Covering numbers: arithmetics and dynamics for rotations and interval exchanges

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Abstract

We study a particular case of the two-dimensional Steinhaus theorem, giving estimates of the possible distances between points of the form $k\alpha$ and $k\alpha + \beta$ on the unit circle, through an approximation algorithm of β by the points $k\alpha$. This allows us to compute covering numbers (maximal measure of Rokhlin stacks having some prescribed regularity properties) for the symbolic dynamical systems associated to the rotation of argument α acting on the partition of the circle by the points 0, β. We can then compute topological and measure-theoretic covering numbers for exchange of three intervals; in this way, we prove that every ergodic exchange of three intervals has simple spectrum, and build a new class of three-interval exchanges which are not of rank one.

This work is a sequel of [CHE]. In that paper, we proposed to find invariants of dynamical systems which, when computed for the irrational rotation of argument α, are explicitly linked to the arithmetic properties of α. This aim was achieved by the **covering numbers**, see below, whose expression for this rotation uses the continued fraction approximation of α. In the present paper, we show that the covering numbers can really work as ergodic invariants, as we use them to solve open problems of an ergodic nature; these problems concern the exchange of three intervals, and our invariants are reached through a detailed study of the repartition on the circle of points of the form $k\alpha$ and $k\alpha+\beta$.

The covering numbers (see precise definitions in Section 6, and see [FER1], [KIN], [FER2], [FER4], [CHE]) measure the largest Rokhlin stack (that is, the union of disjoint sets B, TB, $\ldots$,$T^{h-1}B$) of arbitrarily large height h, which we can find in the system for a basis B having some prescribed properties. In [CHE], we computed these quantities for the irrational rotation of argument α, asking that B is either equal to an interval, or included in a cylinder for the symbolic dynamics of the rotation on its Sturmian coding (see Section 3); this gave us invariants $F_I(\alpha)$ and $F(\alpha)$; the proof used a precise study of the repartition of the points $k\alpha$ on the circle for $0\leq k\leq h-1$, centered on the famous **three distance theorem**, conjectured by Steinhaus and proved in [SOS1], [SOS2], [SUR] [SWI], see [ALE-BERT] for a recent survey, together with an explicit expression of the distances between adjacent points ([SOS1], [SOS2] or more recently [BERT]).

A natural problem to study now is what happens when we look at the symbolic dynamics of the same rotation on its coding by a partition of the circle by 0 and by one point β, rationally independent from α; this involves looking at the repartition on the circle of both the points $k\alpha$ and the points $l\alpha+\beta$ for $0\leq k\leq h-1$ and $0\leq l\leq h-1$; in this case, there is a bi-dimensional Steinhaus theorem ([GEE-SIM]), also called **five distance theorem** (see [ALE-BERT]) but no explicit expression of these distances. We give here a method, using the graph of words, to compute these distances, which, together with an **approximation algorithm** of β by α, inspired from [SLA], allows us, in many cases, to have precise estimates of the distances.

These estimates are used to compute the covering numbers $F_C(\alpha,\beta)$ and $F(\alpha,\beta)$, where we ask respectively that B is either equal to or included in a cylinder for the symbolic dynamics of the rotation on its coding by the β-partition: we can give qualitative remarks on the behaviour of these invariants, and their relations with the quantities $F_I(\alpha)$ and $F(\alpha)$. In particular, though the coding by the β-partition is intrinsically different from the Sturmian coding, there are many nontrivial situations (that is, when β is not a multiple of α) where $F_C(\alpha,\beta) = F_I(\alpha)$, though the two quantities can also be different; as for $F(\alpha,\beta)$, it can vary between $F_I(\alpha)$ and $F(\alpha)$, both extreme values and some intermediate values being taken for nontrivial situations; the different β satisfying the above properties are determined through their aproximation algorithm by α.

Now, the dynamics of a rotation of argument α on an interval of length β are intrinsically linked to the dynamics of an **exchange of three intervals** T defined by α and β, through a process of **induction** ([KAT-STE], [RAU2], see Section 9). Such dynamical systems are simple to define but their study is quite involved: they have been suggested by Arnold ([ARNOL]) and used in [KAT-STE] to give examples of (at that time) surprising spectral properties; they were then studied in depth by Veech ([VEE2]), who, through far-reaching geometric arguments, proved their unique ergodicity in almost all cases; the same result has been re-proved later using more elementary methods, rather related to ours as they use also the graph of words, by Boshernitzan ([BOS]). These results are closely linked to our invariants: in [KAT-STE] it is stated (though the proof is not written) that when α has unbounded partial quotients and β is well approximated by the convergents of α then $F_I = 1$; one by-product of Veech's theory is that $F_I = 1$ for almost all exchange of intervals. But, to our knowledge, such obvious questions as when two exchange of three intervals are measure-theoretically isomorphic are still unsolved. The question of whether a nontrivial exchange of three intervals, associated to α and β as above, can be measure-theoretically isomorphic to a rotation has received a partial answer: the answer is negative when α has unbounded partial quotients and β is badly approximated by some convergents of α ([KAT-STE], see the discussion at the end of Section 9).

Here, without any Veech-type machinery, we are able to compute some covering numbers, and, using the fact that they are ergodic invariants, to answer some of these questions. Namely, we prove that for exchanges of three intervals $F_I(T) = F_I(\alpha)$; this implies that $F_I(T) = 1$ if and only if

α has unbounded partial quotients, precising the results of Katok, Stepin and Veech; this implies also that every ergodic exchange of three intervals has simple spectrum, and cannot be measure-theoretically isomorphic to the dynamical system associated to the Morse sequence. Then, we prove that $F_D(T) = F(\alpha, \beta)$, $F_D(T)$ being defined by asking B and its iterates to have arbitrarily small diameter; this gives new examples of three-interval exchanges which are induced by the same rotation but are not topologically conjugate; this gives also a lower bound for the measure-theoretic covering number $F^\star(T)$ (see precise definitions in Section 6), that is the measure of the largest Rokhlin stack whose levels approximate every partition. To find an upper bound for $F^\star(T)$ is much more difficult, but we have been able to find, for any α with bounded partial quotients, a nonempty Cantor set of numbers β such that $F^\star(T) < 1$; this implies that, for these α and β, T is not measure-theoretically isomorphic to any three-interval exchange with an α' with unbounded partial quotients, nor to a rotation, and has a non-discrete spectrum.

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Throughout this paper, $0 < \alpha < 1$ is an irrational number and R_α is the irrational rotation defined on the torus $\mathbb{T}_1$ by $R_\alpha x = x + \alpha$ modulo 1. Except otherwise stated, every quantity is considered modulo 1.

1 Codings and Rauzy graphs

We will study the combinatorics of sequences obtained as codings of irrational rotations on the unit circle.

The **coding** of the orbit of a point x under the rotation of angle α with respect to the partition in intervals I_0, I_1 is the sequence $(u_n)_{n\in\mathbb{N}}$ defined on the finite alphabet $\{0, 1\}$ as follows:

$$u_n = k \Leftrightarrow x + n\alpha \in I_k.$$

A **factor** of **length** l of the sequence $(u_n)_{n\in\mathbb{N}}$ is any word of the form $u_p \dots u_{p+l-1}$. We shall see in Section 2 that the set of factors of the sequence defined above is independent of the point x.

The **Rauzy graph** Γ_n of words of length n of a sequence with values in a finite alphabet is an oriented graph (see, for instance, [RAU1] or [ARNOU-RAU]), which is a subgraph of the de Bruijn graph of words. Its vertices are the factors of length n of the sequence and the edges are defined as follows: there is an edge from U to V if V follows U in the sequence, i.e., more precisely, if there exists a word W and two letters x and y (not necessarily distinct) such that $U = xW$, $V = Wy$ and xWy is a factor of the sequence (such an edge is labelled by xWy). Thus there are $p(n+1)$ edges and $p(n)$ vertices, where $p(n)$ denotes the **complexity function**: $p(n)$ is the number of distinct factors of length n. Let us note that the graphs of words of a sequence are always connected; furthermore, they are strongly connected if and only if this sequence is recurrent, i.e. if every factor appears at least two times, or equivalently if every factor appears an infinite number of times in this sequence. In particular, a coding of rotation is always recurrent.

We call **right extension** (respectively **left extension**) of a factor $w = w_1 \dots w_n$ a factor of the sequence of the form $w_2 \dots w_n x$ (respectively $x w_1 \dots w_{n-1}$), where x denotes a letter (the extensions of a factor w have the same length as w). A factor having more than one right (respectively left extension) is called **right** (respectively **left**) **special factor**. Let U be a vertex of the graph. Let us denote by U^+ the number of edges of Γ_n which have U as origin and U^- the number of edges of Γ_n which have U as ending. In other words, U^+ (respectively U^-) counts the number of right (respectively left) extensions of U. Let us note that

$$p(n+1) - p(n) = \sum_{|U|=n} (U^+ - 1) = \sum_{|U|=n} (U^- - 1).$$

We can deduce from the first-differences of the complexity function some information on the topology of the graph of words. In all what follows, a **branch** of the graph Γ_n denotes a longest sequence of maximal length $(U_1, \dots, U_m)$ of adjacent edges of Γ_n, possibly empty, satisfying

$$U_i^+ = 1, \quad \text{for} \quad i < m, \quad U_i^- = 1, \quad \text{for} \quad i > 1.$$

If the sequence is recurrent, a branch begins either with a left special factor or with an extension of a right special factor.

Furthermore, if U and V are two vertices linked by an edge such that $U^+ = 1$ and $V^- = 1$, then the two factors U and V have the same frequency. Let us recall that the **frequency** $f(B)$ of a factor B of a sequence is the limit, if it exists, of the number of occurrences of this block in the first k terms of the sequence divided by k.

Indeed, let us write $U = xW$ and $V = Wy$, where x and y are letters. As $U^+ = 1$, U has as unique right extension Wy. Similarly, V has as unique left extension xW. Thus $f(U) = f(Uy) = f(xWy) = f(xV) = f(V)$, where f denotes the frequency. Therefore, the edges of a branch have the same frequency and the frequencies of factors of given length belong to the set of frequencies of the branches of the corresponding graph.

2 The five distance theorem

In this section, we suppose β is not a multiple of α.

Let P_n be the partition of the circle by the points $k\alpha$, $\beta + l\alpha$, for $-n+1 \leq k, l \leq 0$; let Q'_n be the partition of the circle by the points $k\alpha$, $\beta + l\alpha$, $0 \leq k \leq n-1$, $0 \leq l \leq n-1$. For such partitions, we shall speak in the obvious sense of **intervals** of the partition, and of **points** of the partition to denote the endpoints of the intervals; the notion of **neighbours** is also unambiguous.

