Generating discrete planes with substitutions

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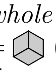
Abstract. Given a finite set S of unimodular Pisot substitutions, we provide a method for characterizing the infinite sequences over S that allow to generate a full discrete plane when, starting from a finite seed, we iterate the multidimensional dual substitutions associated with S. We apply our results to study the substitutions associated with the Brun multidimensional continued fraction algorithm.

1 Introduction

The study of Pisot substitutions has been initiated by Rauzy [18] and has led to many developments in several domains, including combinatorics on words, symbolic dynamics, fractal topology and number theory [12,6].

Dual substitutions, introduced by Arnoux and Ito [1] have proven to be a very powerful combinatorial tool in several contexts (see, e.g., [6]). Intuitively, a 3-letter substitution σ acts on broken lines made of translated unit vectors in $\mathbb{Z}^3$, and its dual $\mathbf{E}_1^\star(\sigma)$ acts on 2-dimensional unit faces in $\mathbb{Z}^3$; see Definition 2.4.

A striking fact is that the image by a dual substitution of a discrete plane remains a discrete plane. This link between substitutions and discrete planes leads us to our main concern: given a *finite* patch $\mathcal{V}$ (a *seed*) of a discrete plane, iterating dual substitutions starting from $\mathcal{V}$ yields finite patches of increasing size. When does this procedure generate a *whole* discrete plane?

A finite seed of particular interest is $\mathcal{U} :=$  (the largest pattern included in every discrete plane, see Remark 2.6). The above question with $\mathcal{V} = \mathcal{U}$ has many equivalent formulations and implications, which constitutes our main motivation for this work, as described in Section 1.1.

Our results Let $\sigma_1^{\mathsf{Brun}}, \sigma_2^{\mathsf{Brun}}, \sigma_3^{\mathsf{Brun}}$ be the substitutions associated with the Brun continued fraction algorithm. In this paper, we obtain in Theorem 5.2:

- The existence of a *finite* seed $\mathcal{V}$ from which iterating $\mathbf{E}_1^\star(\sigma_{i_n}^{\mathsf{Brun}})$ generates a whole discrete plane, for every Brun-admissible sequence $(i_n)_{n\geqslant 0}$.
- A characterization the sequences $(i_n)_{n\geqslant 0}$ for which the seed $\mathcal{V} = \mathcal{U}$ is not sufficient to generate a whole discrete plane when iterating the $\mathbf{E}_1^\star(\sigma_{i_n}^{\mathsf{Brun}})$.

The methods we use are generic and allow the study of other families than the Brun substitutions. Note that the above properties are easy to check for a given *single* substitution σ [6]. Our main contribution is to extend such results to *infinite* families of substitutions obtained as arbitrary products from a finite set.

1.1 Motivation and applications

The work presented in this article is motivated by the following consequences (and equivalent formulations) of our results. Establishing these links in detail will be the subject of a forthcoming article.

Multidimensional Sturmian sequences One-dimensional Sturmian sequences can be defined as the coding (in $\{1,2\}^{\mathbb{Z}}$) of the discretization of a line in the plane. They can also be described as the infinite sequences generated by iterating the substitutions $1 \to 1, 2 \to 21$ and $1 \to 12, 2 \to 2$ (see [12, Chapter 6]). We generalize this result to two-dimensional Sturmian sequences, that is, discrete planes coded in $\{1,2,3\}^{\mathbb{Z}^2}$. The link between a discrete plane of normal vector $\mathbf{v}$ and substitutions can be made thanks to the multidimensional continued fraction expansion of $\mathbf{v}$ (as it is done for Sturmian sequences with the classical continued fraction algorithm). We use the Brun algorithm for this purpose.

Symbolic dynamics The subshift associated with a unimodular Pisot substitution σ has pure discrete spectrum if and only if the patches generated by iterating the dual substitution $\mathbf{E}_1^\star(\sigma)$ on $\mathcal{U}$ cover balls of arbitrarily large radius [16,6]. The Pisot conjecture states that this is always the case. Our present results allow us to prove this property for some infinite families defined by finite products of substitutions over a finite set. (It remains to prove that, in the case where a whole plane is not generated from $\mathcal{U}$, some arbitrarily large balls are still covered somewhere.)

