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Numeration and discrete dynamical systems

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Abstract This survey aims at giving both a dynamical and computer arithmetic-oriented presentation of several classical numeration systems, by focusing on the discrete dynamical systems that underly them: this provides simple algorithmic generation processes, information on the statistics of digits, on the mean behavior, and also on periodic expansions (whose study is motivated, among other things, by finite machine simulations). We consider numeration systems in a broad sense, that is, representation systems of numbers that also include continued fraction expansions. These numeration systems might be positional or not, provide unique expansions or be redundant. Special attention will be payed to β-numeration (one expands a positive real number with respect to the base $\beta > 1$), to continued fractions, and to their Lyapounov exponents. In particular, we will compare both representation systems with respect to the number of significant digits required to go from one type of expansion to the other one, through the discussion of extensions of Lochs' theorem.

Keywords Discrete dynamical system · Numeration system · Continued fractions · Euclidean algorithm · Lyapounov exponent · Simulation

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1 Introduction: numeration dynamics

Let us first make precise what is meant by *numeration dynamics*, according to the terminology introduced by M. Keane. Numeration emcompasses here a system of representation of numbers together with a way of computing on them, *i.e.*, a way of performing operations and applying elementary functions. These numbers might be natural, rational, real, complex numbers, or else, vectors, polynomials, *etc.* These systems might be positional or not, provide unique expansions or be redundant, and we expect them to reflect the algebraic, arithmetic and computational properties of the numbers.

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When studying numeration systems, *discrete dynamical systems* occur in a natural way. (See Section 1.1 below for more on discrete dynamical systems.) As an example, consider the following question that is inherently of a dynamic nature: if one knows how to represent the number n, how to represent $n+1$, *i.e.*, how to perform the addition by 1? This transformation is usually called "successor function", "odometer " or else "adding machine". It has received considerable attention in the literature from various viewpoints; see *e.g.*, in the context of numeration dynamics [49] and Section 5 in [7], or in the context of automata and language theory [42,13]. Note also that the representation of arbitrarily large numbers requires often the iteration of a recursive algorithmic process. Numeration and dynamical systems are thus closely related such as illustrated in [7,30], which is why numeration dynamics aims at developing a dynamical viewpoint on numeration. Numeration dynamics also enters the framework of *arithmetic dynamics*. Arithmetic dynamics provides explicit expansions which have a dynamical meaning in order to produce symbolic codings of dynamical systems which preserve their arithmetic structure; see [91] for more details.

But numeration is also by essence a computational notion: numeration systems provide algorithmic ways of coding numbers with finitely many symbols. Let us illustrate, through the following examples, the relevance of numeration in computer arithmetic. Let us start with the possibility of performing basic operations or of evaluating functions by the simplest machines, namely finite automata. Under the assumption that the language of a numeration is regular, basic operations such as addition and multiplication by a fixed positive integer, or else normalization or digit set conversion, are known to be performed by finite automata (more precisely, transducers) (see [39–41]; see also [76] concerning the limitations of functions that are computable in on-line arithmetic by a finite automaton). Transducers also allow one to deduce the continued fraction expansion of an homographic image from the continued fraction expansion of the original number (see [87]). The analytic theory of continued fractions has now proved its efficiency for the evaluation of functions (see the classic references [55,97] and also [17, 28] for *a priori* truncation error estimates for continued fraction representations). Lastly, based on the work of Gosper [47], exact real computer arithmetic with continued fractions has also been thoroughly investigated (see [95], and also [60,66,73]).

The aim of this survey is to focus on the interaction between both viewpoints on numeration, namely, between dynamical and computational aspects, in particular through the issues raised by the finite state machine simulations of the discrete dynamical systems underlying these numeration systems. We have no claim for exhaustivity, but we have tried to highlight the wide diversity of the literature related to these interactions.

1.1 Discrete dynamical system

The term *discrete dynamical system* usually refers to a piecewise-continuous mapping $T\colon X \to X$ that acts on a space X that will be usually assumed to be compact. The terminology discrete refers here to the time: we consider *trajectories* (also called *orbits*) of points of X under the discrete-time deterministic action of the mapping T; the (one-sided) orbit of $x \in X$ under the action of T is defined as $\{T^n x \mid n \in \mathbb{N}\}$. Discrete dynamical systems can be of a geometric nature (*e.g.*, $X = [0,1]$), or of a symbolic nature.

Let us briefly describe *symbolic dynamical systems* (for more details, see *e.g.,* [58,69]). It is possible to associate a dynamical system with any infinite word $u = (u_n)_{n\geq 0}$ with values in a finite alphabet $\mathcal{A}$. The mapping acting on sets of infinite words is the shift S that maps an infinite word $(u_n)_{n\geq 0}$ onto this infinite word from which the first letter has been taken away, that is, on the infinite word $(u_{n+1})_{n\geq 0}$. The topology is given by the usual metric on infinite words: two infinite words are close if they coincide on their first terms. We then take as compact set on which the shift acts, the closure in $\mathcal{A}^{\mathbb{N}}$ of the orbit of u, *i.e.*, the closure of $\{S^n u \mid n \in \mathbb{N}\}$. A dynamical system defined on sets of words is said to be *symbolic*.

Let us come back to the general case of a discrete dynamical system $T\colon X \to X$. In order to understand the behavior of trajectories, it is natural to partition the set X into a finite number (say d) of subsets $(X_i)_{1\le i\le d}$: $X = \cup_{i=1}^{d} X_i$. We then *code* the trajectory of a point $x \in X$ with respect to the finite partition $(X_i)_{1\le i\le d}$. One thus associates with each point $x \in X$ an infinite word with values in the finite alphabet $\{1,\dots,d\}$ defined as follows: $\forall n \in \mathbb{N},\ u_n = i$ if and only if $T^n(x) \in X_i$. Coding trajectories allows one to go from geometric dynamical systems to symbolic dynamical systems and backwards, provided the coding has been chosen in a nice way.

