that

$$\forall n \in \mathbb{N},\ u_n = i \Longleftrightarrow R_\alpha^n(x) = n\alpha + x \in I_i \pmod 1,$$

with $I_0 = [0, 1-\alpha[,\ I_1 = [1-\alpha, 1[$ or $I_0 =]0, 1-\alpha],\ I_1 =]1-\alpha, 1]$.

For more on Sturmian words, see the surveys [AS02, Lot02, PF02] and the references therein.

The following lemma is classical for the study of Sturmian words. Its interest for further generalizations is stressed in the survey [BFZ05].

Lemma 1. *The word $w = w_1 \cdots w_n$ over the alphabet $\{0,1\}$ is a factor of the Sturmian word u if and only if*

$$I_w := I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n} \neq \emptyset.$$

Proof. By definition, one has

$$\forall i \in \mathbb{N},\ u_n = i \Longleftrightarrow n\alpha + x \in I_i \pmod 1.$$

One first notes that $u_k u_{k+1} \cdots u_{n+k-1} = w_1 \cdots w_n$ if and only if

$$\begin{cases} k\alpha + x \in I_{w_1} \\ (k+1)\alpha + x \in I_{w_2} \\ \ldots \\ (k+n-1)\alpha + x \in I_{w_n}. \end{cases}$$

One then applies the density of $(n\alpha)_{n\in\mathbb{N}}$ in $\mathbb{R}/\mathbb{Z}$ (recall that α is assumed to be an irrational number). □

One first notes that the condition of Lemma 1 does not depend on x but only on α. One easily checks that the sets $I_{w_1} \cap R_\alpha^{-1} I_{w_2} \cap \cdots R_\alpha^{-n+1} I_{w_n}$ are intervals of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. Furthermore, the factors of u of length n are in one-to-one correspondence with the $n+1$ intervals of $\mathbb{T}$ whose end-points are given by $-k\alpha \bmod 1$, for $0 \leq k \leq n$. This implies that two Sturmian words coding the same rotation have the same factors. Furthermore, Sturmian words have exactly $n+1$ factors of length n, for every $n \in \mathbb{N}$. This is even a characterization of Stumian words:

Theorem 1 (Coven-Hedlund [CH73])**.** *A word $u \in \{0,1\}^{\mathbb{N}}$ is Sturmian if and only if it has exactly $n+1$ factors of length n.*

The function that associates with a word the number of its factors of a given length is called the *complexity function*. For more on this function, see for instance [AS02, All94, Fer99].

More generally, one deduces from Lemma 1 various combinatorial properties of Sturmian words, such as the expression of densities of factors [Ber96], that can be deduced from the equidistribution of the sequence $(n\alpha)_{n\in\mathbb{N}}$. Indeed, the frequency of occurrence of the word w in the Sturmian word u is equal to the length of the interval I_w.

Let us note that Definition 2 can be restated in terms of dynamical systems as follows. A *dynamical system* (X, T) is defined as the action of a continuous and onto map T on a compact space X. An example of a geometric dynamical system is given by $(\mathbb{T}, R_\alpha)$.

Given a dynamical system (X,T), a point $x \in X$, and a partition $\mathcal{P} = \{P_0, \ldots, P_{k-1}\}$ of X, the sequence $u = (u_n)_{n\in\mathbb{N}}$ defined by $u_n = i$ whenever $T^n x \in P_i$, for $n \in \mathbb{N}$, is called a *coding* of the dynamical system (X,T). A Sturmian word is thus a coding of the dynamical system $(\mathbb{T}, R_\alpha)$ with respect either to the two-interval partition $\{I_0 = [0, 1-\alpha[,\ I_1 = [1-\alpha, 1[\}$ or to $\{I_0 =]0, 1-\alpha],\ I_1 =]1-\alpha, 1]\}$.

As another example, let us consider *symbolic dynamical systems*. Let $\mathcal{A}$ be a finite set. We endow $\mathcal{A}^{\mathbb{N}}$ with the product topology of the discrete topology on $\mathcal{A}$. Let $u \in \mathcal{A}^{\mathbb{N}}$. Let $\mathcal{L}(u)$ be the set of its factors. The *shift* S is defined as $S : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}},\ (u_n)_{n\in\mathbb{N}} \mapsto (u_{n+1})_{n\in\mathbb{N}}$. The *symbolic dynamical system* generated by u is (X_u, S) with $X_u := \overline{\{S^n(u);\ n \in \mathbb{N}\}} = \{v \in \mathcal{A}^{\mathbb{N}};\ \mathcal{L}(v) \subset \mathcal{L}(u)\} \subset \mathcal{A}^{\mathbb{N}}$.

Since two Sturmian words coding the same rotation have the same set of factors, then one checks that the symbolic dynamical system generated by a Sturmian word coding the rotation R_α consists of *all* the Sturmian words that code the same rotation.

Note that several combinatorial properties of Sturmian words or of naive arithmetic discrete lines respectively, have been studied and stated independently: for instance, the notion of balance, and the chord property respectively, have been considered in [MH38, MH40, Lot02, PF02] for Sturmian words, and in [Fre74, Ros74, Hun85, Mel05] in discrete geometry. For more details on the connections between Sturmian words and discrete lines, see for instance Chap. 1 of [Jam05b], and more generally, for references on discrete lines, see the surveys [KR04, BCK07]. See [AD03, BDJR08] for complexity like results. See also [Fer08, UW08] for recent results in discrete geometry in connection with continued fractions.

3. Discrete planes

Let us consider now the higher-dimensional case.

Definition 3. Let $\mathbf{v} \in \mathbb{R}^3$, and $\mu, \omega \in \mathbb{R}$. The *arithmetic discrete hyperplane* $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is defined as

$$\mathfrak{P}(\mathbf{v}, \mu, \omega) = \{\mathbf{x} \in \mathbb{Z}^3;\ 0 \le \langle \mathbf{x}, \mathbf{v} \rangle + \mu < \omega\}.$$

If $\omega = ||v||_\infty$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said *naive.* If $\omega = ||v||_1$, then $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said *standard.*

A piece of a naive discrete plane (left) as well as a piece of a standard discrete plane (right) are depicted in the figure below.

Let us see now how to associate with a standard arithmetic discrete plane a *coding* as a two-dimensional word on a three-letter alphabet that plays the role of the Freeman code for arithmetic discrete lines, such as described in Section 2.

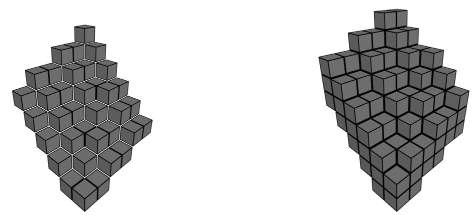

FIGURE 3.1. Left: naive discrete plane. Right: standard discrete plane.

Let $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ stand for the canonical basis of $\mathbb{R}^3$. Let $\mathbf{x} \in \mathbb{Z}^3$ and $i \in \{1,2,3\}$. Let E_1, E_2 and E_3 be the three following faces:

$$\begin{aligned} E_1 &= \left\{\lambda \mathbf{e_2} + \mu \mathbf{e_3};\ (\lambda,\mu) \in [0,1[^2\right\}, \\ E_2 &= \left\{-\lambda \mathbf{e_1} + \mu \mathbf{e_3};\ (\lambda,\mu) \in [0,1[^2\right\}, \\ E_3 &= \left\{-\lambda \mathbf{e_1} - \mu \mathbf{e_2};\ (\lambda,\mu) \in [0,1[^2\right\}. \end{aligned}$$

We call *pointed face* the set $\mathbf{x}+E_i$. The point $\mathbf{x}$ is called the *distinguished vertex* of the face $\mathbf{x}+E_i$. Note that each pointed face includes exactly one integer point, namely, its *distinguished vertex.*

Let $\mathfrak{P} := \mathfrak{P}(\mathbf{v}, \mu, ||v||_1)$ be a standard arithmetic discrete plane. One associates with $\mathfrak{P}$ a so-called *stepped plane* $\mathcal{P}$ defined as the union of faces of integral cubes that connect the points of $\mathfrak{P}$, as depicted in Figure 1.1 (left). By *integral cube*, we mean a translate by a vector with integral entries of the fundamental unit cube $\mathcal{U} = \{\sum_{1\leq i \leq n} \lambda_i \mathbf{e}_i;\ \lambda_i \in [0,1],\ \text{ for all } i\}$ with integral vertices. The stepped plane $\mathcal{P}$ is thus defined as the boundary of the set of integral cubes that intersect the lower open half-space $\{\mathbf{x} \in \mathbb{Z}^3;\ \langle \mathbf{x}, \mathbf{v}\rangle + \mu \leq 0\}$. The vertices (that is, the points with integer coordinates) of $\mathcal{P}$ are exactly the points of the arithmetic discrete plane $\mathfrak{P}$, according for instance to [BV00b].