We consider a coding u of the orbit of a point x under the rotation R_α for the partition $\{[0, \beta[, [\beta, 1[\}$. Let $I_0 = [0, \beta[$ and $I_1 = [\beta, 1[$. A finite word $w_1 \cdots w_n$ defined on the alphabet $\{0, 1\}$ is a factor of the sequence u if and only if there exists an integer k such that

$$x + k\alpha \in I(w_1, \ldots, w_n) = \bigcap_{j=0}^{n-1} R_\alpha^{-j}(I_{w_{j+1}}).$$

As α is irrational, the sequence $(x + n\alpha)_{n \in \mathbb{N}}$ is dense in the unit circle, which implies that $w_1 w_2 \ldots w_n$ is a factor of u if and only if $I(w_1, \ldots, w_n) \neq \emptyset$. The connected components of these sets are bounded by the points of P_n. Furthermore, these sets are connected for n large enough (see [ALE]). *In all the sequel, we suppose n has been chosen large enough for all the*

$I(w_1, \ldots, w_n)$ *to be connected*. The frequency of the factor $w_1 \ldots w_n$ exists and is equal to the density of the set

$$\{k \mid x + k\alpha \in I(w_1, \ldots, w_n)\},$$

which is equal to the length of $I(w_1, \ldots, w_n)$, because of the equirepartition of the sequence $(x + n\alpha)_{n \in \mathbb{N}}$.

For the n we have taken, the complexity function of the sequence u satisfies $p(n+1) - p(n) = 2$ (see for instance [ROT]). Thus there are exactly two right special factors of length n.

Lemma 1 *The two intervals corresponding to the right special factors are the intervals of the partition P_n containing respectively $-n\alpha$ or $-n\alpha + \beta$. The two intervals corresponding to the left special factors are the intervals of the partition P_n containing respectively α or $\alpha + \beta$. The four intervals corresponding to the right extensions of the right special factors are the intervals of the partition P_n touching respectively $-(n-1)\alpha$ or $-(n-1)\alpha + \beta$.*

Proof

Let I be an interval of the partition P_n corresponding to the factor w. Let k be an index of occurrence of w. We thus have $x + k\alpha \in I$ and $x + (k+1)\alpha \in R_\alpha I$. The factor w has two right extensions if and only if the interval $R_\alpha I$ intersects two intervals of P_n, hence if and only if there exists a point of P_n in the interior of $R_\alpha I$; as I is an interval of P_n, this point can only be $(-n+1)\alpha$ or $(-n+1)\alpha + \beta$, hence our first assertion after applying R_α^{-1}; and the two right extensions must correspond to the two intervals of P_n intersecting $R_\alpha I$. The assertion on the left special factor comes from a symmetric reasoning. QED

Lemma 2 *The frequencies of the factors of given length n belong to the set of frequencies of the right special factors of length n and of their right extensions.*

Proof

Let us consider the graph of words of length n. A branch can either end with a right special factor, or begin with a right extension of a right special factor, or begin with a left special factor.

Let s_n be the symmetry of the circle defined by $s_n : x \to \{\beta - (n-1)\alpha - x\}$. The points of the partition P_n are stable under s_n. We have $s_n(R_\alpha^{-k}(I_j)) =$

$R_\alpha^{(-n+1+k)}(I_j)$, for $j = 0, 1$, following the previous notation. The image of $I(w_1, \ldots, w_n)$ by s_n is $I(w_n, \cdots, w_1)$; thus they have the same length. Let I be the interval containing $\beta - n\alpha$ (respectively $-n\alpha$); the interval $s_n(I)$ contains α (respectively $\alpha + \beta$). The right special factors have thus the same frequencies as the left special factors, which proves our lemma. QED

We deduce from these two lemmas, after rotating the picture by $(n-1)\alpha$, the following result:

Proposition 1 *The different lengths of the intervals of Q'_n are the lengths of the two intervals touching* 0*, the lengths of the two intervals touching β, the length of the interval containing $-\alpha$, and the length of the interval containing $\beta - \alpha$.*

Remarks
The length of the interval containing $-\alpha$, is either the sum of the lengths of the two intervals touching 0 (in the case where the frequency of the corresponding right special factor is the sum of the frequencies of its right extensions) or one of the lengths of the intervals touching 0 or β (when one of the right extensions is a left special factor); and the symmetric proposition holds for $\beta - \alpha$. Hence the possible lengths of the intervals of Q'_n are included in the following set: the lengths of the two intervals touching 0, the sum of these two lengths, the lengths of the two intervals touching β, the sum of these two lengths. The reasoning in the proof of Lemma 3 below implies that the length of the interval of Q'_n containing $-\alpha$, resp. $\beta - \alpha$, is the sum of the lengths of the two intervals touching 0, resp. β, whenever neither $(n-1)\alpha$ nor $(n-1)\alpha + \beta$ is a neighbour of $-\alpha$, resp. $\beta - \alpha$, in Q'_n.
In fact, the six lengths in Proposition 1 take at most five different values; this is the two-dimensional Steinhaus theorem ([GEE-SIM]), which we can also call the **five distance theorem**; it is re-proved in [ALE-BERT], using a more precise description of the Rauzy graphs, which may have different topologies, see [ROT] or [FER3].

3 The Sturmian coding

The particular case where $\beta = 1 - \alpha$ is already completely studied: **Sturmian** sequences are defined equivalently, either as sequences of complexity

$p(n) = n + 1$, for every n, or as codings of an irrational rotation of angle α on the unit circle with respect to a partition in two intervals of size α and $1 - \alpha$: a sequence is Sturmian if and only if there exists α irrational in $]0, 1[$ and x such that this sequence is the coding of the orbit of x under the rotation of angle α with respect either to the partition $\{[0, 1 - \alpha[, [1 - \alpha, 1[\}$ or $\{]0, 1-\alpha],]1-\alpha, 1]\}$ ([HED-MOR2]). Let us note that a sequence whose complexity satisfies $p(n) \leq n$, for some n, is ultimately periodic ([HED-MOR1], [COV-HED]). Sturmian sequences have thus the minimal complexity among not ultimately periodic sequences. The reader can consult [BERS] and [LOT] for a recent survey on Sturmian sequences.

In the Sturmian case, there is only one bispecial factor of given length; the graph has three branches and has the following topology (see for example [ARNOU-RAU]). The graph Γ_h contains one right special factor D_h and one left special factor G_h, both with two extensions; D_h and G_h may be the same. There are three branches: the central branch, starting from G_h (it can be reduced to one point, when $G_h = D_h$), and two branches, C_h and L_h, starting from each of the right extensions of D_h.

A complete analysis of the different lengths and frequencies of the branches can be found in [BERT] and [CHE]. Let $[0; a_1, \ldots, a_n, \ldots]$ be the continued fraction approximation of α; let $q_{n+1} = a_{n+1}q_n + q_{n-1}$, $p_{n+1} = a_{n+1}p_n + p_{n-1}$, $p_{-1} = 1$, $p_0 = 0$, $q_{-1} = 0$, $q_0 = 1$. Let $f_n = (-1)^n(q_n\alpha - p_n)$; we have $f_{n-1} = a_{n+1}f_n + f_{n+1}$ for all n.

For given h, let Q_h be the partition of the circle by the points $k\alpha$, $0 \leq k \leq h - 1$. The intervals of the partition Q_h have three possible lengths: this is the three distance theorem, see [SOS1], [SOS2], [SUR], [SWI], [ALE-BERT]. These lengths are also the three possible frequencies of the points of Γ_h, and are computed in [SOS1] (see also [BERT] or [CHE]). We shall need the following results in Section 4.

Let n be fixed; to simplify the notation, we suppose that *n is odd*; let $h = q_n - 1$; then the intervals of the partition Q_h have three possible lengths, f_{n-1}, $f_{n-1} + f_n$, and $2f_{n-1} + f_n$; these values are taken, in particular, respectively by the intervals $[0, f_{n-1} = q_{n-1}\alpha[$, $[-f_{n-1} - f_n = (q_n - q_{n-1})\alpha, 0[$, and $[-\alpha - f_{n-1} - f_n, -\alpha + f_{n-1}[$.

An interval of Q_h is of length at least $f_{n-1} + f_n$ if and only if it is on the circuit of Γ_h beginning with the word corresponding to the interval $[-f_{n-1}-f_n, 0[$ (which is a right extension of D_h) and ending with D_h; this circuit has length q_{n-1} (see [CHE]), hence its points are the $q_{n-1}-1$ first images by T of the interval $[-f_{n-1}-f_n, 0[$: an interval of Q_h is of length at least $f_{n-1}+f_n$ if and only if its right endpoint is $k\alpha$ for some $0 \leq k < q_{n-1}$.

For the same odd n, we look now at some $q_{n-1} < h \leq q_n$; now the three possible lengths are f_{n-1}, $f_{n-1}+f_n+c_hf_{n-1}$ for some $0 \leq c_h \leq a_{n+1}-1$ and $2f_{n-1}+f_n+c_hf_{n-1}$ (this last one is not taken for every value of h); these are taken respectively by the intervals $[0, f_{n-1}[$, $[-f_{n-1}-f_n-c_hf_{n-1}, 0[$, and (when the last one is taken) $[-\alpha-f_{n-1}-f_n-c_hf_{n-1}, -\alpha+f_{n-1}[$; for $h > q_n-q_{n-1}$, we have $c_h = 0$; in particular, the intervals of the partition Q_{q_n}. have lengths f_{n-1} and $f_{n-1}+f_n$. An interval of Q_h is of length at least $f_{n-1}+f_n$ if and only if its right endpoint is $k\alpha$, for some $0 \leq k < q_{n-1}$.

In particular, let $g = q_{n-1}+q_{n-2}-r_n$, for some $2 < r_n < q_{n-2}$. The intervals in the partition Q_g, have three lengths, f_{n-1}, f_{n-2} and $f_{n-1}+f_{n-2}$; the intervals of length $f_{n-1}+f_{n-2}$ are those corresponding to the central branch of Γ_g, and they are those whose right endpoint is $k_1\alpha$ for $q_{n-1}-r_n \leq k_1 \leq q_{n-1}-1$. The three lengths remain the same for the partition Q_{g+1}, and the intervals of length $f_{n-1}+f_{n-2}$ are those whose right endpoint is $k_1\alpha$ for $q_{n-1}-r_n+1 \leq k_1 \leq q_{n-1}-1$.