We have considered so far the notion of dynamical system in a topological context. This notion can be extended to measurable spaces: we thus get measure-theoretic dynamical systems, that is, dynamical systems endowed with a probabilistic structure. We will introduce and discuss measure-theoretic dynamical systems in Section 3.1 by focusing on the Gauss map that produces the partial quotients in continued fraction expansions (see also Section 1.2 for more on the Gauss map).

1.2 Some examples of numeration dynamical systems

Let us use the usual base q numeration, as an illustration of a numeration system that can be described in terms of a dynamical system, where the integer q satisfies $q \ge 2$. But first recall that there are two well-known algorithmic ways of producing the digits $a_i \in \{0,\dots,q-1\}$ of the expansion of a positive integer $N = a_k q^k + \cdots + a_0$ in base q. The greedy algorithm produces the digits of N most significant digit first: take k such that $q^k \le N < q^{k+1}$ and set $a_k := \lfloor N/q^k \rfloor$; one then reiterates the process with N being replaced by $N - a_k q^k$ in order to get the digits in decreasing power order. Now, consider the second generation method. Let the notation y mod q stand for the unique number in $\{0,1,\dots,q-1\}$ which is congruent to y modulo q. The dynamical system $(\mathbb{N}, S_q)$ with

$$S_q\colon \mathbb{N} \to \mathbb{N}, \quad n \mapsto \frac{n - (n \bmod q)}{q}$$

together with the coding map $\psi_q\colon \mathbb{N} \to \{0,1,\dots,q-1\}, \quad n \mapsto n \bmod q$ (which is associated with the natural partition of $\mathbb{N}$ given by the sets $k + q\mathbb{N}$, for $0 \le k \le q-1$), produces the digits least significant digit first: one has $a_{i+1} = \psi_q(S_q^i(N))$ for all i. Taking all sequences of digits produced by considering all integers yields a symbolic dynamical system made of infinite words that all eventually take the value 0.

Similarly, the dynamical system producing the q-ary expansions of positive real numbers is defined as $([0,1], T_q)$, with

$$T_q\colon [0,1] \to [0,1],\ x \mapsto qx - \lfloor qx \rfloor = \{qx\} = qx \pmod 1,$$

together with the coding map $\varphi_q\colon [0,1] \to \mathbb{N},\ x \mapsto \lfloor qx \rfloor$. Indeed, if $x = \sum_{i\ge1} a_i q^{-i}$, then $\lfloor qx \rfloor = a_1 + \sum_{i\ge1} a_{i+1} q^{-i}$, and $\{qx\} = \sum_{i\ge1} a_{i+1} q^{-i}$. One thus has $a_i = \lfloor q T_q^{i-1}(x) \rfloor = \varphi_q(T_q^{i-1}(x))$, for all $i \ge 1$. Note that the admissible expansions produced by T_q never terminate in $(q-1)(q-1)(q-1)\cdots$. When $q = 10$ one recovers the decimal expansion, and the binary one for $q = 2$.

More generally, the so-called *beta-numeration* embraces and extends q-ary numeration. Taking a real number $\beta > 1$, it consists in expanding numbers $x \in [0,1]$ as power series in base β^{-1} with digits in the set $\{0,\dots,\lceil\beta\rceil - 1\}$. The mapping

$$T_\beta\colon x \mapsto \{\beta x\} = \beta x \pmod 1$$

together with the coding map $\varphi_\beta\colon x \mapsto \lfloor \beta x \rfloor$ produces the digits $a_i = \lfloor \beta T_\beta^{i-1}(x) \rfloor = \varphi_\beta(T_\beta^{i-1}(x))$, for $i \ge 1$, which yields the expansion $x = \sum_{i\ge1} a_i \beta^{-i}$. For more on beta-numeration, see *e.g.*, [14,89,30,38, 43]. Such expansions belong to the more general family of so-called f-expansions [88]: one expands real numbers as

$$x = \lim_{n\to\infty} f(a_1 + f(a_2 + f(a_3 + \cdots + f(a_n)\cdots))),\ \text{with } a_i \in \mathbb{N}.$$

We consider now another type of expansion. The Gauss map defined on $[0,1]$ by

$$T_G\colon x \mapsto \{1/x\} \text{ for } x \neq 0, \quad T_G(0) = 0$$

together with the coding map $\varphi_G\colon x \mapsto \lfloor 1/x \rfloor$ produces the partial quotients in the *continued fraction* expansion of a real number $x \in [0,1]$ (see Section 3 for more details, and more generally [57,51,55]). Let $x \in (0,1)$. If $x_1 = T_G(x) = \{1/x\} = \frac{1}{x} - \lfloor \frac{1}{x} \rfloor = \frac{1}{x} - a_1$, then $x = \frac{1}{a_1 + x_1}$. Now, set $a_n = \lfloor \frac{1}{T^{n-1}x} \rfloor = \varphi_G(T^{n-1}x)$ for $n \geq 1$. One has

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \cdots}}}}.$$

We also will use the notation $x = [0; a_1, \cdots, a_n, \cdots]$. The digits a_n are called *partial quotients*. Continued fractions are known to provide good rational approximations of real numbers. Indeed let $p_n/q_n = [0; a_1, \cdots, a_n]$ stand for the n-th truncation of the continued fraction of x. One has $|x - p_n/q_n| < 1/q_n^2$ for all n.

Note that continued fractions extend also to Laurent formal power series with coefficients in a finite field, see *e.g.*, the survey [12] and the references therein. As an application, let us mention some interesting connections with pseudorandom numbers generated by the digital multistep method [78], with low-discrepancy sequences [79], or with stream cipher theory and cryptography [80].

