

Arithmetic distributions of convergents arising from Jacobi-Perron algorithm

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Abstract

We study the distribution modulo m of the convergents associated with the d -dimensional Jacobi-Perron algorithm for a.e. real numbers in $(0, 1)^d$ by proving the ergodicity of a skew product of the Jacobi-Perron transformation; this skew product was initially introduced in [6] for regular continued fractions.

1 Introduction

For an irrational number x , $0 < x < 1$, we denote by $\frac{p_n}{q_n}$ the n -th convergent of x , which is defined by the regular continued fraction expansion coefficients of x . In 1988, H. Jager and P. Liardet [6] studied the distribution properties of the pairs (p_n, q_n) modulo m . These properties were originally considered by P. Szűsz in [16], and then by R. Moeckel [7] who used the ergodicity of geodesic flows over the modular surfaces: more precisely, they proved that given any positive integer $m \geq 2$, for a.e. x , the sequence $\{(p_n, q_n) : n \geq 1\}$ is equidistributed modulo m over the set $\{(p, q) \in \mathbb{Z}_m^2 : \langle p, q \rangle = \mathbb{Z}_m\}$, where $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$ and where the notation $\langle p_1, \dots, p_k \rangle$ stands for the subgroup of \mathbb{Z}_m generated by the elements p_1, \dots, p_k . To prove this property, H. Jager and P. Liardet considered in [6] the group of 2×2 matrices with entries from \mathbb{Z}_m and determinant ± 1 , that is,

$$G(m) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}_m, ad - bc = \pm 1 \right\}.$$

*with the support of ACI Jeunes chercheurs 02 “Combinatoire des mots multidimensionnels, pavages et numération”.

†This work has been done during the second and the third authors' visit at LIRMM, Montpellier.

It is possible to show that $\left\{\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : a \in \mathbb{Z}_m\right\}$ generates $G(m)$. This fact implies the ergodicity of a $G(m)$ -extension (a skew product indeed) of the continued fraction transformation. The equidistribution property of $\{(p_n, q_n) : n \geq 1\}$ modulo m is then an easy consequence of the individual ergodic theorem.

A natural extension of this skew product was then introduced in [3] to deduce the distribution of the approximation coefficients associated with the continued fraction algorithm; these results were also extended to the so-called S -expansions, in the sense of [4]; see also for connected results [1] and [10].

The aim of the present paper is to generalize these equidistribution results to the d -dimensional Jacobi-Perron algorithm. Note that the 1-dimensional Jacobi-Perron algorithm reduces to the regular continued fraction algorithm.

Let us start with the definition of the Jacobi-Perron algorithm. We fix a positive integer $d \geq 2$. Let $X = [0, 1)^d$ be endowed with the Borel σ -algebra \mathbb{B} . We first define the map $T : X \rightarrow X$ by

$$T(\mathbf{x}) = T((x_1, x_2, \dots, x_d)) = \left(\frac{x_2}{x_1} - \left\lfloor \frac{x_2}{x_1} \right\rfloor, \dots, \frac{x_d}{x_1} - \left\lfloor \frac{x_d}{x_1} \right\rfloor, \frac{1}{x_1} - \left\lfloor \frac{1}{x_1} \right\rfloor \right)$$

for $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$ if $x_1 \neq 0$, and $T(\mathbf{x}) = \mathbf{0}$, otherwise; (X, T) is called the d -dimensional Jacobi-Perron algorithm. Notice that there exists a unique absolutely continuous invariant probability measure μ for T which is equivalent to the Lebesgue measure (see for instance [13]).

We put for \mathbf{x} in X with $x_1 \neq 0$

$$\mathbf{k}(\mathbf{x}) = \mathbf{k}^{(0)}(\mathbf{x}) = (k_1, k_2, \dots, k_d) = \left(\left\lfloor \frac{x_2}{x_1} \right\rfloor, \left\lfloor \frac{x_3}{x_1} \right\rfloor, \dots, \left\lfloor \frac{x_d}{x_1} \right\rfloor, \left\lfloor \frac{1}{x_1} \right\rfloor \right),$$

if $x_1 = 0$, we set $\mathbf{k}(\mathbf{x}) = \mathbf{0}$; we similarly define

$$\mathbf{k}^{(s)}(\mathbf{x}) = \left(k_1^{(s)}, k_2^{(s)}, \dots, k_d^{(s)} \right) = \mathbf{k}(T^{s-1}(\mathbf{x})) \quad \text{for } s \geq 1.$$

We then associate (x_1, x_2, \dots, x_d) with the column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}$ and consider the following matrix

$$P = \begin{pmatrix} -k_1 & 1 & 0 & \dots & 0 \\ -k_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -k_d & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}. \quad (1)$$

Then $T((x_1, x_2, \dots, x_d))$ corresponds to $P \begin{pmatrix} x_1 \\ \vdots \\ x_d \\ 1 \end{pmatrix}$. To construct the sequence

$$\left\{ \left(\frac{p_1^{(k)}}{q^{(k)}}, \dots, \frac{p_d^{(k)}}{q^{(k)}} \right) : k \geq 1 - d \right\}$$

of *simultaneous approximation convergents* of \mathbf{x} from the d -dimensional Jacobi-Perron algorithm, we first define $Q^{(0)}$ as the $(d+1) \times (d+1)$ identity matrix I_{d+1} ; we then define recursively $Q^{(n)}$ for $n \geq 1$ as

$$Q^{(n)} := Q^{(n-1)} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & k_1^{(n)} \\ 0 & 1 & \dots & 0 & k_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_d^{(n)} \end{pmatrix}.$$