Topology of Rauzy fractals The periodic sequences of substitutions that fail to generate a whole plane correspond precisely to the finite products of substitutions whose Rauzy fractal does not contain $\mathbf{0}$ as an interior point (see [5,20]). Moreover, our results imply that the Rauzy fractal associated with a finite product of substitutions is always connected.

Number theory Generating arbitrarily large patches of a discrete plane allows us to approximate its normal vector. We hence obtain proofs of convergence for the associated multidimensional continued fraction algorithms if the substitutions have been chosen accordingly, see Corollary 5.3.

Generating a whole discrete plane can be seen as an analog of the *finiteness property* in β-numeration (see the *extended (F) property* in [5]). Such properties have already been proved for some infinite families of algebraic numbers (see for example [13]).

We are also able to associate fractal tiles with every real cubic number field, thanks to a result by Paysant-Le-Roux and Dubois [17] and a similar study of the Jacobi-Perron algorithm.

S-adic systems The study of the dynamical systems and the fractal tiles associated with arbitrary infinite products (S-adic sequences) is still at its beginnings. It is for example not completely understood in what cases an S-adic sequence can be associated with a fractal tile. Our tools provide a starting point for the study of such systems, as initiated in [7].

1.2 Methods

Ito and Ohtsuki [15] initiated the study of the generation of discrete planes with substitutions, while investigating properties of the Jacobi-Perron algorithm. Their main argument is to prove that some topological annuli are preserved under the image by a substitution, and that these annuli grow to a whole discrete plane when the substitutions are iterated.

We use the same approach in Section 4, but with additional combinatorial restrictions (*strong* coverings) introduced in Section 2.3 that are crucial in order to prove the *annulus property* (the fact that the image of an annulus remains an annulus). We introduce a combinatorial criterion, *Property A* (Definition 4.1), which allows for more systematic proofs.

The algorithmic methods developed in Section 3 (the *generation graph*, Definition 3.1) provide powerful tools to manage the complicated behaviour of the growth of the patterns, without having to deal with numerous cases by hand.

The main steps of our argument are:

1. Choose good substitutions and good patterns for strong coverings. Describe all the possible minimal strongly-covered annuli. (Section 2.2.)
2. Construct generation graphs, both to prove that a first annulus is generated, but also to characterize the sequences that fail to generate whole discrete planes. (Section 3.)
3. Prove that annuli are preserved by substitutions. (Section 4.)

Many of the computational tasks performed in Sections 3 and 4 have been performed using the Sage mathematics software.

1.3 Related works

The study of the Jacobi-Perron substitutions in this context was initiated in [15]. Arnoux-Rauzy substitutions have been treated in [3] and [4] (for them the seed $\mathcal{U}$ is always enough). The Modified Jacobi-Perron algorithm is studied in [14] and the substitutions associated with the ordered fully subtractive algorithm have been used in [8] for other (discrete geometrical) purposes.

Our work focuses on the case where the coordinates of the normal vector of the discrete plane are linearly independent over $\mathbb{Q}$. The case of rational vectors has been treated by Fernique [11].

Let us note also that depending on the substitutions studied, our techniques can fail to work if the topological features of the generated patterns are too complicated (for example if many holes appear). This is the case for example with the Selmer algorithm (which is described in [19]).

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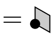
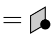
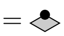
2 Preliminaries

2.1 Discrete planes and substitutions

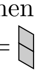
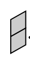
Before defining discrete planes we introduce *faces* $[\mathbf{x}, i]^\star$, which are defined by

$$[\mathbf{x}, 1]^\star = \{\mathbf{x} + \lambda\mathbf{e}_2 + \mu\mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 
$$[\mathbf{x}, 2]^\star = \{\mathbf{x} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_3 : \lambda, \mu \in [0, 1]\} =$$ 
$$[\mathbf{x}, 3]^\star = \{\mathbf{x} + \lambda\mathbf{e}_1 + \mu\mathbf{e}_2 : \lambda, \mu \in [0, 1]\} =$$ 

where $i \in \{1, 2, 3\}$ is the *type* of $[\mathbf{x}, i]^\star$, and $\mathbf{x} \in \mathbb{Z}^3$ is the *vector* of $[\mathbf{x}, i]^\star$.