Another particular case we need is when $a_n \geq 3$ and $g = 2q_{n-1}+q_{n-2}-r_n$, for some $2 < r_n < q_{n-2}$. The intervals in the partition Q_g, have three lengths, f_{n-1}, $f_{n-2}-f_{n-1}$ and f_{n-2}; the intervals of length f_{n-2} are those whose right endpoint is $k_1\alpha$ for $2q_{n-1}-r_n \leq k_1 \leq 2q_{n-1}-1$. The three lengths remain the same for the partition Q_{g+1}, and the intervals of length f_{n-2} are those whose right endpoint is $k_1\alpha$ for $2q_{n-1}-r_n+1 \leq k_1 \leq 2q_{n-1}-1$.

Obviously, similar results hold with the left endpoint, when n is odd.

4 Explicit values for the five distances

In this section, we suppose β is not a multiple of α.

In that case, there are no such formulas for the lengths of the intervals of Q'_h; however, our Proposition 1 allows us to make explicit computations in many cases, as we shall see in the following lemmas, which will be used in Sections 7 and 8. We keep the notation of Section 3; we recall (from Section 2) that Q'_h is the partition of the circle by the points $k\alpha$, $\beta + l\alpha$, $0 \leq k \leq h-1$, $0 \leq l \leq h-1$, whereas Q_h denotes the partition by the points $k\alpha$, $0 \leq k \leq h-1$. The following results could be formulated in a more general form, and some of them can be deduced from [SOS3]; we state here what we shall use in the sequel.

Lemma 3 *Let n be larger than some $n_0(\alpha, \beta)$,*

- *if there does not exist $0 \leq s < q_{n-1}$ such that $0 < (-1)^n(\beta - s\alpha) < f_{n-1}$, then there exists at least one interval in $Q'_{q_n - 1}$ of length exactly f_{n-1},*
- *there is always at least one interval in Q'_{q_n-1} of length at least f_{n-1},*
- *if there exists $0 \leq s < q_{n-1}$ such that $0 < (-1)^n(\beta - s\alpha) = g_n < f_{n-1}$, then the lengths of the intervals of Q'_{q_n} are either (strictly) greater than f_{n-1} or equal to one of the quantities $\{g_n, f_{n-1} - g_n, f_n + f_{n-1} - g_n, f_n + g_n\}$, each of these last four values being actually taken,*
- *if there exists $0 \leq s < q_{n-1}$ such that $0 < (-1)^n(\beta - s\alpha) = g_n < f_{n-1}$, then, if $q_{n-1} < h < q_n$, the lengths of the intervals of Q'_h are either (strictly) greater than f_{n-1} or not greater than the maximum of the quantities $\{g_n, f_{n-1} - g_n, f_n + f_{n-1} - g_n, f_n + g_n\}$.*

Proof
First we fix some n, *odd* and large enough, and take $h = q_n - 1$. Recall (from Section 3) that the lengths of Q_h are f_{n-1}, $f_{n-1} + f_n$, $2f_{n-1} + f_n$. Let us consider the length of the interval of Q_h to which β belongs.

Suppose first that *β is in an interval of Q_h of length f_{n-1}*; then β is necessarily in the interior of this interval, as β is not a multiple of α. So $\beta = k_1\alpha + b_1 = k_2\alpha - b_2$, $b_1 > 0$, $b_2 > 0$, $b_1 + b_2 = f_{n-1}$. No $k\alpha + \beta$ can be closer to β than f_{n-1}, the nearest neighbours of β in Q'_h are $k_1\alpha$ and $k_2\alpha$; the lengths of the interval touching β are b_1 and b_2, and their sum is f_{n-1}.

We check now that this value f_{n-1} is indeed the length of an interval of Q'_h, namely the interval containing $\beta - \alpha$: we can suppose $k_1 > 1$ and $k_2 > 1$, because, if either $k_1 = 1$ or $k_2 = 1$ for arbitrarily large n, we must have $\beta = \alpha$. Hence $(k_1 - 1)\alpha$ and $(k_2 - 1)\alpha$ are neighbours of $\beta - \alpha$, closer than

f_{n-1}; now, if $\beta - \alpha$ has a neighbour in Q'_h of the form $l\alpha$, then $(l+1)\alpha$ has to be a neighbour of β in Q'_h, or else not to appear in Q'_h, which leaves only $l = k_1 - 1$, $l = k_2 - 1$ or else $l = q_n - 2$. But this last case cannot happen for n large enough as $\beta \neq -\alpha$; as for neighbours of $\beta - \alpha$ of the form $\beta + l\alpha$, they are translates by β of neighbours of $-\alpha$ in Q'_h, and we know that these are at distances f_{n-1} and $f_{n-1} + f_n$, since they are also the closest neigbours of 0 in Q_{q_n}. Hence $[(k_1 - 1)\alpha, (k_2 - 1)\alpha[$ is an interval of Q'_h of length f_{n-1}.

We suppose now that β *is in an interval of* Q_h *of length at least* $f_{n-1} + f_n$; then the nearest right neighbour of β in Q_h must be $k_2\alpha$ for some $0 \leq k_2 < q_{n-1}$ (see Section 3).

If $k_2\alpha - \beta > f_{n-1}$, then the nearest right neighbour of β in Q'_h is $\beta + q_{n-1}\alpha = \beta + f_{n-1}$, and hence $[\beta, \beta + q_{n-1}\alpha[$ is an interval of Q'_h of length f_{n-1}. This finishes the proof of our first assertion.

We now have to study the case where $0 < k_2\alpha - \beta = g_n < f_{n-1}$ for some $0 \leq k_2 < q_{n-1}$. From Section 3, β is in an interval of Q_h of length either $f_{n-1} + f_n$ or $2f_{n-1} + f_n$; the nearest right neighbour of β in Q'_h is $\beta + g_n$; its nearest left neighbour is $\beta - f_{n-1} - f_n + g_n$ (of the form $l\alpha$) if β is in an interval of Q_h of length $f_{n-1} + f_n$, and $\beta - f_{n-1} - f_n$ (of the form $l\alpha + \beta$) otherwise. The sums of the two lengths around β in Q'_h is greater than f_{n-1}, and it is the length of the interval of Q'_h containing $\beta - \alpha$, by the same reasoning as above. So we deduce our second assertion from Proposition 1.

We continue to suppose that $0 < k_2\alpha - \beta = g_n < f_{n-1}$, for some $0 \leq k_2 < q_{n-1}$, and look now at $k = q_n$. Then β has to be in an interval of Q_k of length $f_{n-1} + f_n$ (see Section 3), and the nearest neighbours of β in Q'_k are $\beta + g_n$ and $\beta - f_{n-1} - f_n + g_n$. We can now compute the neighbours of 0 in Q'_k, as we can exhibit (left) $\beta + (q_n - k_2)\alpha$ (we can suppose $k_2 \geq 2$ as $\beta \neq \alpha$), at distance $g_n + f_n < f_{n-1} + f_n$; and (right) $\beta + (q_{n-1} - k_2)\alpha$, at distance $f_{n-1} - g_n < f_{n-1}$. Actually, as $0 \leq q_{n-1} - k_2 < q_{n-1}$, this ensures (see Section 3) that $-\beta$ is in an interval of Q_k of length $f_{n-1} + f_n$, which implies that 0 has no nearer neighbour in Q'_k. Hence the intervals of Q'_k have lengths g_n, $f_{n-1} - g_n$, $f_n + g_n$, $f_n + f_{n-1} - g_n$, or possibly the sum $f_n + f_{n-1}$, and each one of the first four possibilities does occur. This is our third assertion.

Let us look now, for the same odd n, at values $q_{n-1} < h \leq q_n$; we suppose still that $0 < k_2\alpha - \beta = g_n < f_{n-1}$ for some $0 \leq k_2 < q_{n-1}$.

The point $k_2\alpha$ is still an endpoint of Q_h, and β is in an interval of Q_h of length at least $f_{n-1} + f_n$ -namely, either $f_{n-1} + f_n$ or $f_{n-1} + f_n + cf_{n-1}$, $c > 0$. The point $(q_{n-1} - k_2)\alpha$ is still an endpoint of Q_h, and $-\beta$ is in an interval of Q_h of length at least $f_{n-1} + f_n$ -namely, either $f_{n-1} + f_n$ or $f_{n-1} + f_n + c' f_{n-1}$, $c' > 0$. We have still four (possibly) useful lengths which are g_n, $f_{n-1} - g_n$, $f_{n-1} + f_n + cf_{n-1} - g_n$, $f_{n-1} + f_n + c' f_{n-1} - (f_{n-1} - g_n)$. Each one of the last two lengths either is strictly greater than f_{n-1} or reduces to one of the four lengths used in the last paragraph; hence our fourth assertion.

A similar reasoning applies for even n. QED

Lemma 4 *For n large enough*

- *if $g = q_{n-1} + q_{n-2} - r_n$, for some $2 < r_n < q_{n-2}$, and if for some $0 \leq t \leq r_n - 2$, we have $|t\alpha - \beta| < b_n < f_{n-1} + f_{n-2}$, there exists an interval of Q'_g, of length at least $f_{n-1} + f_{n-2} - b_n$, and included in an interval of Q_{g+1} of length $f_{n-1} + f_{n-2}$.*

- *if $a_n \geq 3$, $g = 2q_{n-1} + q_{n-2} - r_n$, for some $2 < r_n < q_{n-2}$, and if for some $0 \leq t \leq r_n - 2$, we have $|t\alpha - \beta| < b_n < f_{n-2}$, there exists an interval of Q'_g, of length at least $f_{n-2} - b_n$, and included in an interval of Q_{g+1} of length f_{n-2}.*

Proof

Let n be *odd*; under the hypotheses of the first assertion, and assuming $0 < t\alpha - \beta$, we take $k_1 = q_{n-1} - r_n + 1$, $k_2 = k_1 + t$; then $\beta + k_2\alpha$ is between $k_1\alpha - b_n$ and $k_1\alpha$; which guarantees (from Section 3) that $\beta + k_2\alpha$ is in an interval of Q_g of length $f_{n-1} + f_{n-2}$, and in an interval of Q_{g+1} of length $f_{n-1} + f_{n-2}$; hence, between the nearest left neighbour of $k_1\alpha$ in Q_h, denoted by $k_3\alpha$, and the point $\beta + k_2\alpha$ there can be no endpoint of Q_h ; also, there can be no other $\beta + l\alpha$ as $k_2\alpha$ itself is in an interval of Q_g of length $f_{n-1} + f_{n-2}$. Hence $[k_3\alpha, \beta + k_2\alpha[$ is an interval of Q_g, with the required properties. A similar reasoning applies for $0 < \beta - t\alpha$, and for even n.