We thus set for $n \geq 1$

$$Q^{(n)} = \begin{pmatrix} p_1^{(n-d)} & p_1^{(n-d+1)} & \dots & p_1^{(n-1)} & p_1^{(n)} \\ p_2^{(n-d)} & p_2^{(n-d+1)} & \dots & p_2^{(n-1)} & p_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ p_d^{(n-d)} & p_d^{(n-d+1)} & \dots & p_d^{(n-1)} & p_d^{(n)} \\ q^{(n-d)} & q^{(n-d+1)} & \dots & q^{(n-1)} & q^{(n)} \end{pmatrix}.$$

Let us observe that

$$Q^{(1)} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & k_1 \\ 0 & 1 & \dots & 0 & k_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & k_d \end{pmatrix} = P^{-1}. \quad (2)$$

It is well-known that for any $\mathbf{x} = (x_1, x_2, \dots, x_d) \in X$,

$$\lim_{n \rightarrow \infty} \frac{p_i^{(n)}}{q^{(n)}} = x_i \quad \text{for } 1 \leq i \leq d$$

holds.

In this paper, we prove that for almost every $\mathbf{x} \in X$ the sequences of vectors $\{(q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) : n \geq 1\}$ and $\{(p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) : n \geq 1\}$ are both equidistributed modulo m for any integer $m \geq 2$.

More precisely we put

$$\tilde{\mathbb{Z}}_m^{d+1} = \{(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \mathbb{Z}_m^{d+1} : \langle \alpha_1, \alpha_2, \dots, \alpha_{d+1} \rangle = \mathbb{Z}_m\}$$

and

$$c_m = \#\tilde{\mathbb{Z}}_m^{d+1} \quad (\text{the cardinality of } \tilde{\mathbb{Z}}_m^{d+1}).$$

One easily sees that

$$\begin{aligned} c_m &= \varphi_{d+1}(m) \\ &= \#\{(a_1, a_2, \dots, a_{d+1}) \in \{1, \dots, m\}^{d+1} : \gcd(a_1, \dots, a_{d+1}, m) = 1\}, \end{aligned} \quad (3)$$

where φ_{d+1} denotes the Jordan totient function of order $d+1$; we thus have (see for instance [15] or [11])

$$c_m = m^{d+1} \prod_{p|m} (1 - p^{-(d+1)}),$$

where the notation $\prod_{p|m}$ stands in all that follows for the product over the prime numbers p that divide m . We then have the following:

Theorem 1. *Let $m \geq 2$ be a nonnegative integer. For almost every $\mathbf{x} \in X$ and for any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, we have*

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \pmod{m}\}}{N} \\ &= \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (p_1^{(n)}, p_2^{(n)}, \dots, p_d^{(n)}, q^{(n)}) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \pmod{m}\}}{N} \\ &= \frac{1}{c_m} = \frac{1}{\varphi_{d+1}(m)} = \frac{1}{m^{d+1} \prod_{p|m} (1 - p^{-(d+1)})}. \end{aligned}$$

To prove this theorem, we consider for a given integer $m \geq 2$, the group $G(m)$ defined in a similar way as in [6]:

$$G(m) = \begin{cases} SL(d+1, \mathbb{Z}_m) & \text{if } d \text{ is even,} \\ SL_{\pm}(d+1, \mathbb{Z}_m) & \text{if } d \text{ is odd,} \end{cases}$$

where $SL(d+1, \mathbb{Z}_m)$ stands for the matrices with entries in \mathbb{Z}_m with determinant 1, whereas $SL_{\pm}(d+1, \mathbb{Z}_m)$ stands for the matrices with entries in \mathbb{Z}_m with determinant ± 1 . Let us recall that (see for instance [11] or [9]) that

$$\#SL(d+1, \mathbb{Z}_m) = m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) = m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m).$$

Let C_m denote the cardinality of $G(m)$. Since $SL(d+1, \mathbb{Z}_m)$ is a subgroup of $SL_{\pm}(d+1, \mathbb{Z}_m)$ of index 2 if d is odd and $m \neq 2$, one thus gets

$$C_m = \begin{cases} \begin{aligned} & m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) \\ &= m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m) \end{aligned} & \text{if } d \text{ is even or } m = 2 \\ \begin{aligned} & 2m^{(d+1)^2-1} \prod_{i=2}^{d+1} \prod_{p|n} (1 - p^{-i}) \\ &= 2m^{d(d+1)/2} \prod_{i=2}^{d+1} \varphi_i(m) \end{aligned} & \text{if } d \text{ is odd and } m \neq 2. \end{cases} \quad (4)$$

We identify $Q^{(1)}$ with the matrix with coefficients in \mathbb{Z}_m obtained by reducing modulo m its entries. Here we note that $\det Q^{(1)} = 1$ or -1 if d is respectively even or odd, which implies that $Q^{(1)}$ belongs to the group $G(m)$, whatever may be the parity of d .

We define the map T_m on $X \times G(m)$ by

$$T_m(\mathbf{x}, A) = (T(\mathbf{x}), AQ^{(1)});$$

T_m is said to be a $G(m)$ -extension of the map T .