Let Δ be the antidiagonal plane of equation $x_1 + x_2 + x_3 = 0$ and let π_0 be the orthogonal projection onto Δ. Note that $\pi_0(\mathbb{Z}^3)$ is a lattice in Δ with basis $(\pi_0(e_1), \pi_0(e_2))$, and that $\pi_0(e_3) = -\pi_0(e_1) - \pi_0(e_2)$. If we use this basis for $\pi_0(\mathbb{Z}^3)$, then the restriction of π_0 to $\mathbb{Z}^3$ becomes the following map, also denoted by π_0 by abuse of notation:

$$\pi_0 \colon \mathbb{Z}^3 \longrightarrow \mathbb{Z}^2,\ \mathbf{x} \mapsto (x_1 - x_3, x_2 - x_3).$$

According to [BV00b, ABI02], the restriction of the projection map π_0 to $\mathfrak{P}$ is one-to-one and onto Δ:

$$\forall (m,n) \in \mathbb{Z}^2,\ \exists!\ (\mathbf{x}, i) \text{ such that } \mathbf{x} + E_i \subset \mathcal{P},\ \pi_0(\mathbf{x}) = (m,n). \tag{3.1}$$

Furthermore, the projections of the faces of the stepped plane $\mathcal{P}$ tile the diagonal plane Δ with three kinds of lozenges (see Figure 1.1).

We then provide the stepped plane $\mathcal{P}$ with a two-dimensional coding as follows. The *two-dimensional coding* of the stepped plane $\mathcal{P}$ is the two-dimensional word $U \in \{1,2,3\}^{\mathbb{Z}^2}$ defined, for all $(m,n) \in \mathbb{Z}^2$ and all $i \in \{1,2,3\}$, by

$$U_{m,n} = i \iff \exists\,(\mathbf{x}, i) \text{ such that } \mathbf{x} + E_i \subset \mathcal{P},\ \pi_0(\mathbf{x}) = (m,n).$$

According to (3.1), the value of U at each point (m,m) is well-defined. One checks (e.g., see [BV00b, ABI02, ABS04]) that for $(m,n) \in \mathbb{Z}^2$ and $i \in \{1,2,3\}$, then $U_{m,n} = i$ if and only if:

$$(3.2)\qquad mv_1 + nv_2 + \mu \mod v_1 + v_2 + v_3 \in [v_0 + \cdots + v_{i-1}, v_0 + \cdots + v_i[,$$

by setting $v_0 = 0$.

Let us now introduce an analogue of the dynamical system $(\mathbb{T}, R_\alpha)$ that is coded by the two-dimensional word U. Given two continuous and onto maps T_1 and T_2 acting on X and satisfying $T_1 \circ T_2 = T_2 \circ T_1$, the $\mathbb{Z}^2$*-action* by T_1 and T_2 on X, that we denote by (X, T_1, T_2), is defined as

$$\forall (m,n) \in \mathbb{Z}^2,\ \forall x \in X,\quad (m,n)\cdot x = T_1^m \circ T_2^n(x).$$

As an example, consider a $\mathbb{Z}^2$-action by two rotations on the torus $\mathbb{R}/\mathbb{Z}$, that is, the $\mathbb{Z}^2$-action defined by

$$(m,n)\cdot x = R_\alpha^m R_\beta^n(x) = x + m\alpha + n\beta \mod 1.$$

Given any partition $\{P_1, \cdots, P_d\}$ of the torus and a point x we can define a (two-dimensional) word $U = (U_{m,n})_{(m,n)\in\mathbb{Z}^2} \in \{1,2,\cdots,d\}^{\mathbb{Z}^2}$ coding the orbit of x under this $\mathbb{Z}^2$-action by $U_{m,n} = i$ whenever $R_\alpha^m R_\beta^n x \in P_i$, for $(m,n) \in \mathbb{Z}^2$. The two-dimensional coding given by (3.2) is an example of such a coding.

After a suitable renormalization by $||v||_1$ of the parameters involved, one thus defines two-dimensional Sturmian words as follows:

Definition 4 ([BV00b])**.** Let $U = (U_{m,n})_{(m,n)\in\mathbb{Z}^2} \in \{1,2,3\}^{\mathbb{Z}^2}$. The two-dimensional word U is said to be a *two-dimensional Sturmian word* if there exist $x \in \mathbb{R}$, and $\alpha, \beta \in \mathbb{R}$ such that $1, \alpha, \beta$ are $\mathbb{Q}$-linearly independent and $\alpha + \beta < 1$ such that

$$\forall (m,n) \in \mathbb{Z}^2,\ Um,n = i \Longleftrightarrow R_\alpha^m R_\beta^n(x) = x + n\alpha + m\beta \in I_i \text{ (mod 1)},$$

with

$$I_1 = [0, \alpha[,\ I_2 = [\alpha, \alpha+\beta[,\ I_3 = [\alpha+\beta, 1[$$

or

$$I_1 =]0, \alpha],\ I_2 =]\alpha, \alpha+\beta],\ I_3 =]\alpha+\beta, 1].$$

Let us state now the analogue of Lemma 1. We first consider finite rectangular arrays of consecutive letters, that is, rectangular words

$$W = \begin{bmatrix} w_{0,n-1} & \cdots & w_{m-1,n-1} \\ \vdots & & \vdots \\ w_{0,0} & \cdots & w_{m-1,0}. \end{bmatrix}$$

We say that w has size (m, n). The *rectangular complexity* of the two-dimensional word U is the function $p_U(m, n)$ which associates with each $(m, n) \in \mathbb{N}^2$, m and n being nonzero, the cardinality of the set of rectangular factors of size (m, n) occurring in U.

The analogue of Lemma 1 can be stated as follows: the word $W = \begin{bmatrix} w_{0,n} & \cdots & w_{m,n} \\ \vdots & & \vdots \\ w_{0,0} & \cdots & w_{m,0} \end{bmatrix}$ is a factor of the two-dimensional Sturmian word U if and only if

$$\bigcap_{0 \le i \le m, 0 \le j \le n} R_\alpha^{-i} R_\beta^{-j} I_{w_{i,j}} \neq \emptyset. \tag{3.3}$$

We first deduce that for a given (α, β), then the language of rectangular factors of U is here again the same for every x. We also deduce results concerning the counting of rectangular factors of a given size: there are exaclty $mn + m + n$ factors of size (m, n) in the two-dimensional Sturmian word U. We can not only deduce topological results from (3.3) but also metrical results: the frequencies of rectangular factors of size (m, n) of a two-dimensional Sturmian word take at most $min(m, n) + 5$ values [BV00b]. For more on two-dimensional Sturmian words, see [BV00a, BV00b, BV01, BT04].

Let us note that we have chosen in Definition 4 to restrict ourselves to rationally independent parameters. Usually in arithmetic discrete geometry, parameters are chosen to be integers. The results discussed above can also be obtained for standard arithmetic discrete planes $\mathfrak{P}(\mathbf{v}, \mu, \omega)$, whatever is the value taken by $\dim_{\mathbb{Q}}(v_1, v_2, v_3)$ (which can take the value $1, 2, 3$), either by direct application of Bezout's lemma if the parameters (v_1, v_2, v_3) are coprime integers, or from the density of the sequence $(n\alpha)_{n \in \mathbb{Q}}$, for α being assumed to be irrational. For more details, see the complete study performed in [Jam05b].

Note that we consider here standard arithmetic discrete planes. Recall that when replacing the norm $|| \, ||_1$ by the norm $|| \, ||_\infty$ in the definition of arithmetic discrete planes, one gets naive arithmetic discrete planes. The latter are usually considered in discrete geometry. Both notions are strongly related as shown e.g. in [SDC04], Theo. 1.

4. Functionality

Naive arithmetic discrete planes have been widely studied (e.g., see [DRR95, AAS97, VC99, Gér99b, Gér99a, VC00, Jac01, Jac02, BB02, BB05, BCK07, Kis04, AD08]) and are well known to be *functional*, i.e., in a one-to-one correspondence with the integer points of one of the coordinate planes by an orthogonal projection map. In other words, given a naive arithmetic discrete plane $\mathfrak{P}$ and the suitable coordinate plane, then for any integer point P of this coordinate plane, there exists a unique point of $\mathfrak{P}$ obtained from P by adding a third coordinate.