Let us briefly mention two other classical dynamical numeration systems (see also [7,38,43] and the references therein). *Canonical number systems* allow one to expand numbers in algebraic bases, see *e.g.*, [59,56]. *Signed number systems*, based on the use of negative digits, are known for their applications in computer arithmetic and in cryptography. Indeed, they allow redundant or sparse representations: *redundancy* facilitates arithmetical operations and allows their parallelism by the limitation of the propagation of the carry when performing additions and subtractions (see *e.g.*, [3,74,75,4,77]); *sparse* representations (the non-adjacent signed binary expansion is known to have, on average, only one third of the digits that are different from zero, see *e.g.*, [18]) are used for the multiplication and the exponentiation in cryptography (see [50,48,52] and the references therein).

1.3 Dynamics and computation

Let us come back to dynamical systems. An orbit $(T^n(x))_{n\in\mathbb{N}}$ of the dynamical system (X,T) is said to be *eventually periodic* if there exists n such that $T^n(x) = T^{n+k}(x)$, for $k \in \mathbb{N}$. If $n = 0$, then the orbit is said to be *purely periodic*. In order to define a stronger notion of "finiteness" for an orbit, we will assume that (X,T) has 0 as a fixed point: $T(0) = 0$. The orbit $(T^n(x))_{n\in\mathbb{N}}$ is thus said to be *finite* if there exists n such that $T^n(x) = 0$; this yields $T^{k+n}(x) = 0$ for all $k \in \mathbb{N}$. Finite orbits are thus particular cases of periodic orbits. For instance, in base 10, rational numbers have periodic orbits, whereas decimal numbers have finite orbits.

Note that among the orbits of a dynamical system, periodic ones are particularly interesting from a computational viewpoint. Indeed the orbits produced by a finite state machine simulation of a dynamical system are eventually periodic (the set of representable numbers is finite, we use here finite precision computer arithmetic). By *finite state machine simulation* of the dynamical system (X,T), we mean the following: we consider a finite set $\hat{X}$ (which is a set of finite sequences of, usually, binary digits), a coding map $\varphi\colon X \to \hat{X}$, and a map $\hat{T}$ that acts on $\hat{X}$ ($\hat{T}(\hat{X}) \subset \hat{X}$), whose action is defined as a finite state machine, *i.e.*, the image of $x \in \hat{X}$ by $\hat{T}$ is computed by a finite state machine that takes as input the sequence of digits of x and then outputs the sequence of digits of $\hat{T}(x)$; we also want the behavior of $\hat{T}$ to be close to the behavior of T, that is, $\hat{T} \circ \varphi$ to be close to $\varphi \circ T$. As an example, consider a floating-point simulation (see for instance (1) for the Gauss map). Of course, rounding and truncation errors are then to be taken care of (see Section 3.3 for a discussion).

The following questions are thus natural, both from a computational and an arithmetic viewpoint: what are the finite or the periodic expansions of the dynamical system (X, T)? how to describe them? do periodic expansions have a typical behavior? how far are periodic orbits from typical ones? Assume now we have a finite state machine simulation of the dynamical system (X, T). What can be said concerning the roundoff errors when simulating trajectories? how far are computed orbits from exact ones? are there typical orbits among computable ones? We will come back to these questions in Section 3.3 through the study of the Gauss map.

Contents of the paper

We now have gathered all the tools and notions required for presenting an overwiew of the present paper. Section 2 will illustrate the use of dynamical methods for describing periodic orbits, and thus producing Galois' type theorems. We then will use as a guideline the Gauss map and continued fractions in Section 3 in order to introduce and discuss the notions of ergodicity (see Section 3.1), chaoticity (see Section 3.2), and of numerical simuations of dynamical systems trough a floating-point version of the Gauss map in Section 3.3 (based on [25–27]). Section 4 is devoted to the classic Lochs' theorem [70] which can be summarized under the following form: continued fraction expansions are not significantly more efficient at representing real numbers than the decimal expansion. We conclude this survey by alluding to the multidimensional case in Section 5.

2 Periodic orbits and dynamics

Let us illustrate the efficiency of the dynamical approach by providing a simple dynamical proof of the following well-known fact: the rational numbers having, when reduced, a denominator coprime with 10, have a purely periodic decimal expansion.

We have seen that the digits $a_i \in \{0, \dots, 9\}$ of the decimal expansion of a positive real number $x = \sum_{i \geq 1} a_i 10^{-i}$ are produced by $T_{10}\colon [0,1] \to [0,1]$, $x \mapsto 10x - \lfloor 10x \rfloor = \{10x\}$. We will use the notation $T = T_{10}$ in this paragraph for the sake of simplicity. Let us fix $a/b \in (0,1]$ with b coprime with 10 and let us prove that the decimal expansion of a/b is purely periodic. One has

$$T(a/b) = \frac{10a - \lfloor 10 \cdot a/b \rfloor \cdot \ b}{b} = \frac{(10 \cdot a) \bmod b}{b};$$

recall that $y \bmod b$ stands for the unique number in $\{0, 1, \dots, b-1\}$ which is congruent to y modulo b. More generally, for any positive integer k, the denominator of $T^k(a/b)$ is equal to b and its numerator belongs to $\{0, 1, \dots, b-1\}$. The sequence of numerators of $T^k(a/b)$, for $k \in \mathbb{N}$, is eventually periodic since taking a finite number of values. We want to prove that this sequence is purely periodic. Assume that this sequence is produced by iterating a one-to-one map; the fact that it is eventually periodic together with injectivity implies pure periodicity. However, the map T is not one-to-one on $(0, 1)$. But, luckily enough, this is the case of the mapping $\widetilde{T}_b\colon x \mapsto 10 \cdot x \bmod b$ that produces the numerator of $T(x/b)$, for any $x \in \{0, 1, \dots, b-1\}$: we use the fact that b and 10 are coprime. This implies the expected result, that is, the sequence $\widetilde{T}_b^k(a/b)$, for $k \in \mathbb{N}$, is not only eventually periodic but purely periodic. We thus have proved that the decimal expansion of a rational number having, when reduced, a denominator coprime with 10 is purely periodic. Note that the converse also holds true.