We define the probability measure δ_m on $G(m)$ by $(\frac{1}{C_m}, \dots, \frac{1}{C_m})$. Then it is easy to see that $\mu \times \delta_m$ is an invariant probability measure for T_m . Our question is whether $(T_m, \mu \times \delta_m)$ is ergodic or not. In Section 2, we show that the set of matrices of the form (2) (reduced modulo m) generates $G(m)$. Then in Section 3, we prove the ergodicity of T_m , from which we deduce the following proposition and then Theorem 1 (in the same way as in [6]):

Proposition 1. *For a.e. $\mathbf{x} \in X$ and any $A \in G(m)$,*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : Q^{(n)} \equiv A \pmod{m}\} = \frac{1}{C_m}.$$

Finally we have the following

Corollary 1. *For a.e. $\mathbf{x} \in X$ and any $a \in \mathbb{Z}_m$*

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : q^{(n)} \equiv a \pmod{m}\} = \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)}.$$

In all that follows, we simply denote by $0, 1, \dots, m-1$ the elements of \mathbb{Z}_m if it is clear that the elements are in \mathbb{Z}_m according to the context. In this case, one has obviously $m-1 = -1$.

2 Basic properties of $G(m)$

We first define

$$\Gamma_m = \{A \in SL(d+1, \mathbb{Z}) : A \equiv I_{d+1} \pmod{m}\}.$$

Then it is well-known that

$$SL(d+1, \mathbb{Z}_m) \cong \Gamma_m \backslash SL(d+1, \mathbb{Z})$$

e.g., see G. Shimura [15], p. 21. From this property, it easily follows that

$$SL_{\pm}(d+1, \mathbb{Z}_m) \cong \Gamma_m \backslash GL(d+1, \mathbb{Z}).$$

We respectively say that a $(d+1) \times (d+1)$ matrix with \mathbb{Z} (or \mathbb{Z}_m)-entries of the form (2) is a J-P matrix, and that a matrix of the form (1) is a J-P* matrix; a J-P matrix is the inverse of a J-P* matrix.

In the sequel of this section, we show that the monoid generated by the set of J-P matrices with \mathbb{Z}_m -entries is equal to $G(m)$:

Theorem 2. *For any $B \in G(m)$, there exist J-P matrices A_1, A_2, \dots, A_s such that*

$$B = A_1 A_2 \cdots A_s.$$

For this purpose, we first need some notation and some preliminary lemmas. We put

$$\Delta(k_1, k_2, \dots, k_d) := \begin{pmatrix} k_1 & 1 & 0 & \dots & 0 \\ k_2 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_d & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix};$$

in particular,

$$\Delta = \Delta(0, 0, \dots, 0) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

We thus have

$$(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1})\Delta = (\mathbf{a}_{d+1}, \mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d), \quad (5)$$

where $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1}$ are either elements in \mathbb{Z}_m or $(d+1)$ -dimensional vectors with \mathbb{Z}_m -entries. Let us notice that the matrices $\Delta(k_1, k_2, \dots, k_d)$ are J-P* matrices.

Lemma 1. *For any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, there exist J-P matrices A_1, A_2, \dots, A_s with \mathbb{Z}_m -entries such that*

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) = (0, \dots, 0, 1)A_1A_2 \dots A_s,$$

where s depends on $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$.

Proof of Lemma 1. We define the following natural order \prec on \mathbb{Z}_m by

$$0 \prec 1 \prec \dots \prec m-1.$$

Let $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{(d+1)}$. We denote by α^* the element in \mathbb{Z}_m such that $\langle \alpha^* \rangle = \langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle$. Let us prove by induction on α^* (considered then as an element in $\{1, \dots, m\}$) that there exists a finite number of J-P* matrices $\Delta_1, \dots, \Delta_t$ and $(\alpha'_1, \dots, \alpha'_d) \in \mathbb{Z}_m^d$ such that

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \Delta_1 \dots \Delta_t = (1, \alpha'_1, \dots, \alpha'_d).$$

If $\alpha^* = 1$, then $\langle \alpha_1, \alpha_2, \dots, \alpha_d \rangle = \mathbb{Z}_m$ and there exist $k_1, \dots, k_d \in \mathbb{Z}_m$ such that $\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} = 1$. We thus have

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \Delta(k_1, k_2, \dots, k_d) = (1, \alpha_1, \dots, \alpha_d).$$

Suppose now that $\alpha^* \neq 1$. Since $\langle \alpha_1, \alpha_2, \dots, \alpha_{d+1} \rangle \neq \langle \alpha^* \rangle$, there exist $k_1, \dots, k_d \in \mathbb{Z}_m$ such that $0 \prec \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1} \prec \alpha^*$. We thus have

$$(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \Delta(k_1, k_2, \dots, k_d, 1) = \left(\sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \dots, \alpha_d \right).$$

We can now conclude inductively since $\langle \sum_{i=1}^d k_i \alpha_i + \alpha_{d+1}, \alpha_1, \dots, \alpha_d \rangle = \mathbb{Z}_m$.

Now we have

$$(1, \alpha'_1, \dots, \alpha'_d) \cdot \Delta(-\alpha'_d, 0, \dots, 0) \cdot \Delta(0, -\alpha'_{d-1}, 0, \dots, 0) \cdots \Delta(0, \dots, 0, -\alpha'_1) \\ = (0, \dots, 0, 1).$$

Since a J-P* matrix is the inverse of a J-P matrix, we get the assertion of this lemma. \square

The following lemmas are essential and easily proved.