The aim of this section is first to show how to extend the notion of functionality for naive arithmetic discrete planes to a larger family of arithmetic discrete planes. Secondly, we deduce from the functionality a suitable coding of a dynamical system acting on the torus, in order to get information on local configurations, according to the strategy described in Section 2 and 3. The results we present here are from [BFJ05, BFJP07].

Instead of projecting on a coordinate plane, we introduce in Section 4.1 a suitable orthogonal projection map on a plane along a direction $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$, in some sense dual to the normal vector of the discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$. By dual, we mean here

$$\langle \boldsymbol{\alpha}, \mathbf{v} \rangle = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = \omega,$$

so that the projection of $\mathbb{Z}^3$ and the points of the discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ are in one-to-one correspondence (see Lemma 2 below). We then introduce in Section 4.2 the notion of local configurations and (m, n)-cubes which will play the role of factors.

One interest of the notion of functionality is to reduce a three-dimensional problem to a two-dimensional one, allowing a better understanding of the combinatorial and geometric properties of arithmetic discrete planes: this allows us, first, to recode in Section 4.3 arithmetic discrete planes by a two-dimensional word over the two-letter alphabet $\{0, 1\}$ (similarly as explained in Section 3), and second, to exhibit from this coding many geometric properties of arithmetic discrete planes (set of local configurations, enumeration of (m, n)-cubes, statistical properties...). This is the object of Section 4.4.

4.1. **Functional vectors.** An arithmetic discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is said to be *rational* if parameters $\mathbf{v}, \mu, \omega$ belong to $\mathbb{Z}$ or have integer entries. One easily checks that one can choose parameters satisfying $\mathbf{v} \in \mathbb{Z}^3$, $\mu \in \mathbb{Z}$, $\omega \in \mathbb{N}$ and $\gcd(v_1, v_2, v_3) = 1$.

Let $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ be an arithmetic discrete plane, and let $\boldsymbol{\alpha} \in \mathbb{Z}^3$ be such that $\gcd\{\alpha_1, \alpha_2, \alpha_3\} = 1$. Let $\pi_{\boldsymbol{\alpha}} : \mathbb{R}^3 \longrightarrow \{\mathbf{x} \in \mathbb{R}^3,\ \langle \boldsymbol{\alpha}, \mathbf{x} \rangle = 0\}$ be the affine orthogonal projection map onto the plane $\{\mathbf{x} \in \mathbb{R}^3;\ \langle \boldsymbol{\alpha}, \mathbf{x} \rangle = 0\}$ along the vector $\boldsymbol{\alpha}$.

Lemma 2. [BFJP07] *Let $\boldsymbol{\alpha} \in \mathbb{Z}^3$ be such that $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$. The map $\pi_{\boldsymbol{\alpha}} : \mathfrak{P}(\mathbf{v}, \mu, \omega) \longrightarrow \pi_{\boldsymbol{\alpha}}(\mathbb{Z}^3)$ is a bijection if and only if $|\langle \boldsymbol{\alpha}, \mathbf{v} \rangle| = \omega$.*

By Bezout's lemma, for any rational arithmetic discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$, with $\mathbf{v} \in \mathbb{Z}^3$, $\mu \in \mathbb{Z}$, $\omega \in \mathbb{N}$ and $\gcd\{v_1, v_2, v_3\} = 1$, then there exists a vector $\boldsymbol{\alpha} \in \mathbb{Z}^3$ such that $\langle \boldsymbol{\alpha}, \mathbf{v} \rangle = \omega$. A vector $\boldsymbol{\alpha} \in \mathbb{Z}^3$ is said *functional* if it satisfies conditions $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$ and $\langle \boldsymbol{\alpha}, \mathbf{v} \rangle = \omega$. Hence, any rational arithmetic discrete plane has *functional* vectors.

We will make in all that follows the following assumption: there exists a functional vector $\boldsymbol{\alpha} \in \mathbb{Z}^3$ for which *there exists* $i \in \{1, 2, 3\}$ *such that* $\alpha_i = 1$, *say* $\alpha_3 = 1$. Note that, since $\omega = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$, then the hypothesis $\alpha_3 = 1$ is equivalent to $\omega \in v_1\mathbb{Z} + v_2\mathbb{Z} + v_3$, i.e., $\omega - v_3 \in \gcd(v_1, v_2)\mathbb{Z}$.

There does not always exist a functional vector $\boldsymbol{\alpha}$ with $\alpha_3 = 1$. Consider for instance the case $\mathbf{v} = (6, 10, 15)$ with $\omega = 20$: it is impossible to express ω as $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$ with one of the α_i's equal to 1.

Let $\Gamma_{\boldsymbol{\alpha}}$ be the lattice obtained by projecting the arithmetic discrete plane $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ on the third coordinate plane along the functional vector $\boldsymbol{\alpha}$. Under the previous assumption ($\alpha_3 = 1$), one has $\Gamma_{\boldsymbol{\alpha}} = \mathbb{Z}\mathbf{e_1} + \mathbb{Z}\mathbf{e_2}$. Indeed, the map $\pi_{\boldsymbol{\alpha}}^{-1} : \Gamma_{\boldsymbol{\alpha}} \to \mathfrak{P}$ satisfies for all $\mathbf{y} \in \Gamma_{\boldsymbol{\alpha}}$ with $\mathbf{y} = y_1\mathbf{e_1} + y_2\mathbf{e_2}$:

$$\pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{y}) = \mathbf{y} - \left\lfloor \frac{v_1 y_1 + v_2 y_2 + \mu}{\omega} \right\rfloor \boldsymbol{\alpha}.$$

Hence this assumption provides an explicit and simple expression of the preimage of a point in $\Gamma_{\boldsymbol{\alpha}}$. We define the *height* $H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})$ at $\mathbf{y}$ as the third coordinate x_3 of $\mathbf{x} = \pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{y}) \in \mathfrak{P}$. One has

$$H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y}) = -\left\lfloor \frac{v_1 y_1 + v_2 y_2 + \mu}{\omega} \right\rfloor. \tag{4.1}$$

4.2. **Local configurations and (m, n)-cubes.** We want now to apply the functionality to the enumeration of (m, n)-cubes and local configurations, generalizing the study performed for naive planes in [VC97, Sch97, Gér99b, Gér99a, VC99, Jac02, AD08]. For the sake of consistency in the notation, we call them here $\mathbf{m}$-cubes with $\mathbf{m} = (m_1, m_2)$ rather than (m, n)-cubes.

Let $\mathfrak{P} := \mathfrak{P}(\mathbf{v}, \mu, \omega)$ be an arithmetic discrete plane satisfying the hypothesis of Section 4.1. Let $\boldsymbol{\alpha} \in \mathbb{Z}^3$ such that $\gcd(\alpha_1, \alpha_2, \alpha_3) = 1$ and $\langle \boldsymbol{\alpha}, \mathbf{v} \rangle = \omega$. We assume that $\alpha_3 = 1$ in all that follows, i.e., $\omega - v_3 \in v_1\mathbb{Z} + v_2\mathbb{Z}$.

Let $\mathbf{m} \in (\mathbb{N}^\star)^2$ be given. By $\mathbf{m}$-cube we mean a local configuration in the discrete plane that can be observed thanks to $\pi_{\boldsymbol{\alpha}}$ through an $\mathbf{m}$-window in the functional lattice $\Gamma_{\boldsymbol{\alpha}} = \mathbb{Z}\mathbf{e_1} + \mathbb{Z}\mathbf{e_2}$ (see Figure 4.2). More precisely, the $\mathbf{m}$-*cube* $\mathcal{C}(\mathbf{y}, \mathbf{m})$ of $\mathfrak{P}$ is defined as the following subset of $\mathfrak{P}$:

$$\mathcal{C}(\mathbf{y}, \mathbf{m}) = \left\{ \pi_{\boldsymbol{\alpha}}^{-1}(\mathbf{y} + \mathbf{z});\ \mathbf{z} \in [\![0, m_1 - 1]\!]\mathbf{e_1} + [\![0, m_2 - 1]\!]\mathbf{e_2} \right\}.$$

Two $\mathbf{m}$-cubes $\mathcal{C}$ and $\mathcal{C}'$ are said *translation equivalent* if there exists a vector $\mathbf{z} \in \mathbb{Z}^3$ such that $\mathcal{C}' = \mathcal{C} + \mathbf{z}$.

