MEYER SETS, PISOT NUMBERS, AND SELF-SIMILARITY IN SYMBOLIC DYNAMICAL SYSTEMS

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ABSTRACT. Aperiodic order refers to the mathematical formalisation of quasicrystals. Substitutions and cut and project sets are among their main actors; they also play a key role in the study of dynamical systems, whether they are symbolic, generated by tilings, or point sets. We focus here on the relations between quasicrystals and self-similarity from an arithmetical and dynamical viewpoint, illustrating how efficiently aperiodic order irrigates various domains of mathematics and theoretical computer science, on a journey from Diophantine approximation to computability theory. In particular, we see how Pisot numbers allow the definition of simple model sets, and how they also intervene for scaling factors for invariance by multiplication of Meyer sets. We focus in particular on the characterisation due to Yves Meyer of Pisot and Salem numbers as the parameters of the dilations preserving some Meyer set.

1. Introduction

In his beautiful text from the book devoted to the Abel Laureates [HP19], Y. Meyer recalls: "I read the extraordinary book *Ensembles parfaits et Séries trigonométriques* by J.P. Kahane and R. Salem. I was enthusiastic. The role played by Pisot numbers in the problem of uniqueness for trigonometric series had been discovered by Salem and Zygmund and was detailed in this book. (...) I was fascinated by the interplay between number theory and harmonic analysis which is so profound and beautiful in this result. I decided to extend the work of Salem and Zygmund to the problem of spectral synthesis." Thus Yves Meyer began a journey that would culminate in the two books [Mey70, Mey72], and that would lead him to quasicrystals and tilings via model sets.

We recall that a real algebraic integer $\beta > 1$ is a *Pisot number* (also called a Pisot-Vijayaraghavan number) if all of its other algebraic conjugates λ satisfy $|\lambda| < 1$. Yves Meyer thus started his journey by extending the work of Salem and Zygmund to the problem of *spectral synthesis* for the Cantor set E_β with $\beta > 2$, where

$$E_\beta := \left\{ \sum_i \varepsilon_i \beta^{-i} \text{ with } \varepsilon_i \in \{0,1\} \right\}.$$

The aim of spectral synthesis is to recreate a function from its Fourier coefficients: a compact set Γ is called a set of spectral synthesis if any continuous bounded function f in L^1 with spectrum in Γ (i.e., whose Fourier transform has support in Γ), can be approximated in the weak star topology by the set $\mathcal{P}_\Gamma = \{\sum_{\gamma\in\Gamma} a_\gamma e^{2i\pi\gamma x}\}$ of finite trigonometric polynomials with support in Γ. As was similarly done for the case of sets of uniqueness, namely those sets outside of which a Fourier series uniquely determines a function, Yves Meyer showed

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that Pisot numbers also play a crucial role in the problem of spectral synthesis. Indeed, for $\beta > 2$, Yves Meyer proved the striking fact that E_β is a set of spectral synthesis if β is a Pisot number. Strong spectral synthesis even holds with respect to uniform convergence on compact sets, as stated below.

Theorem 1.1. [Mey72, Theorem V, Chapter VII] *Let $\beta > 2$ be a Pisot number. For each bounded continuous function $f : \mathbb{R} \to \mathbb{C}$ whose spectrum lies in E_β, there exists a sequence $(P_k)_k$ of finite trigonometric sums such that*

- *for each k, the frequencies of P_k (i.e., its spectrum) belong to the set of finite sums $\{\sum_{j=1}^k \varepsilon_j \beta^{-j}$ with $\varepsilon_i \in \{0,1\}\}$,*
- *the sequence $(P_k)_k$ converges uniformly to f on each compact subset of $\mathbb{R}$, and*
- *$\sup_{\mathbb{R}} |P_k]$ converges to $\sup_{\mathbb{R}} |f|$.*

This resolved a longstanding conjecture of Salem and Zygmund and extended the result that Carl Herz had obtained for the triadic Cantor set E_3. In addition, the tools that Yves Meyer developed allowed him to discern the relationship between the arithmetical nature of a set Λ, not necessarily compact, and the properties of the finite trigonometric polynomials with support in Λ. As an illustration, let us start with the simplest case $\Lambda = \mathbb{Z}$. The almost periodic functions with spectrum in $\mathbb{Z}$, i.e., those that are uniform limits of trigonometric polynomials in $\{\sum_{n\in\mathbb{Z}} a_n e^{2i\pi n x}\}$, behave like periodic functions. Yves Meyer introduced a class of sets Λ, considered as *pseudo-lattices*, which have the property that any almost-periodic function with spectrum (frequencies) in Λ behaves like a periodic function. Yves Meyer investigated these sets Λ in the context of Diophantine approximation for the study of Pisot and Salem numbers in the books [Mey70, Mey72]; see also [Mey95, Mey12]. Here, in view of Theorem 1.1, these sets are given by the discrete grids Λ_β (obtained from the sets E_β up to a suitable rescaling) with $\beta > 2$ and

$$\Lambda_\beta = \left\{ \sum_{i\in F} \varepsilon_i \beta^i \text{ with } \varepsilon_i \in \{0,1\}, F \text{ finite} \right\};$$

they provide explicit approximation schemes, underlying Theorem 1.1. They are examples of sets that are now referred to as *Meyer* and *model* sets. Let us briefly recall their definitions.

Definition 1.2 (Delone and Meyer sets). A *Delone set* is a subset Λ of $\mathbb{R}^n$ which is both uniformly discrete and relatively dense, i.e., there exist $r, R > 0$ so that each ball with radius r contains at most one point of Λ and each ball of radius R contains at least one point of Λ. A Delone set Λ is a *Meyer set* if there is a finite set F such that $\Lambda - \Lambda \subset \Lambda + F$, or equivalently, if $\Lambda - \Lambda$ is also a Delone set [Mey72, Lag96, Moo97].

Meyer sets play the role of lattices for crystalline structures. With atoms being located at points of Λ, the set $\Lambda - \Lambda$ corresponds to interatomic distances, and restrictions on interatomic distances produces long-range order.

In [Mey72] Yves Meyer introduced *cut and project schemes* as explicit constructions of particular Meyer sets, i.e., model sets. In order to introduce the idea in the simplest setting, we consider a full rank lattice L in $\mathbb{R}^d$, i.e., a lattice generated by a set of d linearly independent vectors, together with a decomposition of $\mathbb{R}^d = \mathbb{R}^n \times \mathbb{R}^k$, with $d = n + k$, where $\mathbb{R}^n$ is called the *physical space* and $\mathbb{R}^k$ is called the *internal space*. Let $\pi_{\mathtt{phy}}$ and $\pi_{\mathtt{int}}$ denote the natural projections onto $\mathbb{R}^n$ and $\mathbb{R}^k$ respectively. We further assume that the restriction $\pi_{\mathtt{phy}}$ to L is injective and that $\pi_{\mathtt{int}}(L)$ is dense in $\mathbb{R}^k$. We project the lattice $L \subset \mathbb{R}^d$ onto the physical space. The lattice L thus brings some order and regularity with its underlying higher-dimensional periodicity; its projection onto the internal space provides the aperiodic structure.

Given a cut and project scheme, model sets are then formed by projections together with a way of selecting points. This selection is done thanks to an *acceptance window* W that lives in the internal space $\mathbb{R}^k$. It is usually assumed to be compact (or relatively compact) and with non-empty interior. We now can define model sets.

Definition 1.3 (Model sets). A subset Λ of $\mathbb{R}^n$ is called a *model set* if there exist a cut and project scheme $(\mathbb{R}^n \times \mathbb{R}^k, L)$ and a compact set W of $\mathbb{R}^k$ with non-empty interior such that

$$\Lambda = \{\pi_{\mathtt{phy}}(P);\ P \in L,\ \pi_{\mathtt{int}}(P) \in W\}.$$

A model set Λ is a Delone set: the uniform discreteness comes from the compactness of the acceptance window W and the relative denseness from its non-empty interior. One of the main features of model sets is that they provide pure point diffraction (or, in more dynamical terms, pure discrete spectrum), such as highlighted in [Moo].

Meyer established the following powerful connections. For details, see [Mey72, Mey95] and for more about model sets, see for instance the surveys [Moo97, Moo00, LM01] and [BG13, Chap. 7].

Theorem 1.4. *Model sets are Meyer sets. Conversely, if Λ is a Meyer set, then it is a relatively dense subset of a model set, i.e., there exist a model set M and a finite set F such that $\Lambda \subset M + F$.*

The work of Meyer on model sets found a second life with the discovery of quasicrystals. In 1974, Penrose discovered his tilings[1], and in 1981 the connection between these tilings and Meyer's work was made explicit thanks to the work of de Bruijn [dB81], who proved that the vertices in the Penrose tilings formed a model set. Michel Duneau, Denis Gratias, André Katz, and Robert Moody then discovered that the *quasicrystals* described by Dan Shechtman can be modeled using model sets.