The key idea used here to obtain a description of purely periodic orbits is to transform the map T of the dynamical system (that is *a priori* not one-to-one) into an injective one. This method does not only apply to decimal, or more generally q-adic numeration. It also applies *e.g.*, for the characterization of periodic and eventually periodic orbits of continued fractions and β-expansions, as will be explained below.

We first start with the case of continued fractions. The arithmetic characterization of periodic orbits is well-known. Lagrange theorem states that the continued fraction expansion of $x \in (0,1)$ is eventually periodic if and only if either $x \in \mathbb{Q}$, or x is a quadratic number. Furthermore, if x is a quadratic number, and x' stands for its algebraic conjugate, then the continued fraction expansion of x is purely periodic if and only if x is an irrational quadratic number and $x' < -1$: this is Galois' theorem. Note that points having periodic orbits are dense in $[0,1]$. This is also the case of points having finite orbits: these are exactly the rational numbers in $[0,1]$.

By using the same idea as previously, let us give a dynamical and short proof of Galois' theorem that can be considered as being part of the folklore literature on the subject. Let $x \in (0,1)$ with x being a quadratic number. We want to prove that x has a purely periodic expansion if and only if $x' < -1$. We will transform the Gauss map into an invertible map defined on $[0,1] \times (\mathbb{R} \setminus \{0\})$ by transforming the Gauss map into a two-dimensional map, with the Gauss map acting on the first coordinate. We thus introduce a so-called *realization of the natural extension* of the Gauss map; it is defined over $[0,1] \times (\mathbb{R} \setminus \{0\})$ as follows:

$$\widetilde{T}_G(x,y) = (\{1/x\}, 1/y - \lfloor 1/x \rfloor) \text{ for } x, y \neq 0, \quad \widetilde{T}_G(0,y) = (0,0) \text{ for } y \neq 0.$$

The digits that are produced by the Gauss map on the first coordinate are lost when performing T_G, but not for $\widetilde{T}_G$ that stores them thanks to the second coordinate.

In order to describe periodic orbits, one needs to find a stable set under $\widetilde{T}_G$ on which the restriction of $\widetilde{T}_G$ is a bijection. This is the case of the set $A = [0,1] \times (-\infty, -1)$: it is stable by $\widetilde{T}_G$, and $\widetilde{T}_G$ is one-to-one and onto A. Furthermore, if x is a quadratic number, and x' stands for its algebraic conjugate, one has $\widetilde{T}_G(x,x') = (Tx, (Tx)')$, and more generally $\widetilde{T}_G^n(x,x') = (T_G^n x, (T_G^n x)')$, for all n. The notation y' stands for the algebraic conjugate; note that $T_G^n(y)$ is a quadratic number as soon as y is a quadratic number. Lastly, A is an atttractor for conjugate pairs, that is, there exists n such that $\widetilde{T}_G^{n+k}(x,x') \in A$ for all $k \in \mathbb{N}$, with x' being the conjugate of the quadratic number x (this comes from the fact that x and x' have different continued fraction expansions since being distinct).

Assume first that x has a purely periodic expansion. Since A is an attractor, there exists n such that $\widetilde{T}_G^n(x,x') \in A$, and thus $\widetilde{T}_G^m(x,x') \in A$, for $m \geq n$. As A is stable and $\widetilde{T}_G$ is one-to-one, one has $(x,x') \in A$.

Assume now that x has an eventually periodic expansion and that $(x,x') \in A$. Let y be defined as the real number in $[0,1]$ having as continued fraction expansion the purely periodic part of x. For n large enough, one has $\widetilde{T}_G^n(x,x') = \widetilde{T}_G^n(y,y') \in A$. Since $\widetilde{T}_G^n$ is bijective, one gets $x = y$, and consequently, x has a purely periodic continued fraction expansion, which ends the proof of Galois' theorem for continued fractions.

A similar characterization (and proof) holds for continued fractions and formal power series with coefficients in a finite field (see [12]). The same type of methods also applies in the β-numeration case allowing a characterization of periodic orbits for a Pisot number β. An algebraic integer $\alpha > 1$, *i.e.*, a root of a monic polynomial with integer coefficients, is a *Pisot-Vijayaraghavan number* or a *Pisot number* if all its algebraic conjugates λ other than α itself satisfy $|\lambda| < 1$. The analogue of Lagrange theorem has been proved in [14,89]: if β is a Pisot number, then $x \in [0,1]$ has an eventually periodic expansion if and only if $x \in \mathbb{Q}(\beta) \cap [0,1]$. Moreover, an analogue of Galois' theorem is given in [1] (see also the references therein): let β be a Pisot number; a real number $x \in \mathbb{Q}(\beta) \cap [0,1)$ has a purely periodic β-expansion if and only if x and its conjugates belong to an explicit subset in a finite product of Euclidean and p-adic spaces that depends on β; this set (called *generalized Rauzy fractal*) is a graph-directed self-affine compact subset of non-zero measure; the primes p that occur are prime divisors of the norm of β. The proof is here again based on an explicit realization of the natural extension of the β-transformation T_β.

3 Continued fractions

The continued fraction algorithm is obtained by applying the Gauss map $T_G : x \mapsto \{1/x\}$, which is closely related to *Euclid's algorithm*: let us start with two (coprime) positive integers u_0 et u_1; Euclid's algorithm works by subtracting as much as possible the smallest of both numbers from the largest one (that is, one performs the Euclidean division of the largest one by the smallest); this yields $u_0 = u_1\lfloor \frac{u_0}{u_1} \rfloor + u_2$, $u_1 = u_2\lfloor \frac{u_1}{u_2} \rfloor + u_3$, *etc.*, until we reach $u_{m+1} = 1 = \mathrm{pgcd}(u_0, u_1)$. By setting for $i \in \mathbb{N}$, $\alpha_i = \frac{u_i}{u_{i+1}}$ and $a_i = \lfloor \alpha_i \rfloor$, one gets $\alpha_{i-1} = a_{i-1} + \frac{1}{\alpha_i}$ and

$$\alpha_0 = u_0/u_1 = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots + \frac{1}{a_{m-1} + 1/a_m}}}}$$

For general references on continued fractions, see *e.g.*, [15,30,51,57].