The proof of the second assertion is identical. QED

Remark
Though it will not be used in the sequel, we can show a situation where β is not a multiple of α, but the $2q_n$ intervals in Q'_{q_n} have only *two* possible lengths, for infinitely many n.

Suppose α is such that $q_n + q_{n-1}$ is odd for arbitrarily large odd n and let $\beta = \frac{\alpha}{2}$; for such odd n we take $k = q_n$; one right neighbour of β in Q_k is $k_2\alpha$ with $k_2 = \frac{q_n+q_{n-1}+1}{2} > q_{n-1}$, $k_2\alpha - \beta = \frac{f_{n-1}-f_n}{2}$, hence it is the nearest, and β is in an interval of Q_k of length f_{n-1}; let $k_1\alpha = k_2\alpha - f_{n-1}$. The two nearest neighbours of 0 in Q'_k must be $\beta+(q_n-k_2)\alpha$ (left) and $\beta+(q_n-k_1)\alpha$ (right), as they are respectively at distances $\frac{f_{n-1}+f_n}{2}$ and $\frac{f_{n-1}-f_n}{2}$; we check than $-\alpha$ is in an interval of Q'_k of length $\frac{f_{n-1}+f_n}{2}$, and the same for $\beta-\alpha$; there are only two different lengths in Q'_k, $\frac{f_{n-1}+f_n}{2}$ and $\frac{f_{n-1}-f_n}{2}$, both being smaller than f_{n-1}.

5 Approximation algorithm

To be able to precise our estimates, we need to know the best approximations of β by points $s\alpha$; we shall suppose for example that $\beta \leq \frac{1}{2}$, and look at the best left approximations.

Proposition 2 *If we define the sequence of natural integers k_i, starting from $k_0 = 0$, by the property that k_{i+1} is the smallest integer $s > k_i$ such that $k_i\alpha < s\alpha < \beta$, then the k_i are given by the following algorithm, where the f_n are as above, and k_i, n_i, c_i and e_i are uniquely defined by:* $k_0 = 0$;

$$\beta - k_i\alpha = c_i f_{n_i} + f_{n_i+1} + e_i,$$

with $0 < e_i \leq f_{n_i}$, $1 \leq c_i \leq a_{n_i+1}$, *if* $n_i > 0$, *and* $1 \leq c_i \leq a_1 - 1$, *if* $n_i = 0$,

$$k_{i+1} = k_i + q_{n_i}, \quad \textit{if } n_i \textit{ is even},$$

$$k_{i+1} = k_i - c_i q_{n_i} + q_{n_i+1}, \quad \textit{if } n_i \textit{ is odd.}$$

This algorithm is inspired from [SLA], where it is proved that it does give the best left approximations of β by $k\alpha$ (the "gap result" in Section 4 of [SLA] implies exactly that the k_{i+1} defined here is the smallest integer $s > k_i$

such that $k_i\alpha < s\alpha < \beta$); see also for a similar algorithm [SOS2], [SOS3] and see alos [SOS4]. We thus have

$$\beta = \sum_{n_i \ even} c_i f_{n_i} + \sum_{n_i \ odd} (c_i f_{n_i} + f_{n_i+1}),$$

and the sequence $(k_i\alpha)_i$ tends towards β. Note that $\beta - k_{i+1}\alpha$ is equal to e_i if n_i is odd, and to $(c_i - 1)f_{n_i} + f_{n_i+1} + e_i$, if n_i is even. Hence, we may have $n_{i+1} = n_i$; this happens if and only if n_i is even and $c_i > 1$; this will then happen $(c_i - 1)$ times and after that the sequence n_i continues to grow, if β is not a positive multiple of α, so $n_i \to +\infty$.
A similar reasoning applies for $\beta > \frac{1}{2}$; and to get the right approximations, we apply the same algorithm to approximate $1 - \beta$ to the left by multiples of $1 - \alpha$; this defines sequences k'_i, n'_i, c'_i and e'_i, using the a'_n and f'_n of the continued fraction approximation of $1 - \alpha$.

This algorithm gives the successive values $k_{i+1} > k_i$ minimizing $\{k\alpha + \beta\}$. The same quantity is also minimized in [KOM2], but for values $k < q_n$, $n \in \mathbb{N}$; that paper gives another approximation algorithm, and another one still is the algorithm of Ostrowski ([OST]) which appears naturally in the study of the codings of the rotation R_α, see [ARNOU-FER-HUB]. For a survey of the different algorithms related to this approximation problem, see [KOM1]. These different algorithms are deeply connected and correspond to numeration systems analogous to the Ostrowski system (see also [SOS2], [SOS3]).

6 Covering numbers

The notions of **rank** and **covering numbers** have been very useful to ergodicians for the last twenty years; see [FER4] for a recent survey of related matters.

For a measure-theoretic dynamical system (X, T, μ), (μ being a probability invariant by T), the (measure-theoretic) **covering number** $F^\star(T)$ ([KAT-SAT], [FER1], [KIN]) is defined by:

Definition 1 *$F^\star(T)$ is the supremum of all real numbers z such that for every measurable partition $P = \{P_1, \ldots, P_r\}$ de X, for every $\epsilon > 0$, for every* *integer h_0, there exist a subset B of X, an integer $h > h_0$ and a partition $P' = \{P'_1, \ldots, P'_r\}$ of X such that if $A = \cup_{j=0}^{h-1} T^j B$:*

- *B, TB, ..., $T^{h-1}B$ are disjoint,*
- *$\mu(A) > z - \epsilon$,*
- *$\sum_{i=1}^{r} \mu((P_i \Delta P'_i) \cap A) < \epsilon$,*
- *each $P'_i \cap A$ is a union of sets $T^j B$, for some $0 \leq j \leq h-1$.*

$F^\star(T)$ is an invariant for the notion of isomorphism for measure-theoretic dynamical systems.

When $F^\star(T) = 1$, the system is said to be of **rank one** ([ORN-RUD-WEI], formalizing a notion which had first appeared in [CHA]); in that case, the system can be generated by a nested sequence of **Rokhlin stacks** (see [KAL] for example): we can find a sequence of sets B_n and numbers $h_n \to +\infty$, such that B_n, TB_n, ..., $T^{h_n-1}B_n$ are disjoint, and, if we call ϕ_n the partition of X by the sets B_n, TB_n, ..., $T^{h_n-1}B_n$, $X \setminus \cup_{i=0}^{h_n-1} T^i B_n$, the partition ϕ_{n+1} refines the partition ϕ_n for each n, and the σ-algebra generated by the $(\phi_n)_{n\in\mathbb{N}}$ separates points on a set of measure 1.

Irrational rotations, and all systems with discrete spectrum, are of rank one ([deJ]).

Let now (X, T) be a topological dynamical system, defined on the torus $\mathbb{T}_1$ or the interval $[0, 1[$ with the usual topology, **minimal** and **uniquely ergodic**: there is a unique probability invariant by T, denoted by μ. We call **interval** an arc of the torus or a sub-interval of $[0, 1[$, open to the left and closed to the right. The **covering number by intervals** $F_I(T)$ ([CHE]) is defined by:

Definition 2 *$F_I(T)$ is the supremum of all real numbers z such that, for every h_0, for every $\epsilon > 0$, there exist $h \geq h_0$ and an interval B such that*

- *B, TB, ..., $T^{h-1}B$ are disjoint intervals,*
- *$\mu(\cup_{i=0}^{h-1} T^i B) \geq z - \epsilon$.*

The result will not be changed if we take open or closed intervals, or if we ask only that the interiors of the T^jB are disjoint. The system is said to be of **rank one by intervals** whenever $F_I(T) = 1$.

F_I depends a priori both from the topological and the measure-theoretic structure of the system; but the measure-theoretic structure is defined uniquely by the topology, and two uniquely ergodic systems defined on the interval or the torus have the same F_I when they are topologically conjugate (as such a conjugacy also sends intervals into intervals). We say that F_I is an invariant of topological conjugacy in this class of systems.

A related topological invariant, for the same class of systems, is what we may call the **covering number by sets of small diameter**:

Definition 3 *$F_D(T)$ is the supremum of all real numbers z such that, for every h_0 and every $\epsilon > 0$, there exist $h \geq h_0$ and a set B such that*

- *B, TB, ..., $T^{h-1}B$ are disjoint sets of diameter not greater than ϵ,*
- *$\mu(\cup_{i=0}^{h-1} T^i B) \geq z - \epsilon$.*

Note that if the T^jB can be taken of arbitrarily small diameter, they allow us to approximate every partition P in the sense of Definition 1: we approximate first P by a partition Q whose atoms are unions of intervals (or, more generally, whose atoms have regular enough boundaries), choose a Rokhlin stack with small enough diameter δ, and define P'_1 as the union of levels which intersect Q_1, P'_2 as the union of levels which intersect Q_2 and not Q_1, and so on; the points in $\cup P'_i \Delta(Q_i \cap A)$ have to be δ-close to the boundary of atoms of Q, and the measure of this set is smaller than ϵ if δ is small enough. Hence

$$F_D(T) \leq F^\star(T).$$

Also, if the T^jB are intervals, they have to be of length not greater than $\frac{1}{h}$, and then have arbitrarily small diameter, hence

$$F_D(T) \geq F_I(T).$$

For a uniquely ergodic topological system, and a Borelian partition $P = \{P_1, \ldots, P_r\}$, we define the **covering number by cylinders** $F_C(T, P)$ ([CHE]) and the **symbolic covering number** $F(T, P)$ ([FER2], [CHE]):

Definition 4 *$F_C(T,P)$ is the supremum of all real numbers z such that, for every h_0, for every $\epsilon > 0$, there exist $h \geq h_0$ and a sequence w_j, $0 \leq j \leq h-1$, $1 \leq w_j \leq r$, such that the set $B = \cup_{j=0}^{h-1} T^{-j} P_{w_j}$ satisfies*

- *B, TB, ..., $T^{h-1}B$ are disjoint,*
- *$\mu(\cup_{i=0}^{h-1} T^i B) \geq z - \epsilon$.*

Definition 5 *$F(T,P)$ is the supremum of all real numbers z such that, for every h_0, for every $\epsilon > 0$, there exist $h \geq h_0$, a sequence w_j, $0 \leq j \leq h-1$, $1 \leq w_j \leq r$, and a subset B of $\cup_{j=0}^{h-1} T^{-j} P_{w_j}$ such that*

- *B, TB, ..., $T^{h-1}B$ are disjoint,*
- *$\mu(\cup_{i=0}^{h-1} T^i B) \geq z - \epsilon$.*

The quantities $F_C(T,P)$ and $F(T,P)$ are associated with the symbolic dynamical system defined as the shift on the sequences $PN(x)$ defined by $PN(x)_n = i$ if $T^n x \in P_i$, $n \in \mathbb{N}$; they are both topological invariants in the class of minimal and uniquely ergodic symbolic systems, as topological conjugacies in that class are finite codes.