A quasicrystal is a physical solid whose atoms are arranged as a translation invariant lattice which does not satisfy the crystallographic restriction. Shechtman's discovery of quasicrystals in 1984 [SBGC84] mobilised the material sciences and physics to provide suitable models for the description of quasicrystals. Mathematicians came to realise that the theory of aperiodic tilings, with roots in both the theory of Wang tiles (Section 6), and also in recreational mathematics, where R. Penrose's aperiodic tilings displayed pentagonal symmetry and thus could not satisfy the crystallographic restriction, provided a context in which to study mathematical models of quasicrystals. In particular, one studies these physical objects by looking at an appropriate cut and project set of their atom locations, and this leaves us with Meyer sets.

Nowadays the term *aperiodic order*, which first appeared in Moody's work [Moo97], refers to the study of mathematical models of quasicrystals, continuous or discrete. As with the study of quasicrystals, aperiodic order is the study of mathematical objects which simultaneously have long-range order and aperiodicity. The connections to Meyer's concepts and to aperiodic tilings are now well recognised and established, in particular thanks to R. V. Moody [Moo00], and the fundamental contributions to aperiodic order brought by Yves Meyer developed in his founding books [Mey70, Mey72] continue to nourish the field. See also [GQ, Moo, OU] in this

[1]A *tiling* is a covering of a surface (usually a Euclidean plane) using copies of geometric tiles, placed next to each other, without holes or overlaps.

volume. This area has now exploded and covers a wide range of domains such as illustrated by the books [Sen95, BM00, KLS15] and the book series *Aperiodic order* by M. Baake and U. Grimm, in particular [BG13].

The notion of long-range order is crucial here: crystallographers now mean an ordering of atoms which produces a diffraction pattern with sharp bright spots, such as confirmed by the general definition of crystals adopted by the Crystallographic Union in 1992. These diffraction patterns translate mathematically to the study of the spectrum of point sets. In a nutshell, aperiodic order goes with diffraction and pure discrete spectrum, as highlighted in the contribution by R. Moody in this volume [Moo]. But the beauty and the strength of aperiodic order is that it goes well beyond spectral considerations.

There is in particular a further crucial aspect here, namely the prevalence of self-similarity in aperiodic order. In mathematics, a self-similar object is exactly or approximately similar to a part of itself, or it can be divided into smaller copies of itself; hence it looks roughly the same on any scale and is invariant upon being scaled larger or smaller. It can be that parts of a figure are small replicas of the whole, or else a self-similar group. The paradigmatic example is a compact set K which is the solution of an iterated function system, i.e., $K = \cup S_i(K)$, where i belongs to a finite set and where the S_i's are contractions. A classical way to generate self-similar objects is by iterating a rule. For example, one can iterate functions, to obtain fractals such as the Sierpinski gasket, one can substitute a side of a triangle by a polygonal line, such as in the Koch curve. Among the main methods for generating mathematical models of quasicrystals are *substitutions* which will be the main object of Section 4. In aperiodic order we iterate these substitutions to obtain self-similar sequences, points sets or tilings. Indeed, in tiling theory, a tile represents an atom and a tiling represents large atomic configurations. In point sets, points are idealisations of atomic positions, and letters are used to code interatomic distances, and, considering specially chosen control points in the tiles allows one to equivalently associate Delone point sets with tilings.

Algebraicity provides a particularly relevant arithmetic expression of self-similarity, and algebraic numbers and particularly Pisot numbers are at the heart of aperiodic order. The aim of this survey is twofold. We want to stress the impact of Pisot numbers in aperiodic order, and also to highlight the explosion of the involved domains, by focusing on an approach based on dynamical systems. The involved mathematical fields cover here an impressive spectrum, with the toolbox of aperiodic order containing Diophantine approximation and arithmetic, together with topological methods via topological dynamics or cohomology, computabillity theory, spectral theory, algebra, discrete geometry and harmonic analysis. There are several reasons that explain this abundance. Firstly, aperiodic order applies to various types of quasiperiodic structures which can serve as models for quasicrystalline structures, whether they are geometric with tilings and point sets, or symbolic with infinite words. Secondly, the two main types of mathematical models for quasicrystals, namely model sets obtained via cut and project schemes, and substitutions, are prominent concepts in the study of dynamical systems, allowing dynamicists to express self-similarity in both arithmetic and geometric terms. As an illustration, consider the role played by substitutions via renormalization in the study of interval exchanges [Yoc06].

In this survey we made the choice to focus on several complementary topics that illustrate the strength of the concepts of cut and project schemes and substitutions in the dynamics of aperiodic order. In Section 2 we see how Pisot numbers allow the definition of simple model sets, and we recall the characterisation, due to Yves Meyer, of Pisot and Salem numbers as the parameters of the dilations preserving some Meyer set: this is Theorem 2.2 and it will serve as a common thread. In Section 3 we discuss how Meyer and model sets are provided by Pisot numbers and beta-numeration. Substitutions, their definitions, stability by multiplication and self-similarity are considered in Section 4. In Section 5 we discuss the spectral properties of substitutions and the Pisot conjecture, simple one-dimensional model sets given by Sturmian words and more exotic acceptance windows provided by Rauzy fractals. In Section 6, we discuss how computabillity enters the picture with the question of the existence of local rules for aperiodic tilings and the domino problem.

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2. Pisot numbers and aperiodic order

Algebraic numbers, and particularly Pisot numbers, are at the heart of aperiodic order. Algebraicity intervenes in two ways: firstly, Pisot numbers allow us to define model sets Λ, and secondly, *scaling invariance*, i.e., the existence of a scaling factor β with $\beta\Lambda \subset \Lambda$, is a step towards self-similarity. This is seen in Theorem 2.2, a striking result by Meyer which has led to a rich theory in the study of associated dynamical systems, some of which we will exposit in later sections. Let us first recall the following definition.

Definition 2.1 (Salem number)*.* A real algebraic integer $\beta > 1$ is a *Salem number* if at least one of its algebraic conjugates λ satisfies $|\lambda| = 1$ while the other conjugates have modulus smaller than 1.

The following result will be a common thread throughout this survey. See also [Lag99] for a similar statement for Delone sets.

Theorem 2.2. [Mey95, Theorem 6] *If Λ is a Meyer set, $\beta > 1$ a real number and if $\beta\Lambda \subset \Lambda$, then β is either a Pisot number or a Salem number. Conversely, for each dimension n and for each Pisot or Salem number β, there exists a model set $\Lambda \subset \mathbb{R}^n$ such that $\beta\Lambda \subset \Lambda$.*

The proof of the first statement goes via the concept of *harmonious sets* which provide a further characterisation by duality of Meyer sets.

Definition 2.3 (Harmonious set)*.* A harmonious set is a subset F of a locally compact abelian group G such that every weak character[2] on the subgroup generated by F may be approximated uniformly by a continuous character on G.

For more on the subject, see [Mey72], [Mey95], [Mey20]. One considers here characters of the additive group generated by the Meyer set Λ, equipped with the discrete topology. The following beautiful result tells us that harmonious sets are non-other than Meyer sets (see [Mey72, Chap. II Section 5], [Mey95, Theorem 4]).

Theorem 2.4. *Let Λ be a Delone set. Then Λ is a Meyer set if and only if Λ is harmonious.*

[2] A weak character is a homomorphism for which no continuity is required.

Using Theorem 2.4, we can prove the first part of Theorem 2.2. We follow here [Mey95]. Take a Meyer set Λ and let $\beta > 1$ be such that $\beta\Lambda \subset \Lambda$. Take a non-zero element λ_0 in Λ. Then the set $\{\beta^k \lambda_0 : k \geq 0\}$ is included in Λ. We then note that a subset of a harmonious set is again a harmonious set, and that the set $\{\beta^k \lambda_0 : k \geq 0\}$ is harmonious in the one-dimensional space $\lambda_0\mathbb{R}$. Since Λ is a Meyer set, then Λ is a harmonious set. We now can conclude by using the following.

Theorem 2.5. [Mey72, Theorem XX, Chapter 1] *Let $\beta > 1$. The set of powers $\{\beta^k,\ k \geq 0\}$ is harmonious if and only if β is a Pisot or a Salem number.*

This sketch of proof is an illustration of the fact that "weak characters pave the road which goes from Bochner to Shechtman", as nicely phrased by Yves Meyer in [Mey20]. He moreover adds: "On the way we are visiting *coherent sets of frequencies, harmonious sets*,(...) and finally we arrive at *model sets*", a visit to coherent sets of frequencies that we will also make in Section 3. We will not say more on the first statement of Theorem 2.2, but rather concentrate on the second one, that is, the construction of the model set related to β. But before providing some elements of the proof of the second statement, we first illustrate it with a paradigmatic and illustrative example of how algebraic numbers provide us with the first and simplest examples of model sets. We later revisit this running example in other settings where aperiodic order is studied. Although it has a simplicity which does not reflect the subtleties needed in the general case, it nevertheless gives the reader a flavour of why these results may be true. See also [GQ] in this volume.