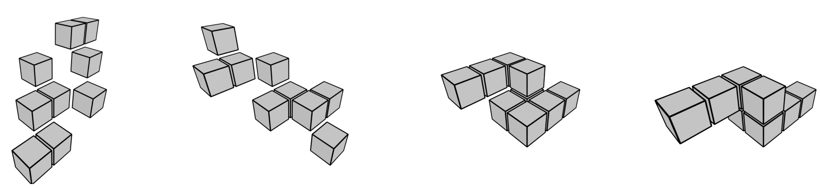

FIGURE 4.1. From left to right: the $(3, 3)$-cube of $\mathfrak{P}(\mathbf{v}, 0, 9)$ (resp. $\mathfrak{P}(\mathbf{v}, 0, 11)$, $\mathfrak{P}(\mathbf{v}, 0, 21)$, $\mathfrak{P}(\mathbf{v}, 0, 37)$) centered at $(0, 0, 0)$, where $\mathbf{v} = 6\mathbf{e_1} + 10\mathbf{e_2} + 15\mathbf{e_3}$, and projected along the vector $-\mathbf{e_1} + \mathbf{e_3}$ (resp. $\mathbf{e_1} - \mathbf{e_2} + \mathbf{e_3}$, $\mathbf{e_1} + \mathbf{e_3}$, $2\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$).

In order to enumerate the different types of $\mathbf{m}$-cubes that occur in $\mathfrak{P}$, that is, the different equivalence classes for the translation equivalence, we represent them as local configurations as follows. An $m_1 \times m_2$-rectangular word $L = [L_{i_1,i_2}]_{(i_1,i_2)\in[\![0,m_1-1]\!]\times[\![0,m_2-1]\!]}$ over the infinite alphabet $\mathbb{Z}$ is called an $\mathbf{m}$-*local configuration* of $\mathfrak{P}$ if there exists $\mathbf{y} \in \mathbb{Z}^2$ such that:

$$L = [H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{z}) - H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})]_{\mathbf{z}\in[\![0,m_1-1]\!]\mathbf{e_1}+[\![0,m_2-1]\!]\mathbf{e_2}}$$

where the height is defined in Equation (4.1).

Let us note that a local configuration is a plane partition. Indeed a *plane partition* of $N \in \mathbb{N}$ is a rectangular word $w = [w_{i_1,i_2}]_{(i_1,i_2)\in[\![0,m_1-1]\!]\times[\![0,m_2-1]\!]}$ over the infinite alphabet $\mathbb{N}$ satisfying $N = \sum_{i,j} w_{i,j}$ and, for all $i_1 \in [\![0, m_1-1]\!]$ and $i_2 \in [\![0, m_2 - 1]\!]$, $\max\{w_{i_1+1,i_2}, w_{i_1,i_2+1}\} \leq w_{i_1,i_2}$.

4.3. **A coding as a two-dimensional word.** Our stategy is now the following: we recode arithmetic discrete planes according to a two-dimensional word $U \in \{0,1\}^{\mathbb{Z}^2}$ over the two-letter alphabet $\{0,1\}$, namely a so-called *generalized Rote word* [Rot94], following the approach of [Vui99, BV01], and in the same flavour of the codings performed in Section 3. Such a two-dimensional word codes a $\mathbb{Z}^2$-action by two rotations with respect to a partition of the one-dimensional torus into two intervals of length 1/2. We then express $\mathbf{m}$-cubes as equivalence classes of rectangular factors of the two-dimensional word U, and show, for every $\mathbf{m} \in \mathbb{N}^2$, that the number of $\mathbf{m}$-cubes in $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ is computed by enumerating points on the one-dimensional torus.

Note that $\mathbf{m}$-cubes are subsets of arithmetic discrete planes whereas $\mathbf{m}$-local configurations are two-dimensional words over an infinite alphabet. To be able to get words over a finite alphabet, let us introduce a two-dimensional word coding in a natural way the parity of the heights $H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})$, for $\mathbf{y}$ in the lattice $\Gamma_{\boldsymbol{\alpha}} = \mathbb{Z}\mathbf{e_1} + \mathbb{Z}\mathbf{e_2}$, according to [Vui99].

Indeed, for a naive discrete plane $\mathfrak{P}$, it is well known that, given two points $\mathbf{x}$ and $\mathbf{x'}$ of $\mathfrak{P}$ such that their projections by $\pi_{\boldsymbol{\alpha}}$ are 4-connected in the functional plane, then $|x_3 - x_3'| \leq 1$. In other words, the difference between the heights of $\mathbf{x}$ and $\mathbf{x'}$ is at most 1.

A quite unexpected fact is that this property holds for any arithmetic discrete plane with $\alpha_3 = 1$. More precisely, it is easy to see that, for all $\mathbf{y} \in \Gamma_{\boldsymbol{\alpha}}$ and $i = 1, 2$, $H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y} + \mathbf{e_i}) - H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y})$ takes only two values, namely $-\lfloor v_i/\omega \rfloor$ and $-\lfloor v_i/\omega \rfloor - 1$. In each case, one of these values is odd, whereas the other one is even; we define E_1 and O_1 to be respectively the even and the odd value taken by $-\lfloor v_1/\omega \rfloor$ and $-\lfloor v_1/\omega \rfloor - 1$; we similarly define E_2 and O_2. It is now natural to introduce the following two-dimensional word of parity of heights by identifying $\Gamma_{\boldsymbol{\alpha}}$ to $\mathbb{Z}^2$:

$$U = (U_{i_1,i_2})_{(i_1,i_2)\in\mathbb{Z}^2} = (H_{\mathfrak{P},\boldsymbol{\alpha}}(\mathbf{y}) \mod 2)_{\mathbf{y}\in\mathbb{Z}^2} \in \{0,1\}^{\mathbb{Z}^2}. \tag{4.2}$$

The two-dimensional word U satisfies, for each $(i_1, i_2) \in \mathbb{Z}^2$

$$U_{i_1,i_2} = 0 \text{ if and only if } v_1 i_1 + v_2 i_2 + \mu \bmod 2\omega \in [0, \omega[.$$

Indeed, one checks that $U_{i_1,i_2} = 0$ if and only if $\left\lfloor \frac{v_1 i_1 + v_2 i_2 + \mu}{\omega} \right\rfloor$ is even, that is, $v_1 i_1 + v_2 i_2 + \mu \bmod 2\omega \in [0, \omega[$.

The word U is a *two-dimensional Rote word*; one-dimensional Rote words have been introduced in [Rot94]; they are defined as the infinite words u over the alphabet $\{0,1\}$ that have exactly $2n$ factors of length n for every positive integer n, and whose set of factors is closed under complementation, i.e., every word obtained by interchanging zeros and ones in a factor of the infinite word u is still a factor of u. Two-dimensional Rote words have been studied for instance in [Vui99, BV01].

Let us now make explicit the connection between local configurations and factors of U.

Let $W = [w_{i_1,i_2}]_{(i_1,i_2) \in [\![0, m_1-1]\!] \times [\![0, m_2-1]\!]}$ be a rectangular word of size $m_1 \times m_2$ over $\{0,1\}$. We define the *complement* $\overline{W}$ of W as follows:

$$\overline{W} = [\overline{w_{i_1,i_2}}]_{(i_1,i_2) \in [\![0, m_1-1]\!] \times [\![0, m_2-1]\!]}, \quad \text{where } \overline{1} = 0 \text{ and } \overline{0} = 1.$$

We introduce the following equivalence relation defined on the set of rectangular factors of U of a given size:

$$V \sim W \text{ if and only if } V \in \{W, \overline{W}\}.$$

There is a natural bijection between the equivalence classes of the relation $\sim$ on the rectangular factors of the two-dimensional word U of size $\mathbf{m} = (m_1, m_2)$ and the $\mathbf{m}$-local configurations of $\mathfrak{P}$; furthermore, the $\mathbf{m}$-local configurations of $\mathfrak{P}$ are in one-to-one correspondence with the translation equivalence classes of $\mathbf{m}$-cubes of $\mathfrak{P}$.

The following result holds, inspired by [Vui99] where it is stated under the assumption $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$. Lemma 3 plays here the role of our key lemma (Lemma 1).