When T is the rotation R_α, μ is the Lebesgue measure, and we have stated above that $F^\star(R_\alpha) = 1$; the techniques in [deJ] imply also that $F_D(R_\alpha) = 1$; we write $F_I(\alpha)$ for $F_I(R_\alpha)$. If $P(\beta)$ is the partition of the torus in two sets, $P(\beta)_0 = [0, \beta[$ and $P(\beta)_1 = [\beta, 1[$, we write $F_C(\alpha, \beta)$ for $F_C(R_\alpha, P(\beta))$, $F(\alpha, \beta)$ for $F(R_\alpha, P(\beta))$ and $F(\alpha)$ for $F(R_\alpha, P(1-\alpha))$. We check that all these quantities remain the same if we take for T the same rotation, but defined on the interval $[0, 1[$, taken as a fundamental domain of $\mathbb{R}/\mathbb{Z}$.

Let $\alpha = [0; a_1, \ldots, a_n, \ldots]$; we denote by v_n the rational number $[0; a_n, \ldots, a_1]$ and by t_n the irrational number $[0; a_{n+1}, \ldots]$; we denote by $a(h)$ the smallest of the (at most three) different lengths of the intervals of Q_h. The following results are proved in [CHE] (the partial result that $F_I(\alpha) \geq \limsup_{n \to +\infty} q_n f_{n-1} \geq \frac{5+\sqrt{5}}{10}$ appears in [GUE]).

Lemma 5 *For an interval B of length $|B|$, the following two conditions are equivalent:*

- *B, $R_\alpha B$, ..., $R_\alpha^{h-1} B$ are disjoint,*

- $|B| \leq a(h)$.

Proposition 3 *If the a_n, t_n, v_n are as above, and q_n and f_n as in Section 3,*

$$F_I(\alpha) = \limsup_{h\to+\infty} ha(h) = \limsup_{n\to+\infty} q_n f_{n-1} = \limsup_{n\to+\infty} \frac{1}{1+t_n v_n} \tag{1}$$

and consequently,

$$F_I(\alpha) \geq \frac{5+\sqrt{5}}{10}$$

for every irrational α.

The quantity $F(\alpha)$ is also computed completely in [CHE]. Namely, we have the following proposition.

Proposition 4 *If the a_n, t_n, v_n are as above, and q_n and f_n as in Section 3, $F(\alpha)$ is given by the maximum of the four quantities*

1. $\limsup_{n\to+\infty} \frac{1}{1+t_n v_n} = \limsup_{n\to+\infty} q_n f_{n-1} = F_I(\alpha)$,
2. $\limsup_{n\to+\infty} \frac{(1+t_n)(1+v_n)}{2(1+t_n v_n)} = \limsup_{n\to+\infty} (q_{n-1}+q_{n-2})(\frac{f_{n-1}+f_{n-2}}{2})$,
3. $\limsup_{n\to+\infty, a_n=3, a_{n+1}=1} \frac{3}{2}\frac{1-v_n}{1+t_n v_n}$
   $= \limsup_{n\to+\infty, a_n=3, a_{n+1}=1} \frac{3}{2}(2q_{n-1}+q_{n-2})f_{n-1}$,
4. $\limsup_{n\to+\infty, a_n\equiv 0 \bmod 3,\ a_{n+1}=2} (\frac{a_n}{3}+t_n)\frac{1+(2-a_n)v_n}{1+t_n v_n}$
   $= \limsup_{n\to+\infty, a_n\equiv 0 \bmod 3, a_{n+1}=2} (2q_{n-1}+q_{n-2})(\frac{a_n}{3}+t_n) f_{n-1}$.

Furthermore, $F(\alpha) = 1$ if and only if α has unbounded partial quotients.

In several cases, $F(\alpha)$ is equal to $F_I(\alpha)$: this is the case when the a_n are unbounded, but also, for example, if all the a_n are greater or equal to 3. $F(\alpha)$ is given by formula 2 for example when α is the golden ratio number, or any $[0; 1, k, 1, k, \ldots]$ or $[0; k, 1, k, 1, \ldots]$. Formula 4 holds for example for $[0; 2, 3, 2, 3, \ldots]$ or $[0; 3, 2, 3, 2, \ldots]$.

Whenever $F(\alpha)$ is given by formula 2, the sets B realizing this quantity are built in the following way: for any $q_{n-1} \leq h \leq q_{n-1} + q_{n-2} - 2$, in any interval B_0 of maximum length of the partition Q_{h+1}, there exists a set B, included in some $\cup_{j=0}^{h-1} R_\alpha^{-j}(P(1-\alpha))_{w_j}$, such that $B, \ldots, R_\alpha^{h-1}B$ are disjoint, and $\mu(B) = \frac{f_{n-1}+f_{n-2}}{2}$. So, as will be formalized in the proof of Proposition 13, if h is chosen close enough to $q_{n-1}+q_{n-2}-2$, $h\mu(B)$ will be close to $F(\alpha)$.

Whenever $F(\alpha)$ is given by formula 3, the sets B realizing this quantity are built in the following way: for any n such that $a_n = 3$, for any $q_{n-1} + q_{n-2} \leq h \leq 2q_{n-1} + q_{n-2} - 2$, in any interval B_0 of maximum length of the partition Q_{h+1}, there exists a set B, included in some $\cup_{j=0}^{h-1} R_\alpha^{-j}(P(1-\alpha))_{w_j}$, such that $B, \ldots, R_\alpha^{h-1}B$ are disjoint, and $\mu(B) = \frac{3f_{n-1}}{2}$. So, if h is chosen close enough to $2q_{n-1} + q_{n-2} - 2$, $h\mu(B)$ will be close to $F(\alpha)$.

Whenever $F(\alpha)$ is given by formula 4, the sets B realizing this quantity are built in the following way: for any n such that a_n is a multiple of 3, $q_{n-1} + q_{n-2} \leq h \leq 2q_{n-1} + q_{n-2} - 2$, in any interval B_0 of maximum length of the partition Q_{h+1}, there exists a set B, included in some $\cup_{j=0}^{h-1} R_\alpha^{-j}(P(1-\alpha))_{w_j}$, such that $B, \ldots, R_\alpha^{h-1}B$ are disjoint, and $\mu(B) = (\frac{a_n}{3} + t_n)f_{n-1}$. So, if n is such that a_n maximizes formula 4 and h is chosen close enough to $2q_{n-1} + q_{n-2} - 2$, $h\mu(B)$ will be close to $F(\alpha)$.

7 Properties of $F_C(\alpha,\beta)$

Proposition 5

$$F_C(\alpha,\beta) = \limsup_{h\to+\infty} hb(h) \leq F_I(\alpha),$$

where $b(h)$ is the largest of the (at most five) different lengths of the intervals of Q'_h which is not larger than $a(h)$.

Proof
We use Lemma 5 and the fact that, for h large enough, every cylinder is an interval of Q'_h. QED

Proposition 6 *If $\beta = r\alpha$ for some $r \in \mathbb{Z}$, $F_C(\alpha,\beta) = F_I(\alpha)$.*

Proof
Then, after a suitable translation, Q'_h is the same as $Q_{h+|r|}$; hence, for n large enough $b(q_n - |r|) = a(q_n) = a(q_n - |r|)$ and $F_C(\alpha,\beta) \geq \limsup_{n\to+\infty}(q_n -$

$|r|)a(q_n) = F_I(\alpha)$. QED

In the remainder of this section, we suppose β is not a multiple of α.

Proposition 7

$$F_C(\alpha, \beta) = \limsup_{n \to +\infty} q_n b_n$$

where $b_n = f_{n-1}$ if there does not exist $0 \leq s < q_{n-1}$ such that $0 < (-1)^n(\beta - s\alpha) < f_{n-1}$ and, if there exists $0 \leq s < q_{n-1}$ such that $0 < (-1)^n(\beta - s\alpha) = g_n < f_{n-1}$, b_n is the largest of the quantities $\{g_n, f_{n-1} - g_n, f_n + f_{n-1} - g_n, f_n + g_n\}$ which is not larger than f_{n-1}.

Proof

Because of Lemma 3 (first, third and fourth assertions), $b(h)$ is not greater than b_n for $q_{n-1} < h \leq q_n$, with equality either for $h = q_n - 1$ or $h = q_n$. Then we use Proposition 5. QED

Our approximation algorithm, together with Proposition 7, allows us, for any given α and β, to compute $F_C(\alpha, \beta)$; we shall use it to derive some explicit properties of these quantities.

Proposition 8 *$F_C(\alpha, \beta) = 1$ if and only if α has unbounded partial quotients.*

Proof

If α has bounded partial quotients, $F_I(\alpha) < 1$ and $F_C(\alpha, \beta) \leq F_I(\alpha)$.

If α has unbounded partial quotients, then $a_n \to +\infty$ on the sequence $n = s_m$, $m \in \mathbb{N}$. From (1), we have $q_n f_{n-1} \geq \frac{1}{1+\frac{1}{a_n a_{n+1}}}$ and hence $q_n f_{n-1}$ tends to one on the sequences s_m and $s_m - 1$.

But, for any n, either $b_n = f_{n-1}$ or $b_{n+1} = f_n$ (the definition of b_n is given in Proposition 7): take n *odd* for example; if $b_{n+1} \neq f_n$, then there exists $0 \leq k_1 < q_n$ such that $0 < \beta - k_1\alpha < f_n$, hence $k_1\alpha$ is the nearest left neighbour of β in $Q_{q_{n+1}}$, but also in Q_{q_n}; if also $b_n \neq f_{n-1}$, the nearest right neighbour of β in Q_{q_n}, $k_2\alpha$, must satisfy $k_2\alpha - \beta < f_{n-1}$, with $0 \leq k_2 < q_{n-1}$. But as $k_2 < q_{n-1}$, then (from Section 3) the interval of Q_{q_n} of right endpoint $k_2\alpha$ is of length $f_n + f_{n-1}$, which is a contradiction with $k_2\alpha - k_1\alpha < f_n + f_{n-1}$; and the same happens for even n.