3.1 Continued fractions and ergodicity

One interest of the dynamical approach is that it provides statistical information concerning the partial quotients that are produced by the Gauss map and more generally, concerning the behavior of orbits, through the notion of *ergodicity*: this yields metric results that hold almost everwyhere with respect to the Lebesgue measure on $[0,1]$. We recall in this section basic ergodic properties of the Gauss map.

More precisely, let us endow the dynamical system $([0,1], T_G)$ with a structure of a measure-theoretic dynamical system. A *measure-theoretic dynamical system* is defined as a system $(X, T, \mu, \mathcal{B})$, where μ is a probability measure defined on the σ-algebra $\mathcal{B}$ of subsets of X, and $T : X \to X$ is a measurable map which preserves the measure μ ($\mu(T^{-1}(B) = \mu(B)$ for all $B \in \mathcal{B}$). Here, we endow $([0,1], T_G)$ with the Gauss measure μ_G which is the Borel probability measure defined as the absolutely continuous measure with respect to the Lebesgue measure by

$$\mu_G = \frac{1}{\log 2} \int \frac{1}{1+x} dx.$$

One checks that this measure is T_G-invariant, *i.e.*, $\mu_G(B) = \mu_G(T_G^{-1}B)$ for every Borel subset B of $[0,1]$. The Gauss map is *ergodic* with respect to the Gauss measure, that is, every Borel subset B of $[0,1]$ such that $T_G^{-1}(B) = B$ has either zero or full measure. This implies that almost all orbits are dense in $[0,1]$ (almost all means that the set of elements x whose orbit is not dense is contained in a set of zero measure). More generally a property is said to hold *almost everywhere* (abbreviated as a.e.) if the set of elements for which the property does not hold is contained in a set of zero measure; this property is said to be *generic* (the points that satisfy this property are then also said to be generic). This helps us to give a meaning to the notion of typical behavior for a dynamical system.

Ergodicity yields furthermore the following striking convergence result. Indeed, measure-theoretic ergodic dynamical system satisfy the *Birkhoff ergodic theorem*, also called *individual ergodic theorem*, which relates spatial means to temporal means.

Theorem 1 (Birkhoff Ergodic Theorem) *Let $(X, T, \mu, \mathcal{B})$ be an ergodic measure-theoretic dynamical system. Let $f \in L^1(X, \mathbb{R})$. Then the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k)_{n \geq 0}$ converges a.e. to $\int_X f \, d\mu$:*

$$\forall f \in L^1(X, \mathbb{R}) \ , \quad \frac{1}{n}\sum_{k=0}^{n-1} f \circ T^k \xrightarrow[n\to\infty]{\mu-a.e.} \int_X f \, d\mu \ .$$

Points for which this convergence property holds for a given f are generic.

For more on ergodic aspects of the Gauss map, see [15,30]; see also [36,37] which focus on spectral properties of transfer operators. More generally, for more about ergodic theory of discrete dynamical systems, the reader is referred to [98]. Lastly, for computational aspects (with respect to numerical simulation) of ergodic theory, see [24], and for algorithmic effectiveness, see for instance [96,44].

3.2 Continued fractions and chaoticity

This section aims at illustrating the chaotic nature of the Gauss map, motivated by the question of its numerical simulation.

A dynamical system is said to be *chaotic* if it is sensitive to initial conditions, if its periodic points are dense, and if it is topologically transitive. A dynamical system is said to be *topologically transitive* if for any pair of open sets $U, V \subset X$, there exists a positive integer n such that $T^n(U) \cap V \neq \emptyset$. This condition can be seen as a topological indecomposability property. A map is said to be *sensitive to initial conditions* if close initial points have divergent orbits, with the separation rate being exponential. In particular, the Gauss map is sensitive to initial conditions: rational initial points form a dense set and are attracted to 0, whereas quadratic irrational points are eventually attracted to a periodic orbit (which is not finite).

The *Lyapounov exponent* measures the exponential rate of separation of orbits. It is defined for a dynamical system (X, T), with T being piecewise differentiable as

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log \left(\prod_{i=0}^{n} |T'(T^i(x))| \right),$$

when this limit exists. Note that the formula given here comes from the chain rule applied to $T^{n+1}(x)$ in order to get its derivative. This allows us to get information on $|T^n(x) - T^n(y)|$. Indeed, intuitively (and as nicely explained in Chap. 9 of [24]), $|T(x) - T(y)|$ is approximatively equal to $T'(x) \cdot |x - y|$ (under suitable hypotheses such as x and y being close), whereas $|T^n(x) - T^n(y)|$ has to be compared with $\prod_{i=0}^{n-1} |T'(T^i x)| \cdot |x - y|$. This implies

$$|T^n(x) - T^n(y)| \sim \exp n\lambda(x) \cdot |x - y|,$$

which allows one to connect the rate of divergence of distinct orbits to the Lyapounov exponent. By applying the ergodic theorem to the Gauss map, one obtains that the Lyapounov exponent is a.e. equal to $\frac{\pi^2}{6\log 2}$. Indeed, one has for a.e. $x \in [0, 1]$

$$-2 \lim_{n\to\infty} \frac{1}{n} \left(\sum_{i=0}^{n} \log(T_G^i(x)) \right) = -\frac{2}{\log 2} \int_0^1 \frac{\log x}{1+x} d\mu = \frac{\pi^2}{6\log 2}.$$

For $\beta > 1$, the Lyapounov exponent of the β-transformation T_β is proven to be equal to $\log \beta$ as a direct application of the formula and of the fact that T_β is piecewise linear (the set of discontinuities is countable).