Lemma 3. *Let $\mathfrak{P} := \mathfrak{P}(\mathbf{v}, \mu, \omega)$ be a rational arithmetic discrete plane with $\omega - v_3 \in v_1\mathbb{Z} + v_2\mathbb{Z}$.*

Let $W = [w_{i_1,i_2}]_{(i_1,i_2) \in [\![0, m_1-1]\!] \times [\![0, m_2-1]\!]}$ be a rectangular word of size $m_1 \times m_2$ over $\{0,1\}$. Let $I_0 = [0, \omega[$ and $I_1 = [\omega, 2\omega[$. Let

$$I_W = \bigcap_{i_1=0}^{m_1-1} \bigcap_{i_2=0}^{m_2-1} \left(I_{w_{i_1,i_2}} - (v_1 i_1 + v_2 i_2) \mod 2\omega \right).$$

The set I_W is a left-closed right-open interval of $[0, 2\omega[$.

- *If $\dim_{\mathbb{Q}}(v_1, v_2, v_3) > 1$ or $\mathfrak{P}$ is rational and $\gcd(v_1, v_2, 2\omega) = 1$, then a rectangular word W over $\{0,1\}$ is a factor of U if and only if $I_W \neq \emptyset$.*
- *Otherwise, if $\mathfrak{P}$ is rational and $\gcd(v_1, v_2, 2\omega) = 2$, then a rectangular word W over $\{0,1\}$ is a factor of U if and only if I_W contains an integer with the same parity as μ.*

4.4. **Enumeration of local configurations.** Let us now investigate the enumeration of $\mathbf{m}$-cubes ($\mathbf{m} = (m_1, m_2)$) occuring in a given arithmetic plane. The number of $(3,3)$-cubes included in a given rational naive arithmetic discrete plane has been proved to be at most 9 in [VC97]. More generally, in [Rev95, Gér99b], the authors proved that a rational naive arithmetic discrete plane contains at most $m_1 m_2$ $\mathbf{m}$-cubes (to be more precise, translation equivalence classes of $\mathbf{m}$-cubes). In [Gér99b] local configurations which are non-necessarily rectangular are also considered. In the following theorem, we show that this property also holds in our framework. For the sake of simplicity, we omit to mention that we consider translation equivalence classes of $\mathbf{m}$-cubes:

Theorem 2. *Let* $\mathfrak{P} := \mathfrak{P}(\mathbf{v}, \mu, \omega)$ *be a discrete plane with* $\omega - v_3 \in v_1\mathbb{Z} + v_2\mathbb{Z}$. *Let* $\mathbf{m} = (m_1, m_2) \in (\mathbb{N}^\star)^2$. *Then,* $\mathfrak{P}$ *contains at most* $m_1 m_2$ $\mathbf{m}$*-cubes. More precisely, one has:*

(1) *If* $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 1$, $\mathbf{v} \in \mathbb{Z}^3$, $\mu \in \mathbb{Z}$, $\omega \in \mathbb{Z}$ *and* $\gcd(\mathbf{v}) = 1$, *then* $\mathfrak{P}$ *contains at most* ω $\mathbf{m}$*-cubes for every* $\mathbf{m} = (m_1, m_2) \in (\mathbb{N}^\star)^2$. *Moreover, for* m_1 *and* m_2 *large enough,* $\mathfrak{P}$ *contains exactly* ω $\mathbf{m}$*-cubes.*
(2) *Let us assume* $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 2$. *Let* $(p_1, p_2) \in \mathbb{Z}^2$ *be a generator of the lattice of periods of the two-dimensional word* U. *Then* $\mathfrak{P}$ *contains at most* $m_1|p_2| + m_2|p_1| - \min\{m_1, |p_1|\}\min\{m_2, |p_2|\}$ $\mathbf{m}$*-cubes for* $(m_1, m_2) \in \mathbb{N}^2$.
(3) *If* $\dim_{\mathbb{Q}}(v_1, v_2, v_3) = 3$, *then* $\mathfrak{P}$ *contains exactly* $m_1 m_2$ $\mathbf{m}$*-cubes for every* $\mathbf{m} = (m_1, m_2) \in (\mathbb{N}^\star)^2$.

Let us note that the bounds for m_1 and m_2 upon which the previous results hold (cases (1) and (2) in Theorem 2) can be explicitly computed in terms of $\mathbf{v}$ and ω. The proof is a direct application of Lemma 3. For more details, see [BFJP07].

We thus can establish that the computation of the frequency of occurrence of an $\mathbf{m}$-cube of $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ can be reduced to the calculation of the length of an interval of the torus $\mathbb{R}/\omega\mathbb{Z}$. For more details, see [BFJP07]. We also investigate in [BFJP07] the closure of the set of $\mathbf{m}$-cubes of $\mathfrak{P}(\mathbf{v}, \mu, \omega)$ under the action of a particular geometric transformation: the *centrosymmetry*.

5. Stepped surfaces

Let us generalize the codings as two-dimensional words introduced in Section 3 for arithmetic discrete planes to more general discrete objects, namely the *functional stepped surfaces*, such as introduced in [Jam04]. See also [Jam05a, JP05, Jam05b, ABFJ07].

A *functional discrete surface* is defined as a union of pointed faces E_i, for $i = 1, 2, 3$ (defined in Section 3) such that the orthogonal projection π_0 onto the antidiagonal plane $\Delta\colon x_1 + x_2 + x_3 = 0$ induces an homeomorphism from the discrete surface onto Δ.

As done for functional arithmetic discrete planes, one then provides a discrete surface with a coding as a two-dimensional word over a three-letter alphabet [Jam04, JP05]. Indeed, let $\mathcal{S}$ be a functional stepped surface. One has

$$\mathcal{S} \cap \mathbb{Z}^3 = \{\mathbf{x};\, \exists\, i \text{ such that } \mathbf{x} + E_i \subset \mathcal{S}\}.$$

Furthermore, given $(m_1, m_2) \in \mathbb{Z}^2$, there exists a unique face $\mathbf{x}+E_i \subset \mathcal{S}$ such that $(m_1, m_2) = \pi_0(\mathbf{x}+E_i)$. The following coding is thus well-defined: a two-dimensional word $U \in \{1,2,3\}^{\mathbb{Z}^2}$ is said to be the *coding* of the functional stepped surface $\mathcal{S}$ if for all $(m_1, m_2) \in \mathbb{Z}^2$ and for every $i \in \{1,2,3\}$:

$$U_{m_1,m_2} = i \iff \exists(\mathbf{x}, i), \text{such that } \mathbf{x} + E_i \in \mathcal{S},\ \pi_0(\mathbf{x}) = (m_1, m_2).$$

We illustrate this with the following figure where a piece of a discrete surface in $\mathbb{R}^3$ is depicted, as well as its orthogonal projection π_0 onto the plane $\Delta\colon x_1 + x_2 + x_3 = 0$, and its coding as a two-dimensional word over a three-letter alphabet.

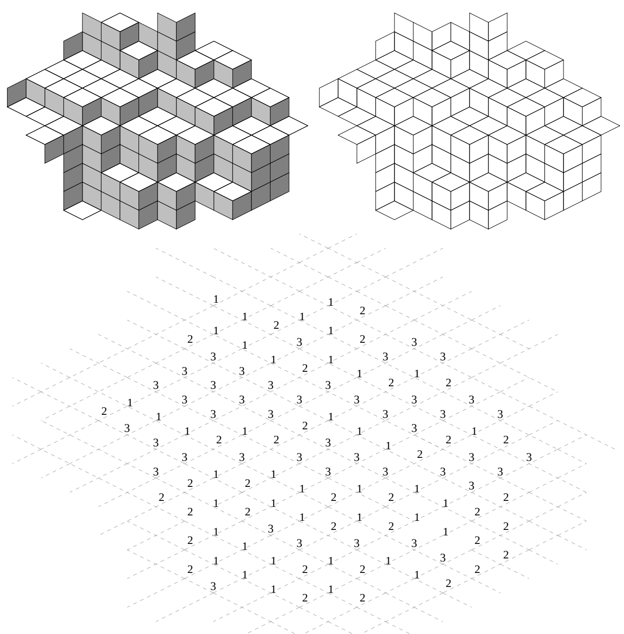

FIGURE 5.1. From discrete surfaces to multidimensional words via tilings

Let us quote the following nice characterization of codings of discrete surfaces [Jam04]. Let $U \in \{1,2,3\}^{\mathbb{Z}^2}$. Then U is a coding of a discrete surface if and only if the factors of U of the shape given in Figure 5.2 are included in the set of factors depicted in Figure 5.2. The main difference between a stepped surface and a stepped plane is thus that it is possible to locally recognize whether a set of points in $\mathbb{Z}^3$ is a subset of the set of vertices of a stepped surface.