Hence $F_C(\alpha,\beta)$ is greater than the $\limsup_{n\to+\infty} q_n f_{n-1}$ on a sequence of s_m or $s_m - 1$, and is equal to 1. QED

For the Sturmian coding, $F_C(\alpha, 1-\alpha) = F_I(\alpha)$; one natural question is to ask whether we can have $F_C(\alpha,\beta) = F_I(\alpha)$ when β is not a multiple of α; surprisingly, this may be the case, and it is nontrivial to find a β with $F_C(\alpha,\beta) \neq F_I(\alpha)$.

Proposition 9 *There exist α and β such that $F_C(\alpha,\beta) < F_I(\alpha)$.*

Proof

We choose α to be periodic of period four, $\alpha = [0; d_1, d_2, d_3, d_4, d_1, d_2, d_3, d_4, \ldots]$, with $d_i \geq c \geq 2$; then the sequence $\tau(n) = q_n f_{n-1} = \frac{1}{1+[0;a_n,\ldots,a_1][0;a_{n+1},\ldots]}$ has a priori four adherence values, reached for $n = 4p+j$, $j = 0,1,2,3$; we choose the d_i such that the highest adherence value, which is $F_I(\alpha)$, is the limit of $\tau(4n)$ and is strictly greater than the limits of $\tau(4n+j)$ for $j = 1,2,3$.

We choose a β through our approximation algorithm: we ask that $c_i = 1$ for every i, and that $n_i = 4(i+1)+1$ for every i; this is possible: we require first that $\beta = f_5 + f_6 + e_0$, $0 < e_0 < f_1$; then $\beta - k_1\alpha = e_0$, and the next requirement is that $e_0 = f_9 + f_{10} + e_1$, $0 < e_1 < f_9$ so that β has to be in the interval $f_5 + f_6 + f_9 + f_{10}, f_5 + f_6 + 2f_9 + f_{10}$ and so on. Note that $k_{i+1} = k_i - q_{n_i} + q_{n_i+1}$, hence $k_{i+1} \leq q_{n_i+2}$.

We denote by C_n the condition

"there exists $0 \leq s < q_{n-1}$ such that $0 < (-1)^n(\beta - s\alpha) = g_n < f_{n-1}$".

We have $b_n \leq f_{n-1}$, and hence $q_n b_n \leq \tau(4p+j)$. Now, for $n = 4p+j$, $j = 1,2,3$, the upper limit of this quantity is strictly smaller than $F_I(\alpha)$.

We have just to look at the cases where $n = 4p$, or else $n_i < n = n_i + 3 < n_{i+1}$. Then $\beta - k_{i+1}\alpha < f_{n-1}$ and $k_{i+1} \leq q_{n_i+2} = q_{n-1}$, hence the condition C_n is satisfied, with $g_n = \beta - k_{i+1}\alpha = f_{n+1} + f_{n+2} + e_{i+1}$, $0 < e_{i+1} < f_{n+1}$. We apply then the formula in Proposition 7; if c is not too small, g_n is close enough to 0 for $b(q_n)$ to be equal to $f_{n-1} - g_n < f_{n-1} - f_{n+2} - f_{n+1}$, and $b(q_n)$ is bounded away from f_{n-1} by some quantity greater than Kf_{n-1}, for some constant K depending on the d_i. Hence $q_n b_n \leq (1-K)\tau_n$; the upper limit of this quantity is strictly smaller than $F_I(\alpha)$. QED

Proposition 10 *For any α, there exists β which is not a multiple of α and for which $F_C(\alpha,\beta) = F_I(\alpha)$.*

Proof
We use the same technique as in last proposition. For a given α, we choose a sequence m_k on which $\tau(m_k) \to F_I(\alpha)$; we suppose that the m_k are even infinitely often, and build a β with $c_i = 1$ and even n_i such that an infinite number of even m_k fall at a place $n = m_k = n_i$; then the condition C_n is not realized, as $\beta - k_i\alpha \geq f_{n_i-1}$ and $k_{i+1} \geq q_{n_i-1}$, so $b_n = f_{n-1}$, hence $F_C(\alpha,\beta) = \lim f_{m_n-1}q_{m_n} = F_I(\alpha)$.

This β cannot be a positive multiple of α as the k_i tend to infinity, and cannot be a negative multiple of α if the consecutives n_i are far enough from each other.

If the m_k are odd, we build another β in the same way, but through its sequence n'_i. QED

Note that numbers β such that $F_C(\alpha,\beta) < F_I(\alpha)$ can be built similarly, at least for any α such that the sequence m_k on which $\tau(m_k) \to F_I(\alpha)$ does not have too small gaps. These α form a set of measure zero as they have bounded partial quotients.

8 Properties of $F(\alpha,\beta)$

Proposition 11 *For every β, $F_I(\alpha) \leq F(\alpha,\beta) \leq F(\alpha)$.*

Proof
The upper bound for $F(\alpha,\beta)$ is clear, as the cylinders for P_β are included in cylinders for $P_{1-\alpha}$.

Conversely, because of the second assertion of Lemma 3, for $h = q_n - 1$, we can find an interval of Q'_h of length at least f_{n-1}; any subinterval B of length f_{n-1} of this interval will be included in a cylinder of P_β, while B, $\ldots$, $R_\alpha^{h-1}B$ are disjoint. This gives our lower bound, through $F(\alpha,\beta) \geq \limsup(q_n - 1)f_{n-1}$. QED

Proposition 12 *For any α, there exists β which is not a multiple of α and such that $F(\alpha,\beta) = F(\alpha)$.*

Proof
If $F(\alpha) = F_I(\alpha)$, every β is convenient.

Suppose then that $F(\alpha)$ is given by formula 2 of Proposition 4. Then $F_I(\alpha) = \lim_{n\to+\infty, n\in J}(q_{n-1} + q_{n-2})(\frac{f_{n-1}+f_{n-2}}{2})$ for some infinite set J of integers. Suppose now that β is such that for an infinite subset J' of J, there exists $0 \leq t \leq r_n - 2$ such that $|\beta - t\alpha| < b_n = \frac{f_{n-1}+f_{n-2}}{2}$ We choose an $n \in J$ and take $g = q_{n-1} + q_{n-2} - r_n$, for some $2 < r_n < q_{n-2}$ such that $\frac{r_n}{q_{n-1}}$ is small when n is large; to fix ideas, we can take $r_n = q_{[n-1-\log n]}$.

We apply the first assertion of Lemma 4 to get an interval B_0' of Q_g' of length at least $f_{n-1} + f_{n-2} - b_n$, which is included in an interval B_0 of Q_{g+1} of maximun length, that is $f_{n-1} + f_{n-2}$.

Let B be a subset of B_0, such that B, $\ldots$,$R_\alpha^{h-1}B$ are disjoint, and of measure $\mu(B) = \frac{f_{n-1}+f_{n-2}}{2}$; the set $B' = B \cap B_0'$ satisfies the same disjunction property, is included in some $\cup_{j=0}^{h-1} R_\alpha^{-j}(P_\beta)_{w_j}$ and has measure at least $\mu(B) - b_n$.

Hence $F(\alpha, \beta) \geq \limsup_{n\to+\infty, n\in J'}(q_{n-1} + q_{n-2})(\frac{f_{n-1}+f_{n-2}}{2} - b_n)$. Because of the hypotheses, this limit will be equal to $F(\alpha)$.

And to find β satisfying this condition, for example for $r_n = q_{[n-1-\log n]}$ and for given J and b_n, is possible with the techniques of last section, by choosing the n_i widely spaced; this will guarantee also that α is not a multiple of β.

If $F(\alpha)$ is given by formula 3 or 4, the same reasoning holds, with the appropriate choice of h and $\mu(B)$, through the second assertion of Lemma 4. QED

With the techniques of last proposition, and when $F(\alpha)$ is given by formula 2, 3 or 4, we can also find some β such that $F_I(\alpha) < F(\alpha, \beta) < F(\alpha)$; this is done by ensuring that $\beta - k\alpha$ is small for some $k < s_n$, where s_n is eq_{n-1} for some fixed e, and that this does not happen for $k < r_n$ for any sequence r_n which is in $o(q_{n-1})$.

In general, we have $F_C(\alpha, \beta) \leq F_I(\alpha) \leq F(\alpha, \beta) \leq F(\alpha) \leq 1$. When one of these quantities is equal to one, every one is equal to one, and this does happen if and only if α has unbounded partial quotients. Among bounded partial quotients, we have seen explicit examples of α and β with $F_C(\alpha, \beta) =$

$F_I(\alpha) < F(\alpha,\beta)$ (for α with rather small partial quotients) or $F_C(\alpha,\beta) < F_I(\alpha) = F(\alpha,\beta)$ (for α with relatively large partial quotients). It is also possible to get $F_C(\alpha,\beta) < F_I(\alpha) \leq F(\alpha,\beta)$, by ensuring simultaneously that $F(\alpha)$ is different from $F_I(\alpha)$ and that the sequence m_k on which $\tau(m_k) \to F_I(\alpha)$ does not have too small gaps, and then chosing a β with widely spaced n_i.

9 Exchanges of three intervals

An **exchange of** s **intervals** is defined in the following way : given s real numbers $l_i > 0$, with $\sum_{i=1}^{s} l_i = 1$, and a permutation π on s letters, let X be the interval $[0,1[$, partitioned into s semi-open intervals I_i, of lengths l_1, ... l_s (in that order), and also into s semi-open intervals J_i of lengths $l_{\pi^{-1}1}$, ... $l_{\pi^{-1}s}$ (in that order); T is the piecewise affine map sending each I_i onto $J_{\pi i}$.

Here, we restrict ourselves to three intervals; then, if π is different from (321), T reduces to an exchange of two intervals, which is a rotation. So our study reduces to the following transformation, depending on two parameters $0 < l < 1$ and $l < m < 1$, and we ask that $1, l, m$ *are rationnally independent*:

- $Tx = x + 1 - l$ if $x \in [0, l[$,
- $Tx = x + 1 - l - m$ if $x \in [l, m[$,
- $Tx = x - m$ if $x \in [m, 1[$.