Lemma 2. For any $(d+1)$ -dimensional vectors with \mathbb{Z}_m -entries $(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_{d+1})$, we have

$$(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_i, \mathbf{a}_{i+1}, \dots, \mathbf{a}_d, \mathbf{a}_{d+1}) \cdot \Delta(0, \dots, 0, \overset{i}{\underset{\vee}{-1}}, 0, \dots, 0) \cdot \Delta^{d-i} \\ \cdot \Delta(0, \dots, 0, \overset{d+1-i}{\underset{\vee}{1}}, 0, \dots, 0) \cdot \Delta^{i-1} \cdot \Delta(0, \dots, 0, \overset{i}{\underset{\vee}{-1}}, 0, \dots, 0) \cdot \Delta^d \\ = (\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{a}_{d+1}, \mathbf{a}_{i+1}, \dots, \mathbf{a}_d, -\mathbf{a}_i).$$

Lemma 3. We have

$$(\mathbf{a}_1, \dots, \mathbf{a}_{d+1}) \cdot \Delta(0, \dots, 0, 1) \cdot \Delta(-1, 0, \dots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \dots, 0, 1) \\ = (\mathbf{a}_d, \mathbf{a}_1, \dots, \mathbf{a}_{d-1}, -\mathbf{a}_{d+1}).$$

In particular, when d is odd

$$(\mathbf{a}_1, \dots, \mathbf{a}_{d+1}) \cdot [\Delta(0, \dots, 0, 1) \cdot \Delta(-1, 0, \dots, 0) \cdot \Delta^{d-1} \cdot \Delta(0, \dots, 0, 1)]^d \\ = (\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_d, -\mathbf{a}_{d+1}).$$

Proof of Theorem 2. Let us fix

$$B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1(d+1)} \\ b_{21} & b_{22} & \cdots & b_{2(d+1)} \\ \vdots & \vdots & \ddots & \vdots \\ b_{(d+1)1} & b_{(d+1)2} & \cdots & b_{(d+1)(d+1)} \end{pmatrix}$$

in $G(m)$. We want to prove that there exist J-P* matrices $\Delta_1, \dots, \Delta_s$ such that

$$B \Delta_1 \cdots \Delta_s = I_{d+1},$$

which implies immediately the desired result. For that purpose, let us prove by induction on $1 \leq j \leq d$ that there exist J-P* matrices $\Delta_1, \dots, \Delta_{s_j}$ such that

$$B_j := B \Delta_1 \cdots \Delta_{s_j} = \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right), \quad (6)$$

where I_j is the $j \times j$ identity matrix. Indeed, if this property holds for $j = d$, then we obtain that there exist J-P* matrices $\Delta_1, \dots, \Delta_{s_d}$ such that

$$B \Delta_1 \cdots \Delta_{s_d} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & \pm 1 \end{pmatrix}.$$

If d is even, then all J-P* matrices are of determinant 1. Thus the $(d+1, d+1)$ -entry of the right hand side is equal to 1. If d is odd and the $(d+1, d+1)$ -entry of the right hand side is equal to -1 , then by Lemma 3 we can reduce it to 1 by application of J-P* matrices. In either case, we get the desired result.

It thus remains to prove the induction property. Let us first prove that it holds for $j = 1$. Since $\det B = \pm 1$, then $\langle b_{11}, b_{12}, \dots, b_{1(d+1)} \rangle = \mathbb{Z}_m$, and there thus exist J-P* matrices $\Delta_1, \dots, \Delta_{s_1-1}$ such that

$$(b_{11}, b_{12}, \dots, b_{1(d+1)}) \Delta_1 \cdots \Delta_{s_1-1} = (0, \dots, 0, 1)$$

by Lemma 1. Thus

$$B \Delta_1 \cdots \Delta_{s_1-1} = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ b_{11}^{(1)} & \cdots & b_{1d}^{(1)} & b_{1(d+1)}^{(1)} \\ \vdots & \ddots & \vdots & \vdots \\ b_{d1}^{(1)} & \cdots & b_{dd}^{(1)} & b_{d(d+1)}^{(1)} \end{pmatrix}.$$

We set

$$B^{(1)} = \begin{pmatrix} b_{11}^{(1)} & \cdots & b_{1d}^{(1)} \\ \vdots & \ddots & \vdots \\ b_{d1}^{(1)} & \cdots & b_{dd}^{(1)} \end{pmatrix}.$$

Since $\det B^{(1)} = \pm 1$, then there exist $k_1, \dots, k_d \in \mathbb{Z}_m$ such that

$$B \Delta_1 \cdots \Delta_{s_1-1} \Delta(k_1, \dots, k_d) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & b_{11}^{(1)} & \cdots & b_{1d}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & b_{d1}^{(1)} & \cdots & b_{dd}^{(1)} \end{pmatrix}.$$

It remains to set $\Delta_{s_1} = \Delta(k_1, \dots, k_d)$ to conclude the proof of the induction property for $j = 1$.

Let us assume now that the induction property holds for $1 \leq j \leq d-1$ (if $d = 1$, the proof is finished); one thus deduces that the determinant of $B^{(j)}$ (defined in (6)) is equal to ± 1 . We set

$$B^{(j)} = \begin{pmatrix} b_{11}^{(j)} & \cdots & b_{1(d+1-j)}^{(j)} \\ \vdots & \ddots & \vdots \\ b_{(d+1-j)1}^{(1)} & \cdots & b_{(d+1-j)(d+1-j)}^{(1)} \end{pmatrix}.$$

Let us divide the induction proof into two steps for clarity issues.