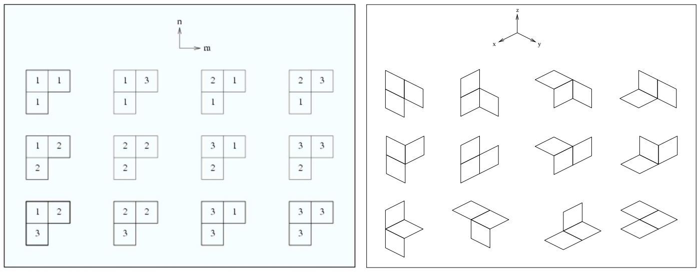

FIGURE 5.2. Permitted factors and their 3-dimensional representation.

6. From discrete to continuous structures

The aim of this section, based on the surveys [Lot05, BS05, BBLT06], is to show how to generate standard arithmetic discrete planes by means of a *generalized substitution*. We work out here in details the example of the *Tribonacci substitution*.

6.1. **The Tribonacci substitution.** Let $\mathcal{A}$ be a finite set. As usual in word combinatorics, we denote by $\mathcal{A}^*$ the set of words over $\mathcal{A}$ and by ε the empty word. The set $\mathcal{A}^*$ endowed with the concatenation map is a free monoid. A *substitution* is an endomorphism of the free monoid $\mathcal{A}^*$. A substitution naturally extends to the set of one-sided words $\mathcal{A}^{\mathbb{N}}$. A *fixed point* of σ is a word $u = (u_i)_{i\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ that satisfies $\sigma(u) = u$.

We consider the *Tribonacci substitution* $\sigma : \{1, 2, 3\}^* \to \{1, 2, 3\}^*$ defined on the letters of the alphabet $\{1, 2, 3\}$ as follows: $\sigma : 1 \mapsto 12,\ 2 \mapsto 13,\ 3 \mapsto 1$. The *Tribonacci word* is the (unique) fixed point of the substitution σ. More precisely, by noticing that $\sigma^j(1)$ is a nontrivial prefix of the word $\sigma^{j+1}(1)$, the sequence of words $1,\ \sigma(1),\ \sigma^2(1),\ \dots, \sigma^n(1),\ \dots$ is easily seen to converge to an infinite word denoted by $\sigma^\omega(1)$. The first terms of this word are

$$1\ 2\ 1\ 3\ 1\ 2\ 1\ 1\ 2\ 1\ 3\ 1\ 2\ 1\ 2\ 1\ \cdots$$

Note that the length, denoted by $|\sigma^j(1)|$, of $\sigma^j(1)$ satisfies the Tribonacci recurrence: $|\sigma^{j+3}(1)| = |\sigma^{j+2}(1)| + |\sigma^{j+1}(1)| + |\sigma^j(1)|$, for every $j \in \mathbb{N}$, hence the terminology.

The Tribonacci substitution has been introduced and studied in [Rau82]. For more results and references on the Tribonacci substitution, see [AR91a, AY81, IK91, Lot05, Mes98, Mes00, PF02].

The *incidence matrix* $\mathbf{M}_\sigma = (m_{i,j})_{1\le i,j\le n}$ of a substitution σ has entries $m_{i,j} = |\sigma(j)|_i$, where the notation $|w|_i$ stands for the number of occurrences of the letter i in the word w. A substitution σ is called *primitive* if there exists an integer n such that $\sigma^n(a)$ contains at least one occurrence of the letter b for every pair $(a, b) \in \mathcal{A}^2$. This is equivalent to the fact that its incidence matrix is primitive, i.e., there exists a nonnegative integer n such that $\mathbf{M}_\sigma^n$ has only positive entries.

If a substitution σ is primitive, then the *Perron-Frobenius theorem* ensures that the incidence matrix $\mathbf{M}_\sigma$ has a simple real positive dominant eigenvalue β, which admits as associated right and left eigenvectors vectors with positive entries. A substitution σ is called *unimodular* if det $\mathbf{M}_\sigma = \pm 1$. A substitution σ is said to be *Pisot* if its incidence matrix $\mathbf{M}_\sigma$ has a real dominant eigenvalue $\beta > 1$ such that, for every other eigenvalue λ, one has $0 < |\lambda| < 1$. The characteristic polynomial of the incidence matrix of such a substitution is irreducible over $\mathbb{Q}$, and the dominant eigenvalue β is a Pisot number (that is, an algebraic integer with all Galois conjugates having modulus less than 1). Furthermore, it can be proved that Pisot substitutions are primitive [PF02].

The *incidence matrix* of the Tribonacci substitution σ is $M_\sigma = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. This matrix is easily seen to be primitive. The characteristic polynomial of M_σ is $X^3 - X^2 - X - 1$; this polynomial admits one positive root $\beta > 1$ (the dominant eigenvalue) and two complex conjugates α and $\overline{\alpha}$, with $|\alpha| < 1$. Hence the Tribonacci substitution is Pisot and the number β is a Pisot number. The matrix M_σ admits as eigenspaces in $\mathbb{R}^3$ one *expanding eigenline* (generated by the eigenvector with positive coordinates $\mathbf{u}_\beta = (1/\beta, 1/\beta^2, 1/\beta^3)$ associated with the eigenvalue β) and a real *contracting eigenplane* $\mathbb{H}_c$. Let $\mathbf{v}_\beta$ be the left eigenvector of $\mathbf{M}_\sigma$ with positive entries normalized so that $\langle \mathbf{v}_\beta, \mathbf{u}_\beta \rangle = 1$. The contracting plane $\mathbb{H}_c$ has equation $\langle \mathbf{x}, \mathbf{v}_\beta \rangle = 0$.

One associates with the Tribonacci word $u = (u_n)_{n \geq 0}$ a broken line starting from 0 in $\mathbb{Z}^3$ and approximating the expanding line generated by $\mathbf{u}_\beta$ as follows. We introduce the *abelianization map* f of the free monoid $\{1,2,3\}^*$ defined by

$$f : \{1,2,3\}^* \to \mathbb{Z}^3, \; f(w) = |w|_1 \mathbf{e}_1 + |w|_2 \mathbf{e}_2 + |w|_3 \mathbf{e}_3,$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ stands for the canonical basis of $\mathbb{R}^3$. Note that for every finite word w, we have $f(\sigma(w)) = \mathbf{M}_\sigma f(w)$.

The *Tribonacci broken line* is defined as the broken line which joins with segments of length 1 the points $f(u_0 u_1 \cdots u_{N-1})$, $N \in \mathbb{N}$ (see Figure 6.1). In other words we describe this broken line by starting from the origin, and then by reading successively the letters of the Tribonacci word u, going one step in direction $\mathbf{e}_i$ if one reads the letter i.

One easily deduces from the fact that σ is a Pisot substitution that the vectors $f(u_0 u_1 \ldots u_N)$, $N \in \mathbb{N}$, stay within bounded distance of the expanding line of $\mathbf{M}_\sigma$, which is exactly the direction given by the vector of probabilities of occurrence of the letters $1, 2, 3$ in u. One can consider this broken line as a discrete approximation of the line generated by the vector $\mathbf{u}_\beta$. It is then natural to try to represent these points by projecting them along the expanding direction onto a transverse plane, that we chose here to be the contracting plane $\mathbb{H}_c$ of $\mathbf{M}_\sigma$.

Let π stand for the projection in $\mathbb{R}^3$ onto the contracting plane along the expanding line generated by the vector $\mathbf{v}_\beta$. We thus define the set $\mathcal{R}_\sigma$ as the closure of the projections of the vertices of the Tribonacci broken line:

$$\mathcal{R}_\sigma := \overline{\{\pi(f(u_0 \dots u_{N-1}));\ N \in \mathbb{N}\}}.$$

The set $\mathcal{R}_\sigma$ is called the *Rauzy fractal* associated with the Tribonacci substitution σ (see Figure 6.1). It can be divided into three pieces, called *basic pieces*, defined for $i = 1, 2, 3$ as

$$\mathcal{R}_\sigma(i) = \overline{\{\pi(f(u_0 \dots u_{N-1}));\ u_N = i,\quad N \in \mathbb{N}\}}.$$

One checks that the Rauzy fractal is a compact set, that is the closure of its interior; it has a non-zero measure, a fractal boundary and it is the attractor of some graph-directed iterated function system [Rau82]. Furthermore, the pieces $\mathcal{R}_\sigma(i)$, for $i = 1, 2, 3$ are disjoint in measure.