We call such T a **nontrivial exchange of three intervals**. T is then known to be minimal and uniquely ergodic; the unique invariant probability is the Lebesgue measure μ.

We extend T to 1 by continuity, $T1 = 1 - m$. We have $l = T^{-1}1$, $m = T^{-1}0$. The point l is a discontinuity: $T^{-1}1$ has two images by T, 1 on the left and $1 - m = T1$ on the right, and if an interval B contains $T^{-1}1$ in its interior, TB is not an interval; the same is true for $T^{-1}0$. T is continuous except at the points l and m.

We define the transformation S on the interval $[0, 1 - l + m[$ by sending affinely

- $[0, l[$ onto $[1-l, 1[$,
- $[l, m[$ onto $[1, 1-l+m[$,
- $[1, 1-l+m[$ onto $[1-m, 1-l[$,
- $[m, 1[$ onto $[0, 1-m[$.

S is an exchange of two intervals, being a translation of $1-l$ on $[0, m[$ and a translation of $-m$ on $[m, 1-l+m[$; hence, it is a rotation on $[0, 1-l+m[$, which we shall consider also on $[0,1[$ after a homothety of ratio $\beta = \frac{1}{1-l+m}$.

Let $J = [0, 1[$; for a point $x \in J$, let $r_i(x)$ be the i-th strictly positive integer m such that $S^m x \in J$ and let $s_i(x)$ be the i-th strictly positive integer m' such that $S^{-m'} x \in J$; we have $T^i x = S^{r_i(x)} x$, $T^{-i} x = S^{-s_i(x)} x$. The map T is called the **first return map**, or the **induced map**, of S on J; for the role of the induction in the study of interval exchanges, see [RAU2]; every exchange of three intervals can be induced by a rotation, but this is not the case for more than three intervals, in which case non-uniquely ergodic examples exist ([KEA], [KEY-NEW]).

Lemma 6 *Let S be any rotation defined on an interval I, and J a sub-interval of length $\beta|I|$; then for every $\epsilon > 0$ there exists h_0 such that, with the above notation, for every $h > h_0$ and every $x \in J$*

$$\beta - \epsilon < \frac{h}{r_h(x)} < \beta + \epsilon,$$

$$\beta - \epsilon < \frac{h}{s_h(x)} < \beta + \epsilon.$$

Proof

The ergodic theorem applies everywhere and uniformly to the indicator function of J; hence for every $h > h_0$ and every $x \in I$,

$$\beta - \epsilon < \frac{1}{h} \# \{0 \le i \le h-1; S^i x \in J\} < \beta + \epsilon$$

and

$$\beta - \epsilon < \frac{1}{h} \# \{0 \le i \le h-1; S^{-i} x \in J\} < \beta + \epsilon.$$

this implies the lemma. QED

Proposition 13 *We have*

$$F_I(T) = F_I\left(\frac{1-l}{m}\right)$$

Proof
Let $\alpha = \frac{1-l}{1-l+m}$, $\beta = \frac{1}{1-l+m}$; for these numbers, let Q'_h and Q_h be as in Section 2 and 3; we recall that f_{n-1} (defined in Section 3) is the smallest length of the intervals of Q_h, for $h = q_n - 1$.

Let $h = q_n - 1$. Let B be a sub-interval of $[0, 1-l+m[$ of length $\frac{f_{n-1}}{\beta}$, whose interior does not contain any $S^{-i}0$ or any $S^{-i}1$ for $0 \leq i \leq h-1$; this is always possible by the second assertion of Lemma 3, after a multiplication and a translation, and then B, SB, $\ldots$,$S^{h-1}B$ are disjoint, by Lemma 5. Suppose first that B is included in J. For every $i > 0$, T^iB is a subset of $\cup_{x\in J} S^{r_i(x)}B$, and hence is disjoint from B as long as $r_i(x) \leq h-1$; hence, by Lemma 6, if h' is the largest integer not greater than $h(\beta-\epsilon)$, if $(\beta-\epsilon)h > h_0$, $r_{h'}(x) < h$ and B, TB, $\ldots$,$T^{h'-1}B$ are disjoint. Now, T^iB is not an interval only if some discontinuity of T, namely $T^{-1}0$ or $T^{-1}1$, appears in the interior of T^jB for some $0 < j < i$; this means that $T^{-1-j}0 = S^{-s_{j+1}}0$ or $T^{-1-j}1 = S^{-s_{j+1}}1$ appears in the interior of B, and this does not happen if $s_{j+1} \leq h-1$; but $j+1 \leq h'-1$ is enough to guarantee that this does not happen, and all the T^iB, for $0 \leq i \leq h'-1$, are intervals.
If B is not contained in J, then, because of the hypothesis, B is contained in $[0, 1-l+m[\setminus J$, and we check that SB is a subinterval of J and the same reasoning applies to it. Thence $F_I(T) \geq \limsup_{n\to+\infty} \frac{\beta-\epsilon}{\beta}(q_n-1)f_{n-1}$.

Conversely, let B be a subinterval of J such that B, TB, $\ldots$, $T^{h-1}B$ are disjoint intervals. Then $S^iB \cap J$ does not intersect B as long as $S^iB \cap J$ is included in a union of $T^{j(x)}B$ such that $i = r_j(x)$. Hence, by Lemma 6, if h' is the largest integer not greater than $\frac{h}{\beta+\epsilon}$ and if $h > h_0$, than $h' < r_h(x)$ for every x and B, SB, $\ldots$, $S^{h'-1}B$ are disjoint. B being an interval, this implies (by Lemma 5), that the length of B is not greater than $\beta a(h')$, where $a(h')$ is the smallest length of the intervals of $Q_{h'}$. Hence $F_I(T) \leq \limsup_{h'\to+\infty} \frac{(\beta+\epsilon)h'-1}{\beta} a(h')$.

Our formulas, together with Proposition 3, imply then that $F_I(T) = F_I(\alpha)$, which proves the proposition, as the value of $F_I(\alpha)$ is not changed if we replace α by $\frac{1}{\alpha} - p$ for an integer p. QED

As a by-product of the proof, we see that, in the definition of $F_I(T)$, the requirement "$B, \ldots, T^{h-1}B$ are interval" is equivalent to the requirement "B is an interval", if T is a rotation or an exchange of three intervals. Note that for a more general system, to reduce the requirement to "B is an interval" in the definition of $F_I(T)$ would not guarantee any more that $F^\star(T) \geq F_I(T)$.

Corollary 1 *T is of rank one by intervals if and only if $\frac{1-l}{m}$ has unbounded partial quotients.*

This makes precise the results in [KAT-STE], where it is stated (without a written proof), that T has good cyclic approximations (a property which implies rank one) if $\frac{1-l}{m}$ has unbounded partial quotients and $\frac{1-l}{1-l+m}$ is approximated in $o(\frac{1}{q_n})$ by rationals $\frac{p_n}{q_n}$, where the q_n are a subsequence of the denominators of the convergents of $\frac{1}{1-l+m}$; the techniques of that paper imply that rank one by intervals was known in that case. A small by-product of Veech's theory ([VEE2]) implies that T is of rank one by intervals for almost all l and m. No result on absence of rank one by intervals can be deduced from these results.

Corollary 2 *Every ergodic exchange of three intervals has simple spectrum (this means that $\mathcal{L}^2(X)$ is the closed linear space generated by $(U^n f_0)_{n\in\mathbb{Z}}$, for some function f_0, and the operator $Uf = f \circ T$).*

Proof

Because of Propositions 3 and 13, for non-trivial exchanges of three intervals

$$F^\star(T) \geq F_I(T) \geq \frac{5+\sqrt{5}}{10} > \frac{1}{2};$$

because of a result attributed to Katok and proved in [KIN], this relation implies simple spectrum. Other ergodic exchanges of three intervals are isomorphic to irrational rotations, hence have also simple spectrum. QED

The simple spectrum was known only in the cases described above, when the transformation was known to be of rank one.

The following question has been asked to one of the authors by Veech: can any interval exchange system be measure-theoretically isomorphic to the system associated to the *Morse sequence* (see [FER2] or [FER4] for a description)? The general answer is not known (in particular, even nontrivial exchanges of three intervals may have rational eigenvalues, see the discussion at the end of Section 9). However

Corollary 3 *No exchange of three intervals can be measure-theoretically isomorphic to the system associated to the Morse sequence.*

Proof
As above, for every nontrivial exchange of three intervals T we have $F^{\star}(T) \geq \frac{5+\sqrt{5}}{10} = 0,7\ldots$, while for the Morse system T' we have $F^{\star}(T') = \frac{2}{3}$ ([FER2]). The result is trivially true if T is not ergodic, or if T is an irrational rotation. QED

Proposition 14 *We have*

$$F_D(T) = F\left(\frac{1-l}{1-l+m}, \frac{1}{1-l+m}\right)$$

Proof
We keep the notation of the proof of Proposition 13. Let B be a subset of $[0, 1-l+m[$ such that $B, \ldots, S^{h'-1}B$ are disjoint, and such that B is included in an interval E whose interior does not contain any $S^{-i}0$ or any $S^{-i}1$ for $0 \leq i \leq h'-1$. The definition of $F(\alpha, \beta)$ imply that, for h' arbitrarily large, we can find such a B such that $h'\beta\mu(B)$ is close to $F(\alpha, \beta)$.

Applying Lemma 6 as in the proof of Proposition 13 (after replacing it by SB if necessary, we can suppose $B \subset J$) we have $B, \ldots, T^{h-1}B$ disjoint, for some h close to $\beta h'$, and B is included in an interval E, whose length is arbitrarily small if h is large enough, such that no discontinuity of T appears in the interior of T^jE for any $0 \leq j < h-1$. Hence the diameter of each T^jB, for $0 \leq j \leq h-1$, is smaller than the length of E, and thus we can approximate every partition in the sense of Definition 1. Hence, as h is arbitrarily large, $F_D(T)$ is greater than $h\mu(B)$, which gives our lower bound.

Conversely, suppose all the T^jB have a diameter not greater than ϵ for $0 \leq j \leq h-1$, and let E be the smallest interval containing B. If in the interior of E there is any point of the form $T^{-s}e$, $e=0$ or 1, $1 \leq s \leq h-1$, then $T^{s+1}B$ must contain two points which are separated by $Te-e$, which is a fixed quantity, larger than ϵ if ϵ is small; hence the interior of E contains no such point. Then, by Lemma 6, the interior of E does not contain any $S^{-i}0$ or any $S^{-i}1$, for $0 \leq i \leq h'-1$, and for some h' close to $\frac{h}{\beta}$. Hence, for h arbitrarily large, $h\mu(B)$ must be not greater than $F(\alpha,\beta)$, which gives our upper bound. QED

Hence the examples in Section 8 allow us to find examples of exchanges of three intervals which are induced by the same rotation but are not topologically conjugate; also they give nontrivial exchanges of three intervals with $F^\star(T) \geq F_D(T) > F_I(T)$.