Example 2.6. Consider the golden ratio φ, which is the largest root of $x^2 - x - 1 = 0$, and its algebraic conjugate φ', and define the lattice

$$\mathcal{L} := \{m(1,1) + n(\varphi, \varphi') : m, n \in \mathbb{Z}\} = \{(m + n\varphi, m + n\varphi') : m, n \in \mathbb{Z}\}.$$

In the first coordinate (the physical space), the lattice $\mathcal{L}$ projects injectively by $\pi_{\texttt{phy}}(\mathcal{L})$ onto $\mathbb{Z}[\varphi]$ in $\mathbb{R}$, and in the second (the internal space), its projection $\pi_{\texttt{int}}(\mathcal{L})$ is dense in $\mathbb{R}$. Now take $W = [-1, \varphi - 1]$; then

$$\Lambda := \{m + n\varphi : m, n \in \mathbb{Z}, -1 \leq m + n\varphi' \leq \varphi - 1\}$$

is a model set; see Figure 2.6.

Moreover, this model set is scaling invariant under multiplication by φ. We use the fact that φ is a quadratic unit. Indeed, assume that $m + n\varphi \in \Lambda$. As $\varphi^2 = \varphi + 1$, we have $\varphi(m + n\varphi) = n + (m + n)\varphi$. Similarly, $n + (m + n)\varphi' = \varphi'(m + n\varphi')$. Multiplying the given inequality $-1 \leq m + n\varphi' \leq \varphi - 1$ by φ' and using $\varphi\varphi' = -1$ and $\varphi + \varphi' = 1$ yields $\varphi - 1 = -\varphi' \geq n + (m+n)\varphi' \geq -1 - \varphi' \geq -1$, proving the stability of Λ under multiplication by φ.

Finally, it can be shown that this cut and project scheme generates both a one-dimensional tiling of $\mathbb{R}$ using two distinct tiles, and if we label these tiles A and B, we obtain an *infinite word*, i.e., an infinite concatenation of the letters A and B. We will see in Section 4 that this discrete object is the fixed point of a *substitution*, and a discrete model of self-similar aperiodicity. See also [GQ, OU] in this volume.

More generally, given two distinct irrational parameters ν, ϵ (which are not necessarily algebraic) and a bounded acceptance interval W, one similarly gets a one-dimensional model set $\{a + b\nu \mid a, b \in \mathbb{Z}, a + b\epsilon \in W\} \subset \mathbb{Z}[\nu]$. As in the example above, such cut and project schemes, obtained by selecting points with integer coordinates located within a bounded

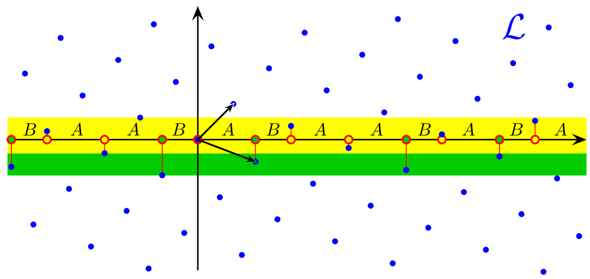

FIGURE 1. A model set generated by the golden mean φ; it consists of the circled points on the horizontal axis. The tile length of A is φ and that of B is 1.

distance of a line, and thus having one-dimensional physical and internal spaces, lead to one-dimensional tilings. Depending on whether or not the length of the window belongs to $\mathbb{Z}[\epsilon]$, the lengths in the tiling take two or three values, according to the size of the acceptance window [GMP03], and if we code the intervals with distinct letters, we obtain two- or three-letter infinite words. Such codings correspond either to Sturmian words (see Section 5.2), in the two interval case, or to codings of so-called three-interval exchanges in the three-interval case. Special attention has been given to the combinatorial properties of this particularly rich class of infinite words: see for instance [GMP03, GMP06] and the survey [PM07]. Such a model set has a priori no scale invariance. Imposing the requirement of self-similarity implies that ϵ is a quadratic integer and $\nu = -\epsilon'$, i.e., the algebraic conjugate of ϵ and the scaling factor γ must be a quadratic Pisot number in the same algebraic field $\mathbb{Q}(\epsilon)$. This brings us back to Meyer's connection between self-similar model sets and Pisot numbers in Theorem 2.2.

Pisot numbers allow the extension of such a simple and seminal construction to higher dimensions, which is a key point in the proof of the second statement of Theorem 2.2, which we now summarise from [Mey95], and which is illustrated with Example 2.6. Briefly, when β is a Pisot or a Salem number, the construction of a model set is algebraically realised by taking the lattice $\mathcal{L}$ to be the ring of algebraic integers in $\mathbb{Q}(\beta)$ and the canonical embeddings σ_i associated with the (other) algebraic conjugates of β provide the projection $\pi_{\texttt{int}}$ onto the internal space, with the projection $\pi_{\texttt{phy}}$ onto the physical space being the identity. The acceptance window is given by the conditions $|\sigma_i(x)| \leq 1$. The model set is thus the set of algebraic integers λ in $\mathbb{Q}(\beta)$ such that $|\sigma_i(\lambda)| \leq 1$ for all $i \geq 2$. This approach by Yves Meyer is now classical and has had a particularly fruitful influence on subsequent constructions.

3. Pisot numbers and beta-numeration

Arithmetic examples of Meyer sets which clearly have scaling invariance are provided by the sets $\Lambda_\beta = \left\{\sum_{i\in F} \varepsilon_i \beta^i \text{ with } \varepsilon_i \in \{0,1\}, F \text{ finite}\right\}$ that we met in the introduction, for $\beta > 2$ Pisot. These grids have remarkable properties that lead to the notion of a *coherent set of frequencies* (see [Mey72, p.110]).

Definition 3.1 (Coherent set of frequencies). A set Λ of real numbers is a *coherent set of frequencies* if there exist a constant $C > 0$ and a compact set K of real numbers such that

$$\sup_{\mathbb{R}} |P(x)| \leq C \sup_{K} |P(x)|$$

for all finite trigonometric sums P with frequencies in Λ.

For more on the contributions of Yves Meyer in this setting, see the enlightening text by A. Cohen in [HP19] and see also [Mey20] which revisits the relationship between coherent sets of frequencies and Bochner's property.

The following result has to be compared with Theorem 2.5.

Theorem 3.2. [Mey72, Proposition 7, Chapter 8] *Let $\beta > 2$. The following are equivalent:*

- *the number β is a Pisot number;*
- *the set Λ_β is harmonious;*
- *the set Λ_β is a coherent set of frequencies.*

Note that this statement does not mean that Λ_β is a Meyer set since it is not relatively dense ($\beta > 2$), and hence not Delone. Indeed, consider the strictly increasing sequence (x_n) of numbers that have at least one representation as an element of Λ_β, and then its *first difference sequence* $(x_{n+1} - x_n)$. The following holds.

Proposition 3.3. [FW02, Lemma 1.1] *If $\beta > 2$, and (x_n) is an increasing sequence of numbers that have at least one representation as an element of Λ_β, then its first difference sequence takes arbitrarily large values.*

More generally, sets of finite sums of the form $\Lambda_F = \{\sum_i \varepsilon_i \beta^i \mid \varepsilon_i \in F\}$, for specific finite sets F of non-negative integers, when β is assumed to be a Pisot number, have been widely considered. Researchers in arithmetic and numeration came indeed to notions similar to Meyer sets independently via base expansions. In particular, the study of the first difference sequence was initiated in the fundamental work [EJK90] for $\Lambda_F = \Lambda_\beta$ with $1 < \beta < 2$ and $F = \{0, 1\}$. The study of the limit inferior of the first difference sequence has attracted much attention. In particular, $\liminf x_{n+1} - x_n > 0$ if and only if β is a Pisot number; see [Bug96] and also [EJK98].

Returning to the golden mean Example 2.6, if we take $\beta = \varphi$ and $F = \{0, 1\}$, one can check that $\Lambda_F = \mathbb{Z}[\varphi]$ and that the entries of the first difference sequence are either 1 or $\varphi - 1$. Thus Λ_F is nothing other than our lattice $\mathcal{L}$ and the entries 1 and $\varphi - 1$ of the difference sequence give us the lengths of the tiles that we saw in Figure 2.6, when scaled by a factor of φ.