Step 1. Let us first prove that we can find J-P* matrices $\Delta_{s_j+1}, \dots, \Delta_{s_j+t}$ such that $B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t}$ is equal to

$$\left(\begin{array}{ccc|c|ccc} & & \mathbf{0} & & & & \mathbf{0} \\ \hline & 0 & \dots & 0 & 1 & & 0 \\ \hline g_{1(l+1)}^{(j+1)} & \dots & g_{1(d-j)}^{(j+1)} & g_{1(d+1-j)}^{(j+1)} & & g_{11}^{(j+1)} & \dots & g_{1l}^{(j+1)} \\ \vdots & \ddots & \vdots & \vdots & & \vdots & \ddots & \vdots \\ g_{(d-j)(l+1)}^{(j+1)} & \dots & g_{(d-j)(d-j)}^{(j+1)} & g_{(d-j)(d+1-j)}^{(j+1)} & & g_{(d-j)1}^{(j+1)} & \dots & g_{(d-j)l}^{(j+1)} \\ \hline & & & & I_j & & & \\ \hline & & & & \mathbf{0} & & & \end{array} \right) \quad (7)$$

for some l , $0 \leq l < d - j + 1$.

According to the proof of Lemma 1, we can find $(d - j + 1) \times (d - j + 1)$ J-P* matrices $\Delta(k_1^{(1)}, \dots, k_{d-j}^{(1)}), \dots, \Delta(k_1^{(t)}, \dots, k_{d-j}^{(t)})$ such that

$$(b_{11}^{(j)}, \dots, b_{1(d+1-j)}^{(j)}) \Delta(k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdots \Delta(k_1^{(u)}, \dots, k_{d-j}^{(u)}) = (0, \dots, 0, 1).$$

Now

$$\left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, *, \dots, *) = \left(\begin{array}{c|c|c} \mathbf{0} & I_j & \mathbf{0} \\ \hline * & \mathbf{0} & * \end{array} \right),$$

and one checks more generally that for $0 \leq v \leq d - j$

$$\begin{aligned} & \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, *, \dots, *) \cdot \Delta(\underbrace{*0, \dots, 0}_j, *, \dots, *) \cdots \\ & \cdots \Delta(\underbrace{*, \dots, *}_v, \underbrace{0, \dots, 0}_j, *, \dots, *) = \left(\begin{array}{c|c|c} \overbrace{\mathbf{0} \cdots \mathbf{0}}^{v+1} & I_j & \mathbf{0} \\ \hline \underbrace{* \cdots *}_{v+1} & \mathbf{0} & * \end{array} \right). \end{aligned}$$

One thus gets that if $u \leq d - j + 1$, then

$$\begin{aligned} & \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdot \Delta(\underbrace{k_1^{(2)}, 0, \dots, 0}_j, k_2^{(2)}, \dots, k_{d-j}^{(2)}) \cdots \\ & \cdots \Delta(k_1^{(u)}, \dots, k_{d-j}^{(u)}, \underbrace{0, \dots, 0}_j) \end{aligned}$$

has the desired form (7).

Now if $u \geq d - j + 2$, then using (5), one gets

$$\left(\begin{array}{c|c} \mathbf{0} & I_j \\ \hline * & \mathbf{0} \end{array} \right) \Delta^j = \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & * \end{array} \right),$$

and suitable insertions of Δ^j such as

$$\begin{aligned}
& \left(\begin{array}{c|c} I_j & \mathbf{0} \\ \hline \mathbf{0} & B^{(j)} \end{array} \right) \Delta(\underbrace{0, \dots, 0}_j, k_1^{(1)}, \dots, k_{d-j}^{(1)}) \cdot \Delta(k_1^{(2)}, \underbrace{0, \dots, 0}_j, k_2^{(2)}, \dots, k_{d-j}^{(2)}) \\
& \cdots \Delta(k_1^{(d-j+1)}, \dots, k_{d-j}^{(d-j+1)}, \underbrace{0, \dots, 0}_j) \cdot \Delta^j \cdot \Delta(\underbrace{0, \dots, 0}_j, k_1^{(d-j+2)}, \dots, k_{d-j}^{(d-j+2)}) \\
& \cdots \Delta(k_1^{(2(d-j+1))}, \dots, k_{d-l+j}^{(2(d-j+1))}, \underbrace{0, \dots, 0}_j) \cdot \Delta^j \cdot \Delta(\underbrace{0, \dots, 0}_j, k_1^{2(d-j+1)+1}, \dots, k_{d-j}^{2(d-j+1)+1}) \\
& \cdots \Delta(k_1^{(u)}, \dots, k_{d-l+j}^{(u)}, \underbrace{0, \dots, 0}_j, k_{(d-l+j+1)}^{(u)}, \dots, k_{d-j}^{(u)})
\end{aligned}$$

provide the desired form (7), which ends the proof of Step 1.

Step 2. By (5),

$$\begin{aligned}
& B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t} \cdot \Delta^{l+j} \\
& = \left(\begin{array}{c|cccc|c|cccc} & & & \mathbf{0} & & I_j & & \mathbf{0} & & \\ \hline & 0 & \cdots & 0 & 1 & 0 & 0 & \cdots & 0 & \\ g_{1(l+1)}^{(j+1)} & \cdots & g_{1(d-j)}^{(j+1)} & g_{1(d+1-j)}^{(j+1)} & & & g_{11}^{(j+1)} & \cdots & g_{1l}^{(j+1)} & \\ \vdots & \ddots & \vdots & \vdots & \vdots & \mathbf{0} & \vdots & \ddots & \vdots & \\ g_{(d-j)(l+1)}^{(j+1)} & \cdots & g_{(d-j)(d-j)}^{(j+1)} & g_{(d-j)(d+1-j)}^{(j+1)} & & & g_{(d-j)1}^{(j+1)} & \cdots & g_{(d-j)l}^{(j+1)} & \end{array} \right) \cdot \Delta^j \\
& = \left(\begin{array}{c|cccc} I_j & & & \mathbf{0} \\ \hline & 0 & \cdots & 0 & 1 \\ \mathbf{0} & g_{11} & \cdots & g_{1(d-j)} & g_{1(d-j+1)} \\ & \vdots & \ddots & \vdots & \vdots \\ & g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} & g_{(d-j)(d-j+1)} \end{array} \right).
\end{aligned}$$