To compute $F^\star(T)$ seems to be a difficult problem, as it involves looking at measurable sets, with no further assumptions of regularity; the last proposition implies that

$$F^\star(T) \geq F\left(\frac{1-l}{1-l+m}, \frac{1}{1-l+m}\right);$$

an upper bound is then necessary only in the case where α has bounded partial quotients. We are able to show a partial result in this direction:

Proposition 15 *Let $\alpha = \frac{1-l}{1-l+m}$, $\beta = \frac{1}{1-l+m}$. If α has bounded partial quotients, then there exists a nonempty Cantor set $K(\alpha)$, such that, if $\beta \in K(\alpha)$, $F^\star(T) < 1$.*

Proof
Suppose α has bounded partial quotients. Let C_1 and C_2 be such that $q_{n+1} \leq C_1 q_n$ for all n and the length of the smallest interval in Q_h is greater than $\frac{C_2}{q_n}$, for $q_{n-1} < h \leq q_n$ (this is possible through the estimates in Section 3). Let (for example) $C_3 = \frac{C_2}{3}$ and r_n be the largest integer smaller or

equal to $\frac{q_n-1}{2}$. We then define $K(\alpha)$ by deleting from the circle every interval $[s\alpha - \frac{C_3}{q_n}, s\alpha + \frac{C_3}{q_n}]$ for $n \geq 1$ and $-r_n \leq s \leq r_n$. The construction of $K(\alpha)$ ensures the following property: there exists a constant C_4 such that for every ϵ, the smallest $k \geq 0$ such that either $k\alpha \in [\beta-\epsilon, \beta+\epsilon]$ or $-k\alpha \in [\beta-\epsilon, \beta+\epsilon]$ is at least $\frac{C_4}{\epsilon}$.

Let now $\beta \in K(\alpha)$ be fixed, and suppose $F^{\star}(T) = 1$. Then (see Section 6) we can find an increasing sequence of Rokhlin stacks ϕ_n which ultimately separate every points on a set X of measure one. This means that, if we restrict ourselves to X (which we shall do in all the sequel), the diameter of the level of ϕ_n containing a point x is a decreasing function of n, and has to tend to zero; hence, for every fixed ϵ and every h arbitrarily large, we can find a set B such that B, ..., $T^{h-1}B$ are disjoint, $\mu(\cup_{j=0}^{h-1}T^jB) > 1-\epsilon$, and at least $h(1-\epsilon)$ of the levels have a diameter not greater than ϵ (strictly speaking, the diameter of the intersection of these levels with X). Hence, starting from the basis B , there are h_0 levels of diameter greater than ϵ, then l_1 levels of diameter not greater than ϵ, h_1 levels of diameter greater than ϵ, ...,l_k levels of diameter not greater than ϵ, h_k levels of diameter greater than ϵ, with $h_0 + \ldots + h_k < h\epsilon$.

We choose $1 \leq i \leq k$, and look at the i-th group of levels of diameter not greater than ϵ; we call them B', ...,$T^{l_i-1}B'$. Let E be the smallest interval containing B' (strictly speaking, $B' \cap X$, but this gives the same E). If in the interior of E there is any point of the form $T^{-s}e$, $e = 0$ or 1, $1 \leq s \leq l_i - 1$, then $T^{s+1}B'$ must contain two points which are separated by $Te - e$, which is a fixed quantity, larger than ϵ if ϵ is small, hence this contradicts the hypotheses. Hence the interior of E contains no such point. Hence, using Lemma 6 as in the proof of Proposition 14, either l_i is smaller than some fixed L_0 or the interior of E cannot contain $S^{-j}1$ or $S^{-j}0$, for $0 \leq i \leq l'_i$ where l'_i is close to $\frac{l_i}{\beta}$; hence, using the bound on the lengths of the intervals of Q_{l_i}, $|E| < \frac{C_5}{l_i}$ for some constant C_5. Also, B', ...,$T^{l_i-1}B'$ are disjoint; hence, if $l_i > L_0$, the proof of Proposition 14, using again Lemma 6, implies that $\mu(B') \leq \frac{C_6}{l_i}$ for l_i large enough, for any constant $C_6 > F(\alpha, \beta)$, hence for some $C_6 < 1$ because of Propositions 4 and 11, as α has bounded partial quotients.

Suppose now $1 \leq i \leq k-1$; then $T^{l_i}B'$ has diameter greater than ϵ, hence there must be $T^{-1}e$ in the interior of $T^{l_i-1}E$, for $e=0$ or 1 (and there cannot be both $T^{-1}0$ and $T^{-1}1$ because $T^{l_i-1}E$ has diameter not greater than ϵ). So $T^{l_i}B'$ is included in $([e-|E|,e+|E|]\cap[0,1])\cup([Te-|E|,Te+|E|]\cap[0,1])$, with points in each of these intervals.. Now, if there are no discontinuity of T both in the first h_i images of $]e-|E|,e+|E|[\cap[0,1]$ and in the first h_i images of $]Te-|E|,Te+|E|[\cap[0,1]$, then $T^{l_i+h_i}B'$ contains points at distance less than $|E|<\epsilon$, respectively from $T^{h_i-1}e$ and $T^{h_i}e$; but for all x, Tx and x are at least separated by a fixed distance Δ; hence if ϵ is small enough, $T^{l_i+h_i}B'$ cannot have diameter less than ϵ.

Hence there is a discontinuity of T either in the first h_i images of $]e-|E|,e+|E|[\cap[0,1]$ or in the first h_i images of $]Te-|E|,Te+|E|[\cap[0,1]$. Hence there exists $f=0$ or 1 and $1\leq s\leq h_1$ such that either $T^{-1-s}f$ is in $]e-|E|,e+|E|[\cap[0,1]$ or $T^{-1-s}f$ is in $]Te-|E|,Te+|E|[\cap[0,1]$. But this is a matter of approximation of α, β or $-\beta$ by numbers $k\alpha$; for any δ, the first return time, positive or negative, under S, of 0 into $]-\delta,\delta[$ or of β into $]\beta-\delta,\beta+\delta[$ is at least $\frac{C_7}{\delta}$, for some constant C_7, because α has bounded partial quotients; and the first return time, positive or negative, under S, of 0 into $]-\beta-\delta,-\beta+\delta[$ or into $]\beta-\delta,\beta+\delta[$ is at least $\frac{C_4}{2\delta}$ because $\beta\in K(\alpha)$. But if $\frac{1}{\delta}$ is large enough, this means, by Lemma 6, that the first return time, positive or negative, under T, of any f into $]e-\delta,e+\delta[\cap[0,1]$ or $]Te-\delta,Te+\delta[\cap[0,1]$ are at least $\frac{C_8}{\delta}$.

Hence, if $1\leq i\leq k-1$, $h_i\geq\frac{C_8}{|E|}$, hence $h_i\geq C_9l_i$ (even if $l_i\leq L_0$ if ϵ and hence $|E|$ are small enough). Hence $\sum_{i=0}^{k-1}l_i$ is smaller than $\frac{1}{C_9}h\epsilon$; which means $l_k\geq h(1-C_{10}\epsilon)$; but the first level B' of the k-th group of levels of diameter not greater than ϵ satisfies, as showed above, $\mu(B')<\frac{C_6}{l_k}$, for $C_6<1$; hence the total measure of the stack of height h is at most $\frac{C_6}{1-C_{10}\epsilon}<1$ if ϵ is small enough, while it should be close to 1 for any fixed ϵ and h large enough. Hence $F^\star(T)$ cannot be equal to 1. QED

There are not many known examples of ergodic interval exchanges which are not of rank one; one example appears in [OSE], with as many as thirty intervals, and there is Example 4 in [GOO], which can be seen as an exchange of nine intervals; both these examples do not have simple spectrum, hence cannot be of rank one. Our examples, with β in $K(\alpha)$, are the only ones to be on three intervals, or to have simple spectrum. Note that β will be in $K(\alpha)$ if our approximation algorithms for β and $-\beta$, left and right, by $k\alpha$, give n_i, n'_i, etc... with bounded gaps.

In view of previous results,

Corollary 4 *If α has bounded partial quotients, and $\beta \in K(\alpha)$, T is not measure-theoretically isomorphic to any three-interval exchange with an α' which has unbounded partial quotients, nor to any irrational rotation, and has a non-discrete spectrum.*

It is proved in [KAT-STE] that when α has unbounded partial quotients, and, for a subsequence of the denominators q_n of the convergents of α, we have both $|\alpha - \frac{p_n}{q_n}| < o\left(\frac{1}{q_n^2}\right)$ and $|\beta - \frac{r}{q_n}| > \frac{c}{q_n}$ for a constant c and every integer r, then T is not isomorphic to a rotation, as it has a continuous spectrum (as both this condition and the one we state after Corollary 1 are required only on a subsequence, they are compatible, and thus Katok and Stepin produced the first known systems with simple continuous spectrum); our Corollary 4 proves that T is not isomorphic to a rotation, for completely different reasons, in a subcase of the case where α has bounded partial quotients.

The general question about isomorphism between a non-trivial three-interval exchanges and a rotation is still open; we recall the little-known fact that there are examples, due to Veech, of non-trivial three-interval exchanges with non-continuous spectrum: for every α with unbounded partial quotients and β in a continuum $K'(\alpha)$, the map taking value 1 on $[0, \beta[$ and -1 on $[\beta, 1]$ is a coboundary for the rotation R_α ([VEE1]), and this implies immediately that -1 is an eigenvalue for the induced map on $[\beta, 1]$. However, the only knwon examples have rational eigenvalues; we conjecture that no non-trivial exchange of three intervals can be measure-theoretically isomorphic to a rotation; in all the cases where α has bounded partial quotients we suppose this could be proved through the absence of rank one of the interval exchange, but a forthcoming paper ([BOS-NOG]) will prove that whenever α has bounded partial quotients then the three-interval exchange has continuous spectrum; in any case, new reasons would have to be found for dealing with the cases where α has unbounded partial quotients and β is well approximated by all convergents of α.

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