The case where $\beta \geq 2$ and $F = \{1, \cdots, m\}$ has also been widely studied. See e.g. [AK13], and the references therein, for the study of set of all differences $\sum \varepsilon_i \beta^i$ where $\varepsilon_i \in \{0, 1, \pm 1, \cdots, \pm m\}$ in relation with singular Bernoulli convolution distributions; note that this latter set gives information about $\Lambda_F - \Lambda_F$; in particular the absence of accumulation points in this set is proved to be equivalent to β being Pisot or $\beta \geq m + 1$ [AK13, Theorem 1.1]. In the language of aperiodic order, this gives the following:

Proposition 3.4. *Let $\beta > 1$. For $F = \{0, 1, \cdots, m\}$, the set $\{P(\beta);\ P \in F[X]\}$ is a Meyer set if and only if $\beta \leq m + 1$ and β is a Pisot number.*

Let us provide a sketch of the proof of Proposition 3.4. We first briefly recall how the Pisot property intervenes here as being crucial in guaranteeing the uniform discreteness of $\Lambda_F = \{P(\beta);\ P \in F[X]\}$ for F a finite set of integers. This is a classical argument known as Garsia's Lemma; see [Gar62, Lemma 1.51] or [Sol97, Lemma 6.6]. If β is assumed to be Pisot, there exists C such that $|P_1(\beta^{(i)}) - P_2(\beta^{(i)})| \leq C$ for any algebraic conjugate $\beta^{(i)} \neq \beta$ and any $P_1, P_2 \in F[X]$. Next, since $\prod_i (P_1 - P_2)(\beta^{(i)})(P_1 - P_2)(\beta) \in \mathbb{Z}$ and is non-zero for $P_1, P_2 \in F[X]$ with $P_1(\beta) \neq P_2(\beta)$, we deduce a positive uniform lower bound for $P_1(\beta) - P_2(\beta)$ with $P_1(\beta) \neq P_2(\beta)$, from which we conclude uniform discreteness.

It remains to consider the property of relative denseness where the cases $\beta \leq m+1$ and $\beta > m+1$ have to be distinguished by [FW02, Lemma 1.1]. Indeed, if $\beta > m+1$, the difference sequence $(x_{n+1} - x_n)$ takes arbitrarily large values (this extends Proposition 3.3), which prevents relative denseness (see the proof of [EK98, Lemma 2.1] such as stressed in [FW02]). Next if $\beta \leq m+1$ and β is a Pisot number, the difference sequence can take only finitely many distinct values; it has even a strong combinatorial structure such as described in [FW02], where it is shown that the difference sequence can be generated by a substitution over a finite alphabet (see Section 4.1).

Lastly, assume that $\Lambda_F = \{P(\beta);\ P \in F[X]\}$ is a Meyer set. As $\beta\Lambda_F \subset \Lambda_F$, we deduce from Theorem 2.2 that β is a Pisot or a Salem number. By the same proof as that of Theorem 2.2 and also by Theorem 2.4, Λ_F is harmonious, and so Λ_β is also harmonious, from which we deduce that β is a Pisot number, by Theorem 3.2.

Thus researchers have studied the sets Λ_F for over fifty years, due to their connections to quasi-crystals, infinite Bernoulli convolutions and expansions in non-integer bases. Dynamicists have also studied these objects, via the *beta-transformation*. The *beta-numeration* is a numeration system that allows the representation (in a greedy way) of real numbers as sums of powers of a real number β, with $\beta > 1$, in the same way as real numbers can be expanded in decimal numeration. It was introduced and studied by A. Renyi [Rén57] and W. Parry [Par60] in dynamical terms. Indeed, given a positive irrational number $\beta > 1$, consider the β-transformation T_β acting on the unit interval defined by $T_\beta(x) = \{\beta x\}$. Digits of x in its β-expansion are then obtained as $a_i = \lfloor \beta T_\beta^{-i} x \rfloor$, for $i \geq 1$ (hence $a_i \leq \lfloor \beta \rfloor$), which yields $x = \sum_{i \geq 1} a_i \beta^{-i}$. The beta-numeration has been particularly well studied when β is assumed to be a Pisot number, both from a dynamical and a combinatorial viewpoint; there are indeed specific properties that occur when β is a Pisot number. Returning to Example 2.6 where β is the golden mean, digits are in $\{0, 1\}$ and the product of two consecutive digits in the β-expansion equals 0 (since we use the greedy representation). For Pisot numbers β, the appropriate generalisation of this statement is that the finite sequences of digits that occur in the possible representations have a simple expression in terms of finite automata (see e.g. [BS02]). From a dynamical viewpoint, points in the interval are replaced by the sequence of coefficients in their beta-expansion.

Moving from the unit interval to $\mathbb{R}^+$, one can also expand non-negative real numbers. In this context, the countable set of *beta-integers* $\mathbb{Z}_\beta$ plays a particular role: elements of $\mathbb{Z}_\beta$ are the integer parts of beta-expansions, that is, they are polynomials in β with coefficients in $\{0, \cdots, \lfloor \beta \rfloor\}$ with combinatorial constraints on their digits determined by the algebraicity of β. In this dynamical setting, the constraints are given on the digits ε_i that can appear when expressing the beta-expansion of a number. These constraints are imposed to guarantee uniqueness of expansions, and are particularly easy to express when β is an algebraic integer. For example, returning to Example 2.6, the equation $\varphi^2 = \varphi + 1$ tells us that a fragment $\varphi^{n+1} + \varphi^n$ of φ-expansion can be replaced by φ^{n+2}. For this example then we consider $\mathbb{Z}_\varphi$ to be polynomials in φ with coefficients in $\{0, 1\}$ but where we do not see consecutive monomials. In the case where $1 < \beta < 2$, one sometimes has $\mathbb{Z}_\beta = \Lambda_\beta$, for example in the case of our running example where β is equal to the golden ratio. However in general one only has $\mathbb{Z}_\beta \subset \Lambda_\beta$; see [FS92].

This set $\mathbb{Z}_\beta$ is self-similar, in that $\beta\mathbb{Z}_\beta \subset \mathbb{Z}_\beta$. As stressed by Meyer in [Mey12], "Model sets are non-uniform grids", and beta-numeration has revealed itself as a very efficient tool for the modeling of families of quasicrystals thanks to the beta-grids $\sum \mathbb{Z}_\beta \mathbf{e}_i$, where the $\mathbf{e}_i$

are vectors of the canonical basis (see e.g. [BFGK98, VGG04] and also [Thu89]). The next statement, due to [BFGK98, VGG04], and which also follows from arguments in [Sch80] and [FS92, Proposition 2], is again reminiscent of Theorem 2.2.

Proposition 3.5. *If β is a Pisot number, then $\mathbb{Z}_\beta$ is uniformly discrete, and $\mathbb{Z}_\beta \cup (-\mathbb{Z}_\beta)$ is a Meyer set. Conversely, if $\mathbb{Z}_\beta \cup (-\mathbb{Z}_\beta)$ is a Meyer set, then β has to be a Pisot or a Salem number.*

Furthermore, the set $\mathbb{Z}_\beta$ can be endowed with laws that are close to multiplication and addition. This allows one to recover properties that have the flavour of the algebraic rules of lattices. For some families of Pisot numbers β, mainly Pisot quadratic units and for some cubic Pisot numbers, an internal law can even be produced formalising this quasi-stability under subtraction and multiplication: the set of beta-integers is then endowed with an addition law and a multiplication law, which are compatible with the combinatorial structure which makes it a quasiring [BFGK98]. Let us quote also [ABF05], where a method based on automata is given to find a set F such that $\mathbb{Z}_\beta - \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$.

4. Substitutive dynamics and self-similarity

We have seen in Section 2 that infinite words occur naturally as codings of one-dimensional quasicrystals. There is a particularly simple yet rich combinatorial object that allows the expression of scale invariance and self-similarity properties: this is the notion of a substitution. We define a substitution rigorously below, but roughly speaking, a *substitution* is an inflation rule, either combinatorial or geometric, that, following the cases, replaces a letter by a word, or a tile by a geometric pattern made of a finite union of tiles. Iteration of substitutions generates hierarchically ordered structures such as infinite words, shifts, point sets, or else tilings, that all display strong self-similarity properties.

Substitutions originated with the first works of A. Thue in 1906, with the so-called Prouhet-Thue-Morse substitution, which is an illustrious example of an automatic sequence [AS03]. Substitutions generate symbolic dynamical systems and this is the setting in which we define them, that of symbolic dynamics.