For a survey on chaoticity, see [31,16]. See also [92] for a survey of the numerical method devoted to the computation of Lyapounov exponents for dynamical systems. As an illustration of the chaoticity of the Gauss map, see [25–27]. For a discussion on the number of significant digits required for the numerical estimation of the Lyapounov exponent, see [24]. See also in the same vein Section 4 below.

Since measure-theoretical dynamical systems are defined up to sets of zero measure, the relevance with respect to computation can be questioned. In particular, what can be said concerning the behavior of rational points under the Gauss map? We just know that their orbits are finite, and that they correspond to the application of Euclid's algorithm. We also know that points having finite, or periodic orbits are dense. This does not imply *a priori* that their orbits behave in a generic way. For instance, there exist periodic orbits with arbitrarily large Lyapounov exponents. This will be the subject of next section to discuss these issues.

3.3 The floating-point Gauss map

We will follow here mainly [25–27] devoted to the Gauss map and to its following floating-point simulation

$$\widehat{T}_G(0) = 0,\ \widehat{T}_G(x) = 1/x \bmod 1 \text{ otherwise}, \tag{1}$$

the operations of division and reduction modulo 1 being defined on the floating-point domain (*i.e.*, on the finite set of numbers represented in this fixed-precision system). Note that orbits under the real Gauss map T_G are all finite (they reach 0) since machine-representable numbers are rational numbers. If one looks at the orbits produced by the floating-point Gauss map, it is *a priori* unclear to know whether they also reach 0. Further questions arise in a natural way. How far are calculated orbits from exact orbits? is the chaotic behavior of the Gauss map preserved when working with its floating-point simulation $\widehat{T}_G$? is it possible to define a Lyapounov exponent for the map $\widehat{T}_G$?

These questions occur more generally when working with numerical simulations of dynamical systems. There are two levels of difficulties that have to be handled. First, one has to check that the roundoff errors do not accumulate. One way to handle this problem is to prove that orbits under the simulation of the dynamical system have a counterpart in the exact dynamical system, *i.e.*, that they are uniformly close to exact orbits of T. These orbits are said to *shadow* the simulated orbits; see [84] for more on the concept of shadowing (roughly speaking, approximate orbits of a dynamical system are closely followed by exact orbits). But, a second problem occurs: even if orbits under a floating-point version of the dynamical system are proved to be close to exact orbits, there is no reason for these orbits to be generic with respect to T.

Let us come back to the Gauss map. It is proved in [25, 27] that orbits under the floating-point Gauss map are uniformly close to exact orbits with an explicit construction of the initial point of the exact orbit. The proof relies on "backward error" analysis. Let us quote [26]: "the y whose actual orbit is shadowing the numerical simulation is a quadratic irrational or rational number, and thus is from a set of zero measure and in particular does not have a dense orbit or the correct Lyapounov exponent." Thus, this should indicate that the floating-point Gauss map cannot be used as a good approximation of the real Gauss map. Yet it is in fact a good approximation. "The final resolution of this paradox must somehow account for the fact that the true shadowing orbit behaves like a typical orbit, even though it is not", still by quoting [26]. This comes from the fact that shadowing orbits are usually long, they thus have a tendency to behave like a generic one (even if the Lyapounov exponent is not defined when starting from a rational point!). Furthermore, periods tend to be "long" as indicated in [26]: indeed, according to [59, 35], if N stands for the total number of floating-point numbers in the simulation, then the average length of the period of an orbit is $\sqrt{\pi N/8} + O(1)$ for a map uniformly chosen randomly among all the maps from $\{1, \ldots, N\}$ to itself (each map has probability $1/N^N$ to be chosen). See also [45, 46] in the same vein. Furthermore, the equidistribution results for quadratic irrational numbers obtained in [86], through the use of Parry's prime orbit method [83], confirm this long orbit behavior, and the fact that the periodic orbits capture some kind of genericity: taking averages on periodic orbits yields the usual ergodic limits; if each individual periodic orbit behaves in a non-generic way, the distribution of the quadratic irrational numbers ordered with respect to the lengths of their period follows the Gauss measure.

As quoted from [25], all this even yields "a candidate for 'the worlds' worst' algorithm for computing π. $[\cdots]$. This method is likely worse than nearly any other in existence, since it does **not** converge to the correct value in any particular fixed-precision system, since all orbits are eventually periodic, and the Lyapounov exponent of a periodic orbit is the logarithm of an algebraic number.[$\cdots$]. This method is clearly related to the Monte-Carlo methods, with the roundoff error associated with the floating-point arithmetic playing the part of the random number generator required". (The approximate Lyapounov exponent for the orbit $(\hat{x})_i$ under the floating-point map $\hat{T}_G$ is here defined as $-2\frac{1}{N}\log\left(\sum_{i=0}^{N}\log(\hat{x}_i)|\right)$.)

We find the same kind of paradox within the so-called framework of "dynamical analysis of algorithms" which mixes analysis of algorithms (such as initiated by D. E. Knuth) and spectral study of dynamical systems through their transfer operators, with probabilistic and ergodic methods. In particular, the dynamical analysis of Euclid's algorithm (performed in full details in [23, 72, 5], see also [36, 37, 94]), proves that the orbits of rational points behave indeed in a generic way. This confirms the pertinence of a dynamical approach in contexts where only integer parameters are to be considered, such as in discrete geometry: continued fractions and Euclid's algorithm are indeed known to describe discretizations of lines, and their possible generalizations describe similarly discrete planes (see also Section 5 and [10]).

4 Continued fractions vs. decimal expansions

The aim of this section is to compare in average the level of information required for computing the continued fraction expansion of a positive real number x whose expansion in some numeration system (decimal, binary, base β *etc.*) is given. More precisely, we want to know in average the number of digits in one symbolic representation (here, the continued fraction expansion) that can be obtained from the first n digits in another representation.