We put

$$G = \begin{pmatrix} g_{11} & \cdots & g_{1(d-j)} \\ \vdots & \ddots & \vdots \\ g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} \end{pmatrix}.$$

Since the determinant of G is equal ± 1 , there exist $k'_1, \dots, k'_{d-j} \in \mathbb{Z}_m$ such that

$$\begin{aligned} & \left(\begin{array}{c|ccc} I_j & & & \mathbf{0} \\ \hline & 0 & \cdots & 0 \\ & & & 1 \\ \mathbf{0} & g_{11} & \cdots & g_{1(d-j)} \\ & \vdots & \ddots & \vdots \\ & g_{(d-j)1} & \cdots & g_{(d-j)(d-j)} \\ & & & g_{(d-j)(d-j+1)} \end{array} \right) \cdot \Delta(\underbrace{0, \dots, 0}_j, k'_1, \dots, k'_{d-j}) \\ &= \left(\begin{array}{c|cc} 0 & & \\ \vdots & I_j & \mathbf{0} \\ 0 & & \\ \hline 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \mathbf{0} & & G \\ 0 & & & \end{array} \right). \end{aligned}$$

By applying (5), we get

$$\left(\begin{array}{c|cc} 0 & & \\ \vdots & I_j & \mathbf{0} \\ 0 & & \\ \hline 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & \mathbf{0} & & G \\ 0 & & & \end{array} \right) \cdot \Delta^d = \left(\begin{array}{c|cc} I_j & \mathbf{0} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline 0 & \cdots & 0 \\ \mathbf{0} & & G \\ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & & \end{array} \right).$$

We thus have proved that there exist J-P* matrices $\Delta_{s_j+t+1}, \dots, \Delta_{s_j+t'}$ such that

$$B_j \Delta_{s_j+1} \cdots \Delta_{s_j+t'} = \left(\begin{array}{c|cc} I_j & \mathbf{0} & \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \\ \hline 0 & \cdots & 0 \\ \mathbf{0} & & G \\ \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} & & \end{array} \right).$$

It remains to apply Lemma 2; there thus exist J-P* matrices $\Delta_{s_j+t'+1}, \dots, \Delta_{s_j+1}$ such that

$$B_j \Delta_{s_j+1} \cdots \Delta_{s_j+1} = \left(\begin{array}{c|c} I_{j+1} & \mathbf{0} \\ \hline \mathbf{0} & B^{(j+1)} \end{array} \right),$$

where we put

$$B^{(j+1)} = \begin{pmatrix} g_{12} & g_{13} & \cdots & g_{1(d-j)} & -g_{11} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_{(d-j)2} & g_{(d-j)3} & \cdots & g_{(d-j)(d-j)} & -g_{(d-j)1} \end{pmatrix},$$

which concludes the induction proof. \square

3 Ergodicity of T_m and proof of Theorem 1

3.1 Ergodicity

Let us recall some fundamental facts about Jacobi-Perron algorithm. For an integer vector $\mathbf{a} = (a_1, a_2, \dots, a_d)$ with $a_i \geq 0$ for $1 \leq i \leq d$, we put

$$X_{\mathbf{a}} = \{\mathbf{x} \in X : \mathbf{k}(\mathbf{x}) = \mathbf{a}\}.$$

Then we see that $X_{\mathbf{a}} \neq \emptyset$ if and only if $0 \leq a_i \leq a_d$ for any $1 \leq i \leq d-1$ and $a_d > 0$. For a finite sequence of integer vectors $\{\mathbf{a}^{(l)} = (a_1^{(l)}, a_2^{(l)}, \dots, a_d^{(l)})\}$, $1 \leq l \leq n$ such that $a_i^{(l)} \leq a_d^{(l)}$, $1 \leq i \leq d-1$, and $a_d^{(l)} > 0$ for $1 \leq l \leq n$, we define the cylinder set of rank n by

$$X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}} = \{\mathbf{x} \in X : \mathbf{k}^{(l)}(\mathbf{x}) = \mathbf{a}^{(l)} \text{ for } 1 \leq l \leq n\}.$$

A cylinder set $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}$ is said to be proper (or full) if $T^n(X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}) = X$. It is easy to see that $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}$ is proper if $a_i^{(l)} < a_d^{(l)}$ for all $1 \leq l \leq n$ and $1 \leq i \leq d-1$. The following is essential.

Lemma 4. *For almost every $\mathbf{x} \in X$ there exists a sequence of positive integers $n_1 < n_2 < \dots$ such that $X_{\mathbf{k}^{(1)}(\mathbf{x}) \dots \mathbf{k}^{(n_i)}(\mathbf{x})}$ is proper for any $i \geq 1$.*

This shows the exactness of the dynamical system (X, T, μ) ; the exactness means here that $\bigcap_{n=1}^{\infty} T^{-n}\mathbb{B} = \{\emptyset, X\}$ (μ -mod 0). In particular, (X, T, μ) is ergodic and strong mixing. We refer to F. Schweiger [12] or [14] about the theory of Jacobi-Perron algorithm. Now we will show the ergodicity of T_m .