Symbolic dynamical systems are discrete dynamical systems where the space is a Cantor space. Typical symbolic systems are *shifts* where the space X consists of infinite words with entries from a finite alphabet, and the dynamics is the shift map S moving entries of an infinite word one unit to the left. The set X must be closed in the subspace topology inherited from the product topology on $\mathcal{A}^{\mathbb{Z}}$, where $\mathcal{A}$ is given the discrete topology. Shifts provide representations of dynamical systems: given a dynamical system we can discretise the state space using a finite (or countable) partition, and code trajectories of points as sequences of symbols, as first done by J. Hadamard [Had98] and M. Morse [Mor21], and the Jewett-Krieger theorem gives us very general conditions in which we can represent a dynamical system as a shift.

Substitutions soon outreached in an unexpected way this combinatorial and dynamical setting. Indeed, beginning in the sixties, substitutions were used to input computation in tilings, enabling researchers to produce the first examples of aperiodic tilings, such as Robinson tilings, and thereby proving the undecidability of the domino problem (see Section 6). Also, in the eighties, *Pisot substitutions* attracted much attention in the context of aperiodic order as mathematical models of quasicrystals displaying self-similarity, yielding in particular the Pisot substitution conjecture (see Section 5.1).

4.1. **Substitutions: how dynamicists came to model sets.** Let $\mathcal{A}$ be a finite alphabet and let $\mathcal{A}^+$ be the set of non-empty finite words on $\mathcal{A}$. A (word) *substitution* is a map $\sigma : \mathcal{A} \to \mathcal{A}^+$ which extends to $\sigma : \mathcal{A}^+ \to \mathcal{A}^+$ and $\sigma : \mathcal{A}^{\mathbb{N}} \to \mathcal{A}^{\mathbb{N}}$ via concatenation. A substitution defines a *language*, which is the set of all finite words $w_0 \dots w_k$ which appear as a subword of $\sigma^n(a)$, for $a \in \mathcal{A}$ and $n \in \mathbb{N}$. The shift space X_σ is the set of infinite words all of whose finite subwords belong to the language of σ, and with the shift map $S : X_\sigma \to X_\sigma$, the pair (X_σ, S) is called a (substitution) shift.

Example 4.1. As a prominent example of a word substitution in aperiodic order, consider the *Fibonacci substitution* σ defined on the alphabet $\mathcal{A} = \{A, B\}$ as $\sigma(A) = AB$ and $\sigma(B) = A$. The sequence of finite words $(\sigma^n(A))_n$ are nested, i.e., $\sigma^n(A)$ is a prefix of $\sigma^{n+1}(A)$ for each n, and it converges to the so-called *Fibonacci word*, whose first terms are

$$ABAABABAABAABABAABABAABAABABAABAABABAABABAABAABABA\dots$$

The Fibonacci word is a one-dimensional discrete model of a quasicrystal; indeed, one of the first models that have been considered. For, it codes a one-dimensional tiling obtained as a model set via a selection strip in $\mathbb{Z}^2$ as illustrated in Figure 2.6, noting that the Fibonacci word will be the coding of the right part of the tiling seen there, starting at the origin; see below for how to generate the actual tiling. The Fibonacci word also corresponds to a discretization of a line in discrete geometry [RK01], and from a symbolic dynamics viewpoint, it belongs to the family of Sturmian words which are discussed in Section 5.2. For more on the Fibonacci word, see e.g. [Lot02, Pyt02].

The *incidence matrix* M_σ of a substitution σ is the square matrix whose entries count the number of occurrences of letters in the images of letters. For example, for the Fibonacci substitution σ, we have $M_\sigma = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. This matrix, called the *abelianization* of σ as the order of letters in substitution words is forgotten, allows us to apply tools from linear algebra to analyse the self-similarity properties of the Fibonacci word, as well as its combinatorial and dynamical properties. The Perron–Frobenius theorem tells us that if M_σ is primitive, then it admits a strictly dominant eigenvalue that is positive, and this eigenvalue is called the *inflation factor*, or *expansion factor* of the substitution. Also, under the assumption of primitivity of M_σ, the shift (X_σ, S) has particularly useful ergodic properties, such as unique ergodicity or minimality. For more details, see [Que10].

Analogously, substitutions can also be defined on tiles, and iterating the substitution produces self-affine tilings. Tilings are considered in any dimension and over general combinatorial structures with group actions. Two main natural models of tilings are Wang tiles coming from computer science (see Section 6) and shifts coming from dynamical systems. In the tiling setting, we now work in a d-dimensional geometric space, here $\mathbb{R}^d$, rather than in a symbolic setting, with letters and words.

Definition 4.2 (Self-affine tiling). Let $\{T_i; 1 \le i \le m\}$ be a set of tiles (usually assumed to be polygons) in $\mathbb{R}^d$, called *prototiles*. A *self-affine tiling* $\mathcal{T} = \{x_i + T_i; x_i \in \Lambda_i, i \le m\}$ in $\mathbb{R}^d$ is a tiling fixed by a linear expanding map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ that maps every prototile onto a union of prototiles (*expanding* means that all its eigenvalues are greater than one in modulus). The prototiles are translated by vectors in the sets Λ_i in order to form the tiling $\mathcal{T}$. With the map ϕ comes a substitution rule Σ that explains how to divide a tile inflated by ϕ into a union of translated prototiles and an incidence matrix M_Σ which indicates for each prototile M how

$\phi(M)$ is a union of prototiles. More precisely, there exist finite sets $\mathcal{D}_{ij} \subset \mathbb{R}^d$ $(1 \leq i, j \leq k)$ such that for any j $(1 \leq j \leq k)$

$$\Sigma(T_j) = \{u + T_i, u \in \mathcal{D}_{ij},\ i, j \leq k\} \text{ with } \phi T_j = \cup_{i=1}^{k}(T_i + \mathcal{D}_{ij}).$$

Here all sets on the right-hand side must have disjoint interiors; it is also possible for some of the $\mathcal{D}_{ij}$ to be empty. The matrix M_Σ then counts the cardinalities of the sets $\mathcal{D}_{ij}$.

If M_Σ admits a strictly dominant eigenvalue that is positive, then as with substitutions, this eigenvalue is called the inflation factor of the tiling. It equals the absolute value of the determinant of ϕ, which controls the volume expansion. The tiling is said *self-similar* if ϕ is a similarity.

Example 4.3. We revisit Example 2.6 and 4.1. Here $d = 1$. We define a tiling with the Fibonacci substitution $\sigma : A \mapsto AB, B \mapsto A$ by starting with two intervals (prototiles), labelled respectively A and B, with lengths ℓ_A and ℓ_B. To obtain a geometric realisation Σ of the Fibonacci substitution σ using two one-dimensional interval tiles, we thus need to find tiles A and B, with lengths ℓ_A and ℓ_B, to be determined, where

- Σ maps the tile A to the two concatenated tiles AB,
- Σ maps the tile B to the tile A, and
- Σ corresponds to an expansive map on $\mathbb{R}$.

The substitution Σ maps A, whose length is ℓ_A, to two tiles of combined length $\ell_A + \ell_B$, and the tile B of length ℓ_B to a tile A of length ℓ_A. Since we want the substitution rule Σ to act as a map that expands the space with an inflation factor λ, this implies that

$$\lambda \ell_A = \ell_A + \ell_B \text{ and } \lambda \ell_B = \ell_A,$$

i.e.,

$$\lambda(\ell_A, \ell_B) = (\ell_A, \ell_B)\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = (\ell_A, \ell_B)M_\Sigma,$$

so that λ must be a left eigenvector for M_Σ, and since φ is the only expanding eigenvalue of M_Σ, $\lambda = \varphi$, and we can take $\ell_A = \varphi$ and $\ell_B = 1$. We thus set $A = [0, \varphi]$ and $B = [0, 1]$. This defines a self-affine tiling (and even a self-similar one). The matrix M_Σ equals the matrix M_σ, and one has $\mathcal{D}_{11} = \{0\}$, $\mathcal{D}_{12} = \{\varphi\}$, $\mathcal{D}_{21} = \{0\}$, and $\mathcal{D}_{22} = \emptyset$. Iterating this tiling substitution rule gives us a geometric tiling of the line; see Figure 2.6.

Self-affine tilings occur in a wide range of problems in dynamics; for instance, they are related to Markov partitions for hyperbolic maps and radix representations [BBLT06]. More generally, for tilings, not necessarily generated by substitutions, the associated shift dynamics is given by $\mathbb{R}^d$-translations yielding as dynamical systems orbit closures of tilings [Rob04, Sol97]. The closure is taken in the local topology where coincidences on a large ball around the origin up to a small translation are considered. A remarkable topological property here is that the local structure of the tiling space (i.e., the associated dynamical system endowed with the $\mathbb{R}^d$-action) is described as the product of a Cantor set and a Euclidean space. Locally, it looks like a disk crossed with a totally disconnected set. Tiling spaces then yield particularly interesting types of topological spaces obtained as inverse limits of branched manifolds with totally disconnected compact fibers [Sad08]. This makes cohomological methods particularly adapted in this setting, and particularly the Čech cohomology which gives information on how a tiling can be deformed. This goes beyond the case of substitutive tilings. Consider as an illustration the works of A. Forrest, J. Hunton, and J. Kellendonk in [FHK02] for cut and project tilings which, along the way, illustrate the efficiency of topological methods combined with tools from C^*-algebras (see also [Put18] for an elementary exposition). More generally, topological dynamics and topological notions (e.g., maximal equicontinuous factor or proximality relation) have led to beautiful developments, for instance in the computation of the Ellis semigroup [ABKL15, KY20].