We first start with decimal expansions (this is the case that has been first handled in the literature, it also yields the more striking result) and ask for the number of decimal digits required for expanding x in continued fraction. We first fix the notation. Let $x \in (0,1)$ be an irrational number with continued fraction $x = [0; a_1, \cdots, a_n, \cdots]$, and with decimal expansion $x = \sum_{i\geq 1} \frac{\varepsilon_i}{10^i}$, with $\varepsilon_i \in \{0, 1, \ldots, 9\}$ for all $i \geq 1$. For $n \geq 1$, let x_n be the lower n-th decimal approximations of x: $x_n = \sum_{i=1}^{n} \frac{\varepsilon_i}{10^i}$.

If two numbers are sufficiently close, then their respective continued fraction expansions have the same first partial quotients. Let us quantify this. For a fixed non-negative integer n, let $k_n(x)$ be the largest non-negative integer k such that the first k partial quotients of x_n are equal to the first k partial quotients of x. The following classic result by G. Lochs [70] describes the a.e. behavior of the quantity $k_n(x)$ and indicates that the n first decimals determine approximatively n of the first partial quotients, which might seem at first view non-intuitive.

Theorem 2 *[70] For almost every irrational number $x \in [0,1]$ (with respect to the Lebesgue measure)*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{6\log 10 \log 2}{\pi^2} \sim 0.9702. \tag{2}$$

In particular, Lochs has shown in [71] that the first 1000 decimals of π give the first 968 partial quotients of the continued fraction expansion of $\pi - 3$. For refinements of this result, still in the decimal case, see [32–34, 99, 100]. In particular, it is proved in [34] that if x is such that the sequence $(\frac{\log q_n(x)}{n})_n$ converges (we denote as $\beta(x)$ this limit called *Lévy constant*) and if the growth of its partial quotients satisfies $a_n(x) = O(\alpha^n)$, for all $\alpha > 1$, then

$$\lim \frac{k_n(x)}{n} = \frac{\log 10}{2\beta(x)}.$$

This covers in particular the case of quadratic numbers.

Note that we recognize in (2) the Lyapounov exponent λ_G of the Gauss map, *i.e.*,

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\log 10}{\lambda_G}.$$

This is in fact not surprising to have the Lyapounov exponent λ_G intervening in the statement of Lochs' theorem. Indeed, the sensitive dependence on initial conditions (*i.e.*, the fact of having a positive Lyapounov exponent) governs the accuracy of computations and makes it even decrease exponentially fast. As quoted from [24], "Due to the sensitive dependence on initial conditions $[\cdots]$ there is a possibility of obtaining meaningless output after many iterations of a transformation in computer experiment. Once we begin with sufficiently many digits, however, iterations can be done without paying much attention to the sensitive dependence on initial data. The optimal number of significant digits can be given in terms of the Lyapounov exponent." Still following [24], the *divergence speed* for a dynamical system (X, T), and for $0 \le x \le 1 - 10^{-n}$ with a fixed $n \ge 1$, is defined as

$$V_n(x) = \min\{j \ge 1 \mid |T^j(x) - T^j(x + 10^{-n})| \ge 10^{-1}\}.$$

This quantity is related to the Lyapounov exponent: $V_n(x) \sim n/\lambda(x)$, which implies that on average, the number of significant digits for $T(x)$ becomes $n - \lambda(x)$. The computations made in [24] are based on the maximal number of iterations that can be performed with no loss of precision when working with n significant digits, which can be quantified thanks to the Lyapounov exponent.

A natural question is to understand the dependence of Lochs' theorem with respect to the choice of the basis, namely, here, 10. Lochs's theorem was generalized to more general numerations and transformations in [19,29,68,8], where it was shown that these generalizations of Lochs' theorem can be expressed in terms of the ratio of the entropies (*i.e.*, of the Lyapounov exponents) of the maps involved. In particular, the question of the comparison with β-expansions ($\beta > 1$) is thoroughly answered in [8] (thus also covering the case of q-adic expansions). One expands x as $\sum_{i\ge 1} \frac{\varepsilon_i}{\beta^i}$, where $\varepsilon_i \in \{0, 1, \ldots, \lceil\beta\rceil - 1\}$ for all $i \ge 1$. Recall that the Lyapounov exponent λ_β of the β-transformation T_β is equal to $\log\beta$. Lochs' theorem becomes in this more general framework, with $k_n(x)$ being defined in a similar way as in the decimal case:

Theorem 3 *[8] For every $x \in |0, 1]$*

$$\lim_{n\to\infty} \frac{k_n(x)}{n} = \frac{\lambda_\beta(x)}{\lambda_G(x)} = \frac{6 \log 2 \log\beta}{\pi^2},$$

whenever both limit exist simultaneously.

Note that the Lyapounov exponent λ_G of the Gauss map is also expressed as the following limit (when it exists):

$$\lambda_G(x) = -\lim_{n\to\infty} \frac{1}{n} \log|x - p_n/q_n|,$$

where $p_n/q_n = [0; a_1, \cdots, a_n]$. It thus also measures the exponential speed of convergence of the convergents. Theorem 3 makes Lochs' theorem more intuitive since, as underlined in [8], "if x is well approximated by rational numbers, then the amount of information about the continued fraction expansion that can be obtained from its β-expansion is small. Moreover, the larger β is (that is, the more symbols we use to code a number x), the more information about the continued fraction expansion we obtain". Moreover, [8] also provides the Hausdorff dimension of level sets via multifractal analysis and thermodynamic formalism, and proves that a similar result holds for more general Markov maps. Lochs' theorem has also been the object of further extensions for formal power series with coefficients in a finite field (see [67]).