One interesting feature of Rauzy fractal is that it can tile the plane in two different ways [Rau82, IR06]. These two tilings are depicted in Figure 6.2. The first one corresponds to a periodic tiling (a lattice tiling), and the second one to a self-substitutive tiling.

By *tiling* of $\mathbb{R}^d$, we mean here tilings by translation having finitely many tiles up to translation (a tile is assumed to be the closure of its interior): there exist a finite set of tiles $\mathcal{T}_i$ and a finite number of translation sets Γ_i such that

$$\mathbb{R}^d = \cup_i \cup_{\gamma_i \in \Gamma_i} \mathcal{T}_i + \gamma_i,$$

and distinct translates of tiles have non-intersecting interiors; we assume furthermore that each compact set in $\mathbb{R}^d$ intersects a finite number of tiles. By *lattice tiling*, we mean that there exists a lattice Γ such that

$$\mathbb{R}^d = \cup_{\gamma \in \Gamma} \mathcal{T} + \gamma,$$

where $\mathcal{T} = \cup_i \mathcal{T}_i$.

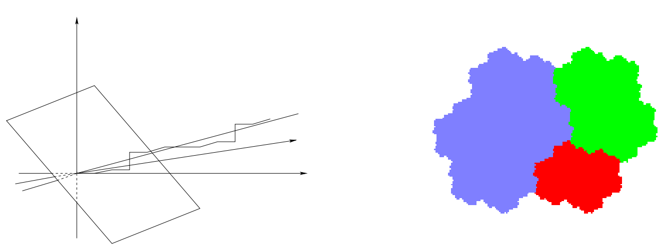

FIGURE 6.1. The Tribonacci broken line and the Rauzy fractal.

6.2. **Discrete planes and tilings.** The self-substitutive Tribonacci tiling depicted in Figure 6.2 has close connections with arithmetic discrete planes. We consider indeed the standard lower arithmetic discrete plane with parameter $\mu = 0$ associated with the left eigenvector $\mathbf{v}_\beta$ that we denote for

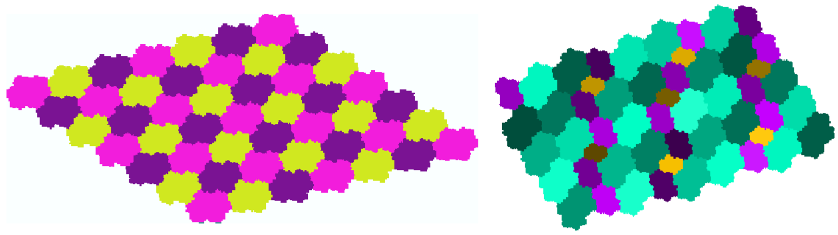

FIGURE 6.2. Lattice and self-substitutive Tribonacci tilings.

short by $\mathfrak{P}_\sigma$. Let us recall that $\mathbf{v}_\beta$ has positive entries. One has

$$\mathfrak{P}_\sigma = \{\mathbf{x} \in \mathbb{Z}^3;\ \ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < ||\mathbf{v}_\beta||_1 = \sum_{i=1,2,3} \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle\}.$$

We also consider the stepped plane $\mathcal{P}_\sigma$ associated with it, such as defined in Section 3. This discretization of the contracting hyperplane $\mathbb{H}_c = \{\mathbf{x} \in \mathbb{Z}^3;\ \langle \mathbf{x}, \mathbf{v}_\beta \rangle = 0\}$ consists in approximating the plane $\mathbb{H}_c$ by selecting points with integral coordinates above and within a bounded distance of the plane $\mathbb{H}_c$. It thus can be considered as the *dual* of the broken line which gives an approximation of the line generated by the eigenvector $\mathbf{u}_\beta$.

It will prove here to be more convenient to use the following set of faces (instead of faces of type E_i introduced in Section 3): for $1 \leq i \leq 3$, we define

$$F_i := \{\sum_{j \neq i} \lambda_j \mathbf{e}_j;\ 0 \leq \lambda_j \leq 1, \ \ \text{for } 1 \leq j \leq 3,\ j \neq i\}.$$

One thus checks that the stepped plane $\mathcal{P}_\sigma$ is spanned as follows:

$$\mathcal{P}_\sigma = \bigcup_{(\mathbf{x},i) \in \mathbb{Z}^3 \times \{1,2,3\},\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \mathbf{x} + F_i. \tag{6.1}$$

This union is a disjoint union up to the boundaries of the faces.

Let us first project the stepped plane $\mathcal{P}_\sigma$ by π onto the contracting space $\mathbb{H}_c$. One gets a first tiling of $\mathbb{H}_c$ by three kinds of lozenges such as illustrated in Figure 1.1. Let us now replace each face $\mathbf{x} + F_i$ by the corresponding basic piece of the Rauzy fractal $\mathcal{R}_\sigma(i)$. Equation (6.1) becomes

$$\mathbb{H}_c = \bigcup_{(\mathbf{x},i) \in \mathbb{Z}^3 \times \{1,2,3\},\ 0 \leq \langle \mathbf{x}, \mathbf{v}_\beta \rangle < \langle \mathbf{e}_i, \mathbf{v}_\beta \rangle} \pi(\mathbf{x}) + \mathcal{R}_\sigma(i). \tag{6.2}$$

According to [Rau82] and [IR06], (6.2) provides also a tiling of the contracting plane $\mathbb{H}_c$, namely the self-substitutive tiling depicted in Figure 6.2.

Let us describe now a generation process for $\mathfrak{P}_\sigma$ based on the notion of generalized substitution due to [AI01], see also [IR06], which explains the terminology *self-substitutive*.

We define $\mathcal{F}^*$ as the set of unions of faces of type $\mathbf{x} + F_i$, for $\mathbf{x} \in \mathbb{Z}^3$, and $i \in \{1, 2, 3\}$. We define the following geometric realization of the substitution

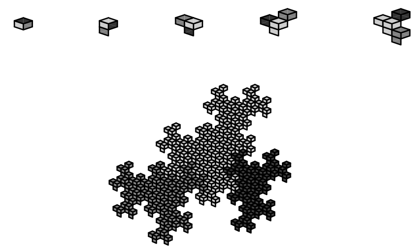

FIGURE 6.3. Generation of $\mathcal{P}_\sigma$ by iterates of $E_1^*(\sigma)$.

σ on the set $\mathcal{F}^*$:

$$\forall(\mathbf{x}, i) \in \mathbb{Z}^n \times \{1,2,3\},\ E_1^*(\sigma)(\mathbf{x}+F_i) = \cup_{\sigma(j)=pis} \mathbf{M}_\sigma^{-1}(\mathbf{x}+\mathbf{l}(p)) + F_j,$$

$$\text{for all } \mathcal{G}_1, \mathcal{G}_2 \subset \mathcal{F}^*,\ \text{then } E_1^*(\mathcal{G}_1 \cup \mathcal{G}_2) = E_1^*(\mathcal{G}_1) \cup E_1^*(\mathcal{G}_2).$$

Theorem 3. [AI01] *Let σ be the Tribonacci substitution. The stepped plane $\mathcal{P}_\sigma$ is stable under the action of $E_1(\sigma)^*$ and contains the unit cube*

$$\mathcal{U} := F_1 \cup F_2 \cup F_3.$$

The iterates $(E_1(\sigma)^*)^n(\mathcal{U})$ all belong to $\mathcal{P}_\sigma$, and they generate larger and larger pieces of the stepped plane $\mathcal{P}_\sigma$. By taking the limit and by projecting by π, one gets

$$\mathbb{H}_c = \lim_{n\to+\infty} \pi(E_1(\sigma)^*)^n(\mathcal{U}).$$

After projection and renormalization, the sequence of pieces $\mathbf{M}_\sigma^n \pi(E_1(\sigma)^*)^n(\mathcal{U})$ is convergent and its limit is equal to the Rauzy fractal:

$$\mathcal{R}_\sigma = \lim_{n\to+\infty} \mathbf{M}_\sigma^n \pi(E_1(\sigma)^*)^n(\mathcal{U}).$$

Hence the vertices of the pieces $(E_1(\sigma)^*)^n(\mathcal{U})$ generate the arithmetic discrete plane $\mathfrak{P}_\sigma$ such as illustrated on Fig. 6.2. For more on generalized substitutions and generation of discrete planes, see [ABI02, ABS04, ABFJ07, Fer06, Fer07].