4.2. **More on self-affine tilings.** In this section, we revisit Meyer's Theorem 2.2 in the setting of self-similar tilings. According to Definition 4.2, we consider an expanding linear map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ and a self-affine tiling $\mathcal{T}$ fixed by ϕ. Dynamicists ask which numbers can appear as inflation factors, or which maps can appear as expanding maps, of self-affine tilings, and which of these are inflation factors for substitutions that lead to quasicrystals. This has led to a vast literature, and as in Theorem 2.2, it is articulated in two steps: firstly, what are the conditions that the inflation factors should satisfy? Secondly, can it be possible to realise a self-similar structure, and even a Meyer or a model set, with each dilation factor satisfying these conditions? We turn to the last question in Section 5.3.

We first introduce two arithmetic definitions.

Definition 4.4 (Perron number). The algebraic integer β is said to be a *Perron number* if $|\beta| > |\alpha|$ for any other of its algebraic conjugates.

For example, the Perron–Frobenius theorem tells us that the dominant eigenvalue of a primitive non-negative matrix is a Perron number.

Definition 4.5 (Pisot family). Let Λ be a finite set of algebraic integers of modulus larger than or equal to 1. The set Λ is said to form a *Pisot family* if for every $\lambda \in \Lambda$ and for every algebraic conjugate λ' of λ with $|\lambda'| \geq 1$, then $\lambda' \in \Lambda$.

In the one-dimensional case, it follows from the work of D. Lind [Lin84] that λ is an inflation factor of a self-similar tiling if and only if λ is a Perron number. In the two-dimensional self-similar case (the dilation is the same in every direction), the expansion map ϕ, when viewed as acting on $\mathbb{C}$, equals λz where λ is a complex Perron number, as shown by W. Thurston [Thu89]. In the case where the expansion map ϕ is diagonalizable, R. Kenyon and B. Solomyak [KS10] [Ken90] have shown that the eigenvalues of ϕ are algebraic integers. The general result (without the diagonalizabiliy condition) has then been established by J. Kwapisz [Kwa16] who proves that, with a suitable notion of Perron, the map ϕ of a self-affine tiling is integral algebraic and Perron.

Now that it is established what Perron numbers have to do with being self-affine (or even being self-similar), Pisot numbers enter into play when further properties are required such as some of those shared by Meyer sets, extending Meyer's Theorem 2.2. In this setting, the connection with the notion of the Pisot family was successfully developed by J.-Y. Lee and B. Solomyak [LS12] for spectral considerations in the case where the map ϕ is diagonalizable over the complex numbers, with all eigenvalues being algebraic conjugates of the same multiplicity. They proved that the associated tiling shift has relatively dense discrete spectrum (i.e., the set of eigenvalues has full rank) if and only if the spectrum of the linear map ϕ forms a Pisot family, which is also equivalent to the discrete set of control points for the tiling to be a Meyer set.

More generally, we will see in Section 5.1 how Pisot numbers enter the picture in this substitutive setting not only as inflation factors, but also for model sets via spectral considerations.

5. Pisot substitutions and aperiodic order

Now that inflation factors are well characterised, the question is to be able to construct self-similar tilings or point sets having this inflation factor that are also Meyer or even model sets. There exist several strategies for such realisations. Among them, Rauzy fractals, discussed in Section 5.3, provide suitable windows for explicit cut and project schemes. But before discussing them, we introduce the so-called Pisot substitution conjecture.

5.1. **Spectral properties and the Pisot conjecture.** The spectral study of substitutive dynamical systems is a fundamental question (see for instance the monographs [Que10, Pyt02]). This is particularly relevant in the context of aperiodic order, as stressed by the contribution by R. Moody in this volume [Moo]. Weak mixing, namely the absence of eigenvalues, i.e., the absence of Bragg peaks, indicates a certain level of disorder and conversely, the existence of spectral eigenvalues provides dynamical factors which consist of group translations. For instance, constant length substitutions yield p-adic factors. In particular, discrete spectrum deals with the possibility of providing substitutive dynamical systems with a representation as a group translation, that is, a dynamical system acting on a space of a geometric nature (such as a torus for instance), that is (measurably) isomorphic to the substitutive system under study.

We have seen through the work of Yves Meyer that the connection with Pisot numbers arose from the inception of quasiorder, and this has led to the following natural definition.

Definition 5.1 (Pisot substitution). If σ is a word substitution whose incidence matrix has a characteristic polynomial which is the irreducible minimal polynomial of a Pisot number, then σ is called a *Pisot substitution.*

From the eighties onwards, Pisot substitutions have attracted much attention in the context of mathematical quasicrystals. We recall that the Fibonacci substitution provided one of the first examples of a one-dimensional quasicrystal and more generally substitutions were considered as promising examples of one-dimensional quasicrystals; as we have seen, they are very simple algorithmic rules that create configurations that display long-range order. However not all substitutions yield quasicrystals. E. Bombieri and J. E. Taylor asked already in [BT86, BT87] which substitutions produce quasicrystals and they highlighted the Pisot algebraic restriction for substitutions to yield pure discrete spectrum. See also [Sol97] where a self-similar tiling is proved to admit a discrete component in its spectrum if and only if its inflation factor is a Pisot number.

This culminated with the *Pisot substitution conjecture*, stated below for substitutions defined on symbols. There exist various formulations, see the survey [ABBS08].

Conjecture 5.2 (Pisot substitution conjecture)**.** *Let σ be a Pisot irreducible substitution (i.e., the characteristic polynomial of its incidence matrix is the minimal polynomial of a Pisot number). Then, the shift (X_σ, S) has pure discrete spectrum, i.e., it is isomorphic, in a measure-theoretic sense, to a translation on a compact abelian group.*

In a nutshell, the Pisot arithmetic condition induces order, where order is expressed here in spectral and dynamical terms as being isomorphic to the simplest dynamical systems, namely group translations. The conjecture is known to be true for the case of substitutions on two letters [BD02, HS03]. The difficulty for a larger alphabet comes from the arithmetic of higher degree algebraic numbers, which is more difficult to manage than for quadratic numbers. Also, in two dimensions, it is easier to identify the occurrences of so-called *coincidences*, the existence of which allows us to project injectively an infinite word onto a group, here the circle; to do so, words are embedded as discrete lines in the plane and coincidences between these lines are to be found, which is easier in a plane (see Example 5.6 for an illustration); generic behaviour is that once there is a coincidence, there are infinitely many occurrences of the coincidence, and this implies that this projection is an injection.

Theorem 5.3. [BD02, HS03] *Two-letter Pisot substitutions have pure discrete spectrum.*

The still open Pisot substitution conjecture, even if solved in the closely related context of beta-numeration by M. Barge [Bar18], shows that important parts of the picture are still to be developed. Once again, one particularly appealing feature concerning the works developed around the Pisot substitution conjecture is that they involve several mathematical approaches and reformulations such as the homological Pisot conjecture, which aims to take into account topological invariance, or the coincidence rank conjecture cohomology, involving topological dynamics, arithmetic, combinatorics, fractal geometry, etc. See as an illustration [ABBS08] or [Thu19].

It follows from algebraic considerations, involving the eigenvalues of the incidence matrix, that Pisot substitution shifts and Pisot tiling dynamical systems must have a non-trivial rotation factor. Criteria for obtaining these dynamical eigenvalues are now well understood; see for instance [Sol07]. The difficult part consists in producing a measurable isomorphism. Fortunately, there is a wide range of algorithmic conditions, called coincidence conditions (such as briefly evoked above), for checking pure discrete spectrum, which date back to the work of M. Dekking on constant-length substitutions, as described in [Que10]. They occur for infinite words, for higher-dimensional and for non-lattice based self-affine tilings, and even in the non-unimodular case. See for instance [Sol97, LM01, ST09, AL11]. Also relevant is the notion of *almost-automorphy*, introduced by Veech [Vee65], which implies discrete spectrum for substitutions, and which incidentally was shown to imply spectral synthesis [Vee69], bringing us back again to Meyer's work.

5.2. **Sturmian words and beyond substitutions.** As noted, the Pisot substitution conjecture has been proved for two–letter irreducible Pisot substitutions (Theorem 5.3). Such substitutions produce model sets. Amongst them are a remarkable and widely studied family, that of the Sturmian substitutions, which generate shifts that belong to the class of Sturmian shifts that we now describe.