5 Toward multidimensional expansions

Let us end this survey by briefly discussing possible higher-dimensional extensions of the Gauss map. Consider as an illustration the following two maps

$$T_0\colon (\alpha,\beta) \mapsto (\{1/\alpha\},\{\beta/\alpha\})$$

$$T_{JP}\colon (\alpha,\beta) \mapsto (\{\beta/\alpha\},\{1/\alpha\}).$$

They are *a priori* very similar but we will see that they display totally different behaviors. The first one is know as *Ostrowski map*. One associates with it two numeration systems, one on integers, and the other one on real numbers (see *e.g.*, [9,6]). Let $\alpha \in (0,1)$ be an irrational number. Let $\alpha = [0; a_1, a_2, \cdots, a_n, \cdots]$ be its continued fraction expansion with convergents $p_n/q_n = [0; a_1, a_2, \cdots, a_n]$. Every integer N can be expanded uniquely in the form

$$N = \sum_{k=1}^{m} b_k q_{k-1},$$

where $0 \le b_1 \le a_1 - 1$, $0 \le b_k \le a_k$ for $k \ge 2$, $b_k = 0$ if $b_{k+1} = a_{k+1}$. Real numbers are expanded according to the base given by the sequence $(\theta_n)_{n\ge 0}$, where $\theta_n = (q_n\alpha - p_n)$. Every real number $-\alpha \le \beta < 1-\alpha$ can be expanded uniquely in the form

$$\beta = \sum_{k=1}^{+\infty} c_k \theta_{k-1},$$

where $0 \le c_1 \le a_1 - 1$, $0 \le c_k \le a_k$ for $k \ge 2$, $c_k = 0$ if $c_{k+1} = a_{k+1}$, $c_k \ne a_k$ for infinitely many even and odd integers.

Ostrowski's numeration system is used to approximate β modulo 1 by numbers of the form $N\alpha$, with $N \in \mathbb{N}$. Indeed, the sequence of integers $N_n = \sum_{k=1}^{n} c_k q_{k-1}$ provides good approximations of $\beta = \sum_{k=1}^{+\infty} c_k \theta_{k-1}$, since $N_n\alpha = \sum_{k=1}^{n} c_k q_{k-1}\alpha \equiv \sum_{k=1}^{n} c_k (q_{k-1}\alpha - p_{k-1})$ mod 1.

Ostrowski's numeration system has many applications. As an example, a fast algorithm for computing a lower bound on the distance between a straight line and the points of a regular grid is given in [65,64], which is used to find worst cases when trying to round correctly the elementary functions in floating-point arithmetic; this is the so-called Table Maker's Dilemma [65], see also [77]. This algorithm is closely related to the algorithm (based on Ostrowski's map T_0) which is presented in [11], and which provides a greedy representation algorithm for double-base number systems. Note that Ostrowski's numeration has deep relations with the study of discrete lines in discrete geometry (via the study of Sturmian words), as discussed in [9].

Pairs of real numbers (α,β) having periodic expansions with respect to Ostrowski's numeration system are characterized in [54] with the same method as the one discussed in Section 2: these are quadratic numbers. In fact, the map T_O is a skew product of the continued fraction transformation, which allows its metrical study (see *e.g.*, [53,54]); in other words, the first coordinate governs the actions made on the second one.

The second map, called *Jacobi-Perron*, is one of the most classic multidimensional continued fraction algorithms. It was introduced by Jacobi, and then later by Perron, in order to characterize cubic numbers as numbers having periodic expansions. Numerical evidence does not support this belief anymore (see [93] for an algorithm aiming at characterizing cubic numbers). Nevertheless, Jacobi-Perron algorithm and Ostrowski algorithms behave in completely different ways, arithmetically or as dynamical systems. Indeed, Jacobi-Perron map "mixes" the actions performed on both coordinates, whereas Ostrowski always performs the Gauss map on the first coordinate.

Continued fractions together with the Gauss map have proved their efficiency in arithmetics, Diophantine approximation, as well as in computer arithmetic. What is the situation for multidimensional continued fractions? What is expected from them? Note first that there is no canonical generalization of continued fractions to higher dimensions (see the discussion and the references in [10]). Several approaches are possible, which are based either on lattice reduction algorithms (see the references in [82]), on best simultaneous approximation properties (see *e.g.*, [61–63]), on Klein polyhedra and sails [2], or else on unimodular multidimensional Euclid's algorithms in the sense of [20, 90]. Jacobi-Perron algorithm enters this latter class, it is said to be a *vectorial algorithm* (according to the terminology in [20]). From an arithmetic viewpoint, a multidimensional continued fraction algorithm is expected to detect linear relations between the parameters, to give algebraic characterizations of periodic expansions, to have "good" properties of convergence, and to provide "good" simultaneous rational approximations. From a dynamical viewpoint, we also would like to have reasonable ergodic properties (concerning ergodic invariant measures, realizations of the natural extension, entropy, Lyapounov exponents, *etc.*), to be able to control the almost everywhere behavior like the a.e. speed of convergence, the distribution of the digits, to understand the "depth" and the number of executions of the algorithm if the parameters are rational, and to be able to perform a dynamical analysis according to the scheme discussed in [94]. In particular, Jacobi-Perron algorithm is known to have an invariant ergodic probability measure equivalent to the Lebesgue measure (see for instance [90]). However, this measure is not known explicitly for Jacobi-Perron (the density of the measure is shown to be a piecewise analytical function in [21]). For a thorough study of the Lyapounov exponents of the Jacobi-Perron algorithm, see [22]. Nevertheless, as underlined in [20], concerning the class of so-called vectorial algorithms to which Jacobi-Perron algorithm belongs, "All continued fraction algorithms which have been proposed since the beginning (Jacobi, 1868), and up to about 1970 belong to this class. [...]. A great disadvantage is that the expansions of vectorial algorithms often converge too slowly or not at all." Yet they are easier to study from an ergodic viewpoint for instance. In particular, the existence of an ergodic absolutely continuous invariant measure allows to understand the way the digits are distributed. We thus can consider these algorithms, and in particular Jacobi-Perron algorithm, as satisfying compromises between efficient computation and sharpness of the provided simultaneous rational approximations.

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