6.3. **Rauzy tilings.** We have seen that two tilings can be associated with the Rauzy fractal, namely, a self-substitutive tiling, and a lattice tiling, as illustrated in Figure 6.2. This latter tiling plays an important role in the spectral study of the substitutive dynamical system (X_σ, S) generated by the Tribonacci word (such as defined in Section 2). Indeed, one of the main incentives behind the introduction of Rauzy fractals is the following result:

Theorem 4. [Rau82] *Let σ be the Tribonacci substitution $\sigma\colon 1 \mapsto 12,\ 2 \mapsto 13, 3 \mapsto 1$. The Rauzy fractal $\mathcal{R}_\sigma$ (considered as a subset of $\mathbb{R}^2$) is a fundamental domain of $\mathbb{T}^2$. Let $R_\beta\colon \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto x + (1/\beta, 1/\beta^2)$. The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the toral translation $(\mathbb{T}^2, R_\beta)$.*

In other words, the Tribonacci word u is a coding with respect to the partition of the two-dimensional torus $\mathbb{T}^2$ by the three pieces $\mathcal{R}_\sigma(i)$, for $i = 1, 2, 3$ of an orbit of a point of $\mathbb{T}^2$ under the action of the translation R_β.

We now can explain what kind of discrete approximation the Tribonacci broken line provides for the line generated by the vector $\mathbf{u}_\beta$. As a consequence of Theorem 4, one proves that the only points in $\mathbb{Z}^3$ whose projection by π belongs to the interior of $\mathcal{R}_\sigma$ are the vertices of the broken line. Hence the broken line is obtained by a selection process which consists in shifting the Rauzy fractal along the direction $\mathbf{u}_\beta$ and selecting points in $\mathbb{Z}^3$ in this stripe. Note that the words associated with broken lines obtained by shifting the unit cube along a given direction are called *billiard words* (see for instance [AMST94, Bar95]). Generalizations of the Tribonacci word are given by the so-called *Arnoux-Rauzy words* [AR91b] and more generally by the family of *episturmian words*. Billiard words and episturmian words are two widely studied families of infinite words in word combinatorics.

The Tribonacci lattice tiling has been widely studied and presents many interesting features. In particular, the Tribonacci central tile has a "nice" topological behavior (0 is an inner point and it is shown to be connected with simply connected interior [Rau82]), which leads to interesting applications in Diophantine approximation [CHM01] where points of the broken line corresponding to $\sigma^n(1)$, $n \in \mathbb{N}$, are proved to produce best approximations for the vector $(\frac{1}{\beta}, \frac{1}{\beta^2})$ for a given norm associated with the matrix $\mathbf{M}_\sigma$. See also [HM06] for a similar study in the case of a family of cubic Pisot numbers with complex conjugates.

Rauzy fractals can more generally be associated with Pisot substitutions (see [BK06, CS01a, CS01b, IR06, Mes00, Mes02, Sie03, Sie04] and the surveys [BS05, PF02]), as well as with Pisot β-shifts under the name of *central tiles* (see [Aki98, Aki99, Aki00, Aki02]), but they also can be associated with abstract numeration systems [BR05], as well as with some automorphisms of the free group [ABHS06]. Theorem 4 is expected to hold in this context: this is the so-called Pisot conjecture.

Conjecture 1. Let σ be a Pisot unimodular substitution. The following equivalent conditions are conjectured to hold:

(1) the symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to a translation on the torus;
(2) (X_σ, S) has a pure discrete spectrum;
(3) the associated Rauzy fractal $\mathcal{R}_\sigma$ generates a lattice tiling, i.e.,

$$\mathbb{K}_\beta = \cup_{\gamma \in \Gamma}(\mathcal{R}_\sigma + \gamma),$$

with the union being disjoint in measure, and Γ being a lattice.

The conjecture holds true for two-letter alphabets [BD02, HS03, Hos92]. Substantial literature is devoted to Conjecture 1 which is reviewed in [PF02], Chap.7. See also [BK06, BK05, BBK06, BS05, IR06] for recent results.

6.4. **Back to stepped planes.** Generalized substitutions (introduced in Section 6.2) are proved in [ABFJ07] to act not only on stepped planes, but also on stepped surfaces. Furthermore, a geometric version of Brun multidimensional continued fraction algorithm acting on stepped surfaces is given in [BF08] in terms of generalized substitutions. This geometric extension of the Brun algorithm is motivated by the discrete plane recognition problem: given a set of points in $\mathbb{Z}^d$, is there a naive arithmetic discrete plane that contains it? A strategy based on multidimensional continued fractions inspired by the one-dimensional Sturmian case is thus given in [BF08, Fer08].

Indeed, in the one-dimensional case, there exists a natural strategy for the recognition problem based on word combinatorics. Let us recall that a substitution is a morphism of the free monoid whereas an S-adic word is an infinite word generated as the limit of an infinite composition of a finite number of substitutions (for more details, see Chap. 12 in [PF02]). Sturmian words are proved to be S-adic words; the rules for the iteration of these substitutions follow the continued fraction of the slope of the line which is coded. We deduce from the combinatorial properties of Sturmian words the following two facts: first, factors 00 and 11 cannot occur simultaneoulsy in a Sturmian word, that is, one of the two letters 0 and 1 occurs as an isolated letter. Hence, up to a prefix of length 1, any infinite Sturmian word can be written as $\sigma_0(v)$ or $\sigma_1(v)$, where v is an infinite word over $\{0, 1\}$, and the substitutions σ_0 and σ_1 are defined as $\sigma_0\colon 0 \mapsto 0$, $\sigma_0\colon 1 \mapsto 10$ and $\sigma_1\colon 0 \mapsto 01$, $\sigma_1\colon 1 \mapsto 1$. Secondly, we use the fact that v is itself a Sturmian word. We can thus reiterate the process. Suppose now we are given a connected union of translates of horizontal and vertical segments with integer vertices and length 1. We apply the previous process to the finite word coding this union of segments by taking care of the boundaries. This corresponds to the method developed in discrete geometry terms in [Wu82, Tro93].

Let us note that the recognition problem is classical and central in the field of discrete geometry for the segmentation of discrete surfaces and for polyhedrization issues, for instance. Indeed, numerous applications can be derived in image analysis and synthesis, volume modeling, pattern recognition, etc. There exist various strategies for the discrete plane recognition problem, as described, for instance, in the survey [BCK07]. These methods are based on linear programming, on computational geometry by performing separability tests, or on the so-called preimage technique which consists in determining the set of parameters of the arithmetic discrete planes that contain the given set of points.

7. Conclusion

Let us conclude by giving a brief list of geometric discretizations that can be described by symbolic codings of dynamical systems.

• Standard arithmetic discrete lines and Sturmian words are particular codings of rotations over the one-dimensional torus $\mathbb{T}$ with respect to a two-interval partition, one interval having as length the parameter of the rotation.

• Similarly, standard arithmetic discrete planes and two-dimensional Sturmian words are codings of a $\mathbb{Z}^2$-action by rotations over the one-dimensional torus $\mathbb{T}$ with respect to a three-interval partition, with two intervals having as respective length the parameters of the $\mathbb{Z}^2$-action.

• More generally, functional arithmetic disrete planes can be coded thanks to generalized Rote words defined as codings of a $\mathbb{Z}^2$-action by rotations over the one-dimensional torus $\mathbb{T}$ with respect to a two-interval partition, with two intervals of the same length. For more examples of codings associated with naive or standard arithmetic discrete planes expressed in terms of dynamical systems, see [Jam05b] where codings by remainders, by umbrellas and by parity of heights are considered.

• In a dual way, we have seen how to associate with the Tribonacci substitution a broken line that can be considered as a discrete line in $\mathbb{R}^3$. A lattice tiling by the Rauzy fractal can then be produced that has close connection with a rotation on the two-dimensional torus $\mathbb{T}^2$.

• Lastly, let us quote [BN07] as an example of a symbolic coding of discrete rotations defined as the composition of Euclidean rotations with a rounding operation, as studied in [NR03, NR04, NR05]. Indeed, it is possible to encode all the information concerning a discrete rotation as two multidimensional words C_α and C'_α called configurations. These configurations C_α and C'_α can be coded by discrete dynamical systems defined by a $\mathbb{Z}^2$-action on the two-dimensional torus $\mathbb{T}^2$. As a consequence, results concerning densities of occurrence of symbols can be deduced.

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