Sturmian shifts are symbolic representations of irrational circle rotations. More precisely, consider the rotation R_α acting on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, with $R_\alpha(x) = x + \alpha \mod 1$, for α irrational. If we code the orbit of a point under the action of R_α using a two-interval partition of semi-open intervals whose lengths are respectively α and $1 - \alpha$, then we obtain a *Sturmian word*, generating a *Sturmian shift* X_α consisting of all Sturmian words of *angle* α, that is, all Sturmian words coding R_α. This seminal class of symbolic dynamical systems, introduced by M. Morse and G. Hedlund in [MH40], laid some of the foundations for symbolic dynamics. For a thorough description of Sturmian words, see [BS02] and [Pyt02, Chapter 6].

The Fibonacci substitution from Section 4.1 is the most cited example of a Sturmian substitution. When it is used to generate a tiling, then as we saw in Figure 2.6 the associated point set is a model set. Sturmian words have provided the simplest examples of cut and project sets: the window is here an interval of unit length and the associated tilings are one-dimensional, with the cut and project scheme relying on two lines as described in Section 2. Phrased in a more geometric way, they code discrete lines in digital geometry [RK01].

There is an impressive literature devoted to the study of Sturmian words and to possible generalisations; let us mention for instance episturmian words, also called Arnoux-Rauzy words, which have attracted a lot of attention [CFM08, AHS16].

One key feature is the scale invariance of Sturmian shifts. Not all Sturmian words are substitutive. Indeed the angle α of a substitutive Sturmian shift X_α is a quadratic number [CMPS93] (see also [BEIR07] for a characterisation of Sturmian words that are fixed points of substitutions obtained in terms of Rauzy fractals discussed in Section 5.3). However Sturmian words can be all generated in terms of sequences of substitutions which we elaborate below. More precisely, Sturmian words are perfectly understood and described via a representation based on substitutions and Euclid's algorithm: the continued fraction expansion of the angle α provides an infinite product of square matrices of size two with non-negative integer entries, each of these matrices can be seen as the incidence matrix of a substitution, and the action of a substitution can be seen as a combinatorial interpretation of a step of Euclid's algorithm, as described below.

Theorem 5.4. [AR91] *We consider the substitutions over the alphabet* $\{0,1\}$ *defined by* $\sigma_0\colon 0 \mapsto 01, 1 \mapsto 1$, *and* $\sigma_1\colon 0 \mapsto 0, 1 \mapsto 10$. *If* α *has continued fraction expansion* $\alpha = [0; a_1+1, a_2, \cdots]$, *then the Sturmian shift* X_α *is generated by the infinite word*

$$\lim_{n\to\infty} \sigma_0^{a_1}\sigma_1^{a_2}\sigma_0^{a_3}\sigma_1^{a_4}\cdots\sigma_0^{a_{2n-1}}\sigma_1^{a_{2n}}(0).$$

Seen through this lens, Sturmian words are described using a renormalization scheme governed by continued fractions via the geodesic flow acting on the modular surface. This crystallises with the study of interval exchanges in relation with the Teichmüller flow through the work, among others, of W. Veech, H. Masur, J.-C. Yoccoz, and A. Avila (see e.g. [Yoc06, Buf14]).

Theorem 5.4 states that a Sturmian word is the limit of an infinite composition of these substitutions; such an approach has been formalised using the language of *S-adic words*, which are infinite words generated when a sequence S of substitutions is applied, as opposed to when a single substitution is iterated [BD14]. The S-adic setting pertains to non-stationary dynamics (i.e., time inhomogeneous dynamics), which consists in working with iterated sequences of transformations drawn according to a further dynamical system, similarly as for random dynamics, random Markov chains and random products of matrices. This formalism allows us to extend the Pisot conjecture beyond algebraicity [BST19]. The Pisot condition is then replaced by the requirement that the second Lyapunov exponent of the infinite associated products of matrices is negative. This extended Pisot conjecture has been proved to hold for large relevant families of systems based on continued fractions expansions, such as the Brun algorithm. As a striking outcome, this yields symbolic codings for almost every translation of $\mathbb{T}^2$ [BST19], paving the way for the development of equidistribution results for the associated two-dimensional Kronecker sequences.

5.3. **Rauzy fractals.** In this section we describe a construction for word substitutions having a Pisot inflation factor that generates suitable and effective windows for model sets; this generalises the Sturmian case for which the window in the internal space is an interval. In fact, it is now well understood that a substitution tiling with pure discrete spectrum is a model set, such as described by Lee in [Lee07]. She shows the equivalence between specific model sets, the so-called inter-model sets[3], and pure point dynamical spectrum in the context of primitive substitution point sets, where the construction of the cut and project scheme involves an abstract internal space. The aim of this section is to present more explicit constructions of these internal spaces as *Rauzy fractals*. This approach appeals in its combination of arithmetic and dynamics.

Recall that for (X_σ, S) to have pure discrete spectrum, it must be isomorphic to a group translation. Thus, given a Pisot substitution σ, one wants to provide a geometric representation for (X_σ, S) as a translation on the torus, or more generally on a locally compact abelian group. And as a candidate for a fundamental domain for the expected translation, one associates with the substitutive dynamical system (X_σ, S) a Rauzy fractal.

Rauzy fractals were first introduced in [Rau82] in the case of the so-called Tribonacci substitution in order to prove the following statement.

Theorem 5.5. [Rau82] *Let σ be the Tribonacci substitution defined over the alphabet $\mathcal{A} = \{A, B, C\}$ as as $\sigma(A) = AB$, $\sigma(B) = AC$ and $\sigma(C) = A$. The symbolic dynamical system (X_σ, S) is measure-theoretically isomorphic to the translation R_β on the two-dimensional torus $\mathbb{T}^2$ defined as $R_\beta : \mathbb{T}^2 \to \mathbb{T}^2$, $x \mapsto x + (1/\beta, 1/\beta^2)$, where β, the Perron eigenvalue for σ, satisfies $\beta^3 = \beta^2 + \beta + 1$.*

We now sketch how to build a fractal from the shift (X_σ, S) as a fundamental domain for the translation of Theorem 5.5.

Example 5.6. Let α be one of the complex roots of $X^3 - X^2 - X - 1$; we have $|\alpha| < 1$. We will associate to a point $x \in X_\sigma$ an α-expansion as follows. The point $x \in X_\sigma$ can indeed be uniquely desubstituted, either as $x = \sigma(y)$, or as $x = S\sigma(y)$ for some $y \in X_\sigma$. We can associate to x an initial digit $\epsilon_0 \in \{0, 1\}$, depending on whether we shift or not. We repeat this procedure, desubstituting y to obtain a second digit ϵ_1. Iterating we obtain a sequence $(\epsilon_n) \in \{0, 1\}^{\mathbb{N}}$, and it can be verified that this sequence satisfies $\epsilon_n\epsilon_{n+1}\epsilon_{n+2} = 0$ for each n. We then project x to $z := \sum_n \epsilon_n \alpha^n$; note the connection to beta-numeration (evoked in Section 3) that appears; this connection is studied in [Thu89]. Geometrically, this amounts to the projection $\pi_{\mathtt{int}}$ along the expanding eigendirection of the matrix M_σ with respect to the decomposition of $\mathbb{R}^3$ into the expanding eigendirection and the contracting eigenplane of the matrix M_σ. The image $\pi_{\mathtt{int}}(X_\sigma)$ of X_σ is the *Rauzy fractal* $\mathcal{R}$; see Figure 5.3. Depending on what we see at the 0-index x_0 of x, we can further specify in which of the three regions $\mathcal{R}_0 = \alpha\mathcal{R}$, $\mathcal{R}_1 = \alpha^3 + \alpha^2\mathcal{R}$ and $\mathcal{R}_2 = \alpha^3 + \alpha^4 + \alpha^3\mathcal{R}$ the point $\pi_{\mathtt{int}}(x)$ lives. The projection is injective outside of the boundaries between these three regions, and shifting inside X_σ corresponds to an exchange from one region to another; see [Mes98] for details. The domain $\mathcal{R}$ is the internal space of a cut and project scheme which generates the Tribonacci tiling.

Rauzy fractals can more generally be associated with Pisot substitutions (see [AI01, BK06, Sie04] and the surveys [BS05, Pyt02]), as well as with Pisot beta-transformations and beta-shifts under the name of central tiles [Aki02]. Rauzy fractals can be defined in a unimodular case [BK06] (when the inflation factor β of a substitution is a Pisot unit) as well as in a p-adic setting [Sie04, Sin06] (the primes p that occur are prime divisors of the norm of β and the Rauzy fractal lives in a finite product of Euclidean and p-adic spaces). Moreover, the geometric properties of Rauzy fractals reflect the self-similarity of the associated substitutive

[3]Inter-model sets are model sets satisfying a topological condition that is less restrictive than the condition for the boundary of having zero measure satisfied by a regular model set.