Theorem 3. *The skew product $(X \times G(m), T_m, \mu \times \delta_m)$ is ergodic.*

Proof. For any non-empty cylinder set $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}$, we see from Proposition 2 in [14] that

$$\sup_{\mathbf{x} \in X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}} |DT^n(\mathbf{x})| < (d+1)^{d+1} \inf_{\mathbf{x} \in X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(n)}}} |DT^n(\mathbf{x})|, \quad (8)$$

where $|DT^n|$ is the Jacobian of T^n . Suppose that \mathcal{M} is a T_m -invariant set of $(\mu \times \delta_m)$ -positive measure. Since T_m acts as T on the first coordinate, the ergodicity of T shows

$$\{\mathbf{x} \in X : (\mathbf{x}, A) \in \mathcal{M} \text{ for some } A \in G(m)\} = X \quad (\mu\text{-mod } 0).$$

Thus there exists $A \in G(m)$ such that $(X \times \{A\}) \cap \mathcal{M}$ has positive $(\mu \times \delta_m)$ -measure. We fix such a set A . By the density theorem and Lemma 4, for a given sequence $\varepsilon_i \searrow 0$ there exists a sequence of proper cylinder sets W_i of rank n_i and $B \in G(m)$ such that for all i

$$\frac{(\mu \times \delta_m)((W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(W_i \times \{A\})} > 1 - \varepsilon_i \quad (9)$$

and

$$A \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(1)} \\ 0 & 1 & \dots & 0 & a_2^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(1)} \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(2)} \\ 0 & 1 & \dots & 0 & a_2^{(2)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(2)} \end{pmatrix} \dots \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(n_i)} \\ 0 & 1 & \dots & 0 & a_2^{(n_i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(n_i)} \end{pmatrix} = B \pmod{m},$$

where $(a_1^{(1)}, a_2^{(1)}, \dots, a_d^{(1)}), \dots, (a_1^{(n_i)}, a_2^{(n_i)}, \dots, a_d^{(n_i)})$ are sequences of integers which define W_i . From (8) we see that (9) implies

$$\frac{(\mu \times \delta_m)(T_m^{n_i}(W_i \times \{A\}) \cap \mathcal{M})}{(\mu \times \delta_m)(T_m^{n_i}(W_i \times \{A\}))} > 1 - (d+1)^{d+1} \varepsilon_i.$$

Since W_i is proper and \mathcal{M} is T_m -invariant, we conclude that

$$(X \times \{B\}) \cap \mathcal{M} = X \times \{B\} \pmod{0} \text{ } ((\mu \times \delta_m)\text{-mod } 0).$$

From Theorem 2, for any $C \in G(m)$ there exist J-P matrices A_1, \dots, A_s such that

$$C = BA_1 \dots A_s \pmod{m}$$

with

$$A_i = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & a_1^{(i)} \\ 0 & 1 & \dots & 0 & a_2^{(i)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & a_d^{(i)} \end{pmatrix} \text{ for } 1 \leq i \leq s.$$

Moreover we can choose A_1, \dots, A_s so that the corresponding cylinder set $X_{\mathbf{a}^{(1)} \dots \mathbf{a}^{(s)}}$

with $\mathbf{a}^{(i)} = \begin{pmatrix} a_1^{(i)} \\ \vdots \\ a_d^{(i)} \end{pmatrix}, 1 \leq i \leq s$, is proper. This means

$$T_m^s(X \times \{B\}) \supset X \times \{C\}$$

and so $\mathcal{M} = X \times G(m) \pmod{0}$. Thus we get the assertion of the theorem. \square

3.2 Proofs

We are now able to give proofs of Proposition 1, Theorem 1, and Corollary 1.

Proof of Proposition 1. Let us recall that C_m denotes the cardinality of $G(m)$. From Theorem 3 and the individual ergodic theorem, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : T_m^n(\mathbf{x}, B) \in X \times \{A\}\} = \frac{1}{C_m}$$

for $(\mu \times \delta_m)$ -a.e. (\mathbf{x}, B) . In particular, it holds for (\mathbf{x}, I_{d+1}) for μ -a.e. \mathbf{x} . Since $T_m^n(\mathbf{x}, I_{d+1}) = (T^n \mathbf{x}, Q^{(n)})$, we get the assertion. \square

Proof of Theorem 1.

For any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$, we denote by $N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})}$ the number of elements in $G(m)$ such that the $(d+1)$ th row is $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$. We will show that

$$N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})} = C_m \cdot m^d, \quad (10)$$

where C_m denotes the cardinality of $SL(d, \mathbb{Z}_m)$ or $SL_{\pm}(d, \mathbb{Z}_m)$ if d is even or odd, respectively. It is easy to see that

$$N_{(0, \dots, 0, 1)} = C_m \cdot m^d. \quad (11)$$

From Lemma 1, we note that there always exists $D \in G(m)$ such that the $(d+1)$ th row is $(\alpha_1, \alpha_2, \dots, \alpha_{d+1})$ for any $(\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \in \tilde{\mathbb{Z}}_m^{d+1}$.

For any matrix E of the form

$$\begin{pmatrix} & * & & \\ & & & \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

ED is of the form

$$\begin{pmatrix} & * & & \\ & & & \\ \alpha_1 & \dots & \alpha_d & \alpha_{d+1} \end{pmatrix}.$$

This implies

$$N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})} \geq N_{(0, \dots, 0, 1)}.$$

On the other hand, for any matrix D' of the form

$$\begin{pmatrix} & * & & \\ & & & \\ \alpha_1 & \dots & \alpha_d & \alpha_{d+1} \end{pmatrix},$$

$D' \cdot D^{-1}$ is of the form

$$\begin{pmatrix} & * & & \\ & & & \\ 0 & \dots & 0 & 1 \end{pmatrix},$$

which implies

$$N_{(\alpha_1, \alpha_2, \dots, \alpha_{d+1})} \leq N_{(0, \dots, 0, 1)}.$$

Thus we have (10).