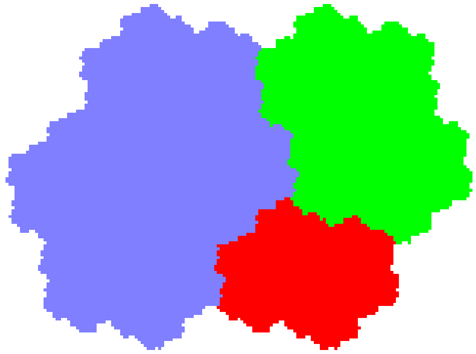

FIGURE 2. The Rauzy fractal

dynamical system: they are solutions of (graph-directed) iterated function systems. Note also that Rauzy fractals have also been studied as quasicrystals [VM01].

Rauzy fractals do not only produce geometric representations of substitutive dynamical systems, but they also have very interesting Diophantine applications. We refer to [IFHY03, AFSS10] for representative examples. Rauzy fractals are particularly convenient for providing arithmetic descriptions of periodic orbits, yielding relevant generalisations of Lagrange's and Galois' theorems for continued fractions. We recall that Lagrange's theorem states that the continued fraction expansion of x is eventually periodic if and only if either $x \in \mathbb{Q}$, or x is a quadratic number. Furthermore, if $x > 1$ is a quadratic number, and x' is its algebraic conjugate, then the continued fraction expansion of x is purely periodic if and only if x is irrational and $-1 < x' < 0$: this is Galois' theorem. Galois' theorem can be proved dynamically by using the Gauss map T acting on $[0, 1]$ as $x \mapsto \{1/x\}$. A key idea used here to provide a dynamical proof of Galois' theorem (and thus to describe purely periodic orbits) is to transform the map T that is not one-to-one into an injective map thanks to a suitable realisation of its natural extension.

The same type of methods can also be applied in the beta-numeration case allowing a characterisation of periodic orbits for a Pisot number β. The condition on algebraic conjugates is then expressed in terms of Rauzy fractals, such as stated below, noting that Galois' theorem expresses the pure periodicity of the continued fraction expansion of a quadratic real number x in terms of the location of x and its algebraic conjugate on the line. Here the location is expressed in terms of belonging to a Rauzy fractal.

Theorem 5.7. *If β is a Pisot number, then $x \in [0, 1]$ has an eventually periodic β-expansion if and only if $x \in \mathbb{Q}(\beta) \cap [0, 1]$. Moreover, a real number $x \in \mathbb{Q}(\beta) \cap [0, 1)$ has a purely periodic β-expansion if and only if the vector composed of x and its conjugates belong to the Rauzy fractal associated with β.*

The analogue of Lagrange's theorem has been proved in [Ber77, Sch80] and the analogue of Galois' theorem in [ABBS08] (see also the references therein). The proof is here again based on an explicit realisation of the natural extension of the beta-transformation T_β.

6. How computability came to model sets

We have seen with Meyer's Theorem 2.2 that algebraicity enters naturally into play for describing self-similarity for aperiodic order. We will see in this section that computability is an increasingly pertinent viewpoint to consider. The tiling setting is the one which is best suited for this discussion with the notion of a local rule, which describes the constraints on how adjacent tiles can be assembled. More generally, local rules naturally model the energetic finite-range interactions between atoms. Aperiodic order emerges when local properties enforce global order together with aperiodicity, and for tilings, this results in the research of local rules.

The notion of local rule is natural when considering Wang tiles. A Wang tile is a unit square with edges marked with symbols (or colors). A tiling by Wang tiles consists in assembling copies of the tiles (by translations only) so that symbols on shared edges match. These tiles were introduced by Wang in 1961 for the study of fragments of first order logics in the context of the *domino problem*, which asks for the existence of an algorithm deciding whether a finite set of Wang tiles can tile the plane. The domino problem is known to be undecidable, as proved by R. Berger in 1966. In dynamical terms, this amounts to ask whether a given two-dimensional shift of finite type is non-empty. This result is based on two ingredients: firstly, the simulation of Turing computations by tilings of the plane (usually via local rules or substitutions), and secondly, the existence of aperiodic sets of tiles (in the sense that they can only produce aperiodic tilings). Since the first examples of aperiodic shifts of finite type were based on hierarchical structures [Ber66, Rob71], substitutive structures have been known to be able to force aperiodicity. But more than that, in dimension $d \geq 2$, under natural assumptions, it is known that most substitution tilings admit local rules (possibly after *decoration*) [Moz89, GS98], as expressed in the following statement that covers all known examples of hierarchical aperiodic tilings.

Theorem 6.1. [GS98] *Every substitution tiling of the d-dimensional Euclidean plane, $d > 1$, can be enforced with finite matching rules, subject to a mild condition.*

After the connection between higher-dimensional substitutions and local rules was established, the following question became natural: can we describe a tiling, which is not necessarily substitutive, in terms of local rules? This question emerged in mathematics as early as the 1990s in the study of quasicrystals [Kat95, Lev88, Le95, Soc90] (see more references in [BF15, FS19]). This has known particularly rich developments for tilings obtained by the cut and project schemes. One important feature here is that it comes down both to computability and algebraic considerations on the associated parameters.

Consider for instance a tiling obtained as an approximation of a d-dimensional space E in $\mathbb{R}^n$, via the cut and project method. When $d = 2$, this yields a rhombus tiling of the Euclidean plane, approximating a real plane embedded in a higher-dimensional space. The question is then how to force tiles to approximate the desired plane E by specifying local rules. The case of lines in the planes ($d = 1$, $n = 2$) corresponds to Sturmian words, which cannot be defined with local rules, in direct contrast with the higher-dimensional case. The study of the connections between the existence of local rules for such planar tilings and the parameters of its slope started with [dB81, Lev88, Le95, Le97, Soc90]. The first conditions on the parameters of cut and projection schemes were of an algebraic nature. In particular, it was proved in [Le97] that a slope enforced by undecorated local rules is necessarily algebraic (this is however not sufficient, see e.g. [BF15, BF20]). However, computability comes into play when the tiles can be decorated (i.e., in dynamical terms, when we go from shifts of finite type to sofic ones). Decorations indeed allow the transfer of information through the tiling, and this was used in [FS19] to prove the following statement.

Theorem 6.2. [FS19, Corollay 2] *A slope can be enforced by colored weak local rules if and only if its slope is computable.*

This computational approach is reminiscent of the proof of the undecidability of the domino problem. Colored local rules are used to encode simulations of Turing computations which check that only planar tilings that approximate the desired planes can be formed. The fundamental tool is the use of so-called simulation theorems, which state that any effective one-dimensional shift can be obtained as the subaction of a two-dimensional sofic shift [Hoc09, AS13, DRS10]). The Penrose tiling is an example where all these viewpoints gather: it can be considered simultaneously as generated by a substitution, it is a model set, thus obtained via a cut and project scheme, and it can be endowed with local rules.

Nourished by this computability viewpoint, aperiodic order has known particularly fruitful developments with the study of tilings, spreading also in the direction of higher-dimensional word combinatorics, symbolic dynamics or else group combinatorics, with symbolic dynamics on groups and the domino problem for them. In this latter setting, the decidability of the domino problem is reinterpreted as a group property, inspired by Higman's embedding theorem which states that any finitely generated and recursively presented group can be embedded as a subgroup of a finitely presented group, such as developed in [Jea16, JV19].

In conclusion, Meyer's Theorem 2.2 and its mathematical successors show that Pisot numbers, and more generally arithmetic and Diophantine approximation, are natural actors of aperiodic order when self-similarity occurs. As an illustration, observe that the most classical combinatorial and dynamical measures of disorder developed for cut and project structures can be evaluated in terms of Diophantine properties such as developed e.g. in [HKWS16, HJKW19] with in particular nice connections with the Littlewood conjecture [HKW18]. However, the last developments of the study of quasiorder show that computability is also a viewpoint that has to be considered, with the possibility of encoding some computation in tilings. We have seen that this has emerged as early as in the 1960s with the introduction of Wang tiles. This concept then crystallised in a striking way 50 years later with the work of M. Hochman [Hoc09] and with the characterisation, obtained by M. Hochman and T. Meyerovitch, of the entropies of multidimensional finite type shifts. If these entropies are logarithms of Perron numbers in the one-dimensional case, M. Hochman and T. Meyerovitch made a fundamental change of perspective by proving that in the higher-dimensional case, these entropies are characterised as the recursive numbers that are right recursively enumerable [HM10].

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