From Proposition 1 together with (10), we have

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (q^{(n-d)}, q^{(n-d+1)}, \dots, q^{(n)}) \equiv (\alpha_1, \alpha_2, \dots, \alpha_{d+1}) \pmod{m}\}}{N} \\ = \frac{C_m \cdot m^d}{C_m} = \frac{1}{c_m} \quad \text{for } \mu\text{-a.e. } \mathbf{x}. \end{aligned}$$

Indeed one easily checks according to (3) and (4) that $\frac{\mathcal{C}_m \cdot m^d}{\mathcal{C}_m} = \frac{1}{c_m}$ holds. Since μ is equivalent to the Lebesgue measure, this holds for a.e. \mathbf{x} with respect to the Lebesgue measure. If we consider the $(d+1)$ th column, then the same argument shows the other equality. This completes the proof of Theorem 1. \square

Proof of Corollary 1. For a given $a \in \mathbb{Z}_m$, let $\Gamma_a(m)$ denote the cardinality of the subset of $G(m)$ of matrices whose $(d+1, d+1)$ -entry is equal to a .

Let us first assume that a and m are coprime. We then deduce from (11) that

$$\begin{aligned}\Gamma_a(m) &= \sum_{(\alpha_1, \dots, \alpha_d): \langle \alpha_1, \dots, \alpha_d, a \rangle = \mathbb{Z}_m} N(\alpha_1, \dots, \alpha_d, a) \\ &= \sum_{(\alpha_1, \dots, \alpha_d)} N(\alpha_1, \dots, \alpha_d, a) \\ &= m^{2d} \cdot \mathcal{C}_m.\end{aligned}$$

Let us assume now that m is a power of a prime divisor p of a . One has $\gcd(\alpha_1, \dots, \alpha_d, a, m) = 1$ if and only if $\gcd(\alpha_1, \dots, \alpha_d, m) = 1$. Hence

$$\begin{aligned}\Gamma_a(m) &= \sum_{(\alpha_1, \dots, \alpha_d): \langle \alpha_1, \dots, \alpha_d, a \rangle = \mathbb{Z}_m} N(\alpha_1, \dots, \alpha_d, a) \\ &= \sum_{(\alpha_1, \dots, \alpha_d): \gcd(\alpha_1, \dots, \alpha_d, m) = 1} N(\alpha_1, \dots, \alpha_d, a) \\ &= m^d \cdot \mathcal{C}_m \cdot \varphi_d(m).\end{aligned}$$

It easily deduced from the Chinese remainder lemma that the functions $m \mapsto \Gamma_a(m)$, $m \mapsto \varphi_d(m)$, and $m \mapsto \mathcal{C}_m$ are arithmetic multiplicative function. Hence one checks that

$$\Gamma_a(m) = \frac{\mathcal{C}_m \cdot \varphi_d(\gcd(m, a)) \cdot m^{2d}}{\gcd(m, a)^d}.$$

It remains now to apply Theorem 1 to obtain the result, that is, for a.e. $\mathbf{x} \in X$

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{1}{N} \#\{1 \leq n \leq N : q^{(n)} \equiv a \pmod{m}\} &= \frac{\Gamma_a(m)}{\mathcal{C}_m} \\ &= \frac{\mathcal{C}_m \cdot m^{2d} \cdot \varphi_d(\gcd(a, m))}{\mathcal{C}_m \cdot \gcd(a, m)^d} \\ &= \frac{m^d \cdot \varphi_d(\gcd(a, m))}{\gcd(a, m) \cdot \varphi_{d+1}(m)} \quad \square\end{aligned}$$

Remark. Let \mathbb{F}_q denote the finite field of cardinality q and let $\mathbb{F}_q[X]$ be the set of polynomials with \mathbb{F}_q -coefficients. We denote by \mathbb{L} the set of formal Laurent power series with negative degree. Since \mathbb{L} is a compact Abelian group, there exists a unique normalized Haar measure m . We can define the Jacobi-Perron algorithm on \mathbb{L}^d for any $d \geq 1$. In this case, m^d is invariant under this algorithm.

Suppose that $\left(\frac{P_1^{(n)}}{Q^{(n)}}, \dots, \frac{P_d^{(n)}}{Q^{(n)}}\right)$ is the n -th convergent of $(f_1, \dots, f_d) \in \mathbb{L}^d$. For any $R \in \mathbb{F}_q[X]$, it is possible to prove the following : for any $A_1, \dots, A_d, A_{d+1} \in \mathbb{F}_q[X]$ such that $A_1, \dots, A_d, A_{d+1}, R$ are relatively prime,

$$\begin{aligned}\lim_{N \rightarrow \infty} \frac{\#\{1 \leq n \leq N : (P_1^{(n)}, \dots, P_d^{(n)}, Q^{(n)}) \equiv (A_1, \dots, A_d, A_{d+1}) \pmod{R}\}}{N} \\ = c_R \quad \text{for } m^d\text{-a.e. } (f_1, \dots, f_d) \in \mathbb{L}^d,\end{aligned}$$

where c_R is a constant depending only on d and R . The proof is essentially the same as that of Theorem 1 of this paper. We refer to K. Inoue and H. Nakada [5] for the study of the rates of convergence for Jacobi-Perron algorithm over \mathbb{L}^d and to R. Natsui [8] for the \mathbb{L} -version of Jager-Liardet's result in the case of continued fractions.

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