Definition 2.1 (Discrete plane). Let $\mathbf{v} \in \mathbb{R}^3_{>0}$. The *discrete plane* $\mathit{\Gamma}_\mathbf{v}$ *of normal vector* $\mathbf{v}$ is the union of faces $[\mathbf{x}, i]^\star$ satisfying $0 \leqslant \langle \mathbf{x}, \mathbf{v} \rangle < \langle \mathbf{e}_i, \mathbf{v} \rangle$.

More intuitively, $\mathit{\Gamma}_\mathbf{v}$ can also be seen as the boundary of the union of the unit cubes with integer coordinates that intersect the lower half-space $\{\mathbf{x} \in \mathbb{R}^3 : \langle \mathbf{x}, \mathbf{v} \rangle < 0\}$.

Remark 2.2. We will often use the arithmetic restrictions of Definition 2.1 in order to simplify the combinatorics of the patterns that appear in a discrete plane of normal vector $\mathbf{v} = (v_1, v_2, v_3)$. For example, if $v_1 \leqslant v_3$ and $v_2 \leqslant v_3$, then $\mathit{\Gamma}_\mathbf{v}$ cannot contain any translate of the two-face pattern $[\mathbf{0}, 1]^\star \cup [(0, 1, 0), 1]^\star =$  or $[\mathbf{0}, 2]^\star \cup [(0, 0, 1), 2]^\star =$ . If moreover $v_1 \leqslant v_2$, then the pattern $[\mathbf{0}, 1]^\star \cup [(0, 1, 0), 1]^\star =$  also never appears.

Definition 2.3 (Substitution). Let $\mathcal{A} = \{1, \dots, n\}$ be a finite set of symbols. A *substitution* is a non-erasing morphism of the free monoid $\mathcal{A}^\star$, *i.e.*, a function $\sigma : \mathcal{A}^\star \to \mathcal{A}^\star$ such that $\sigma(uv) = \sigma(u)\sigma(v)$ for all words $u, v \in \mathcal{A}^\star$, and such that $\sigma(a)$ is non-empty for every $a \in \mathcal{A}$.

The *incidence matrix* $\mathbf{M}_\sigma$ of σ is the matrix of size $n \times n$ defined by $\mathbf{M}_\sigma = (m_{ij})$, where $m_{i,j}$ is the number of occurrences of the letter i in $\sigma(j)$. A substitution σ is *unimodular* if $\det \mathbf{M}_\sigma = \pm 1$.

Definition 2.4 (Dual substitution). Let σ be a unimodular substitution. The *dual substitution* $\mathbf{E}_1^\star(\sigma)$ is defined by

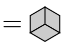
$$\mathbf{E}_1^\star(\sigma)([\mathbf{x}, i]^\star) \ = \bigcup_{(p,j,s) \in \mathcal{A}^\star \times \mathcal{A} \times \mathcal{A}^\star \ : \ \sigma(j) = pis} [\mathbf{M}_\sigma^{-1}(\mathbf{x} + \ell(s)), j]^\star,$$

where $\ell : w \mapsto (|w|_1, \dots, |w|_n) \in \mathbb{Z}^3$ is the *abelianization map* and $|w|_i$ denotes the number of occurrences of i in w. We extend the above definition to any union of faces: $\mathbf{E}_1^\star(\sigma)(P_1 \cup P_2) = \mathbf{E}_1^\star(\sigma)(P_1) \cup \mathbf{E}_1^\star(\sigma)(P_2)$.

Note that for every face $[\mathbf{x}, i]^\star$ we have $\mathbf{E}_1^\star(\sigma)([\mathbf{x}, i]^\star) = \mathbf{M}_\sigma^{-1}\mathbf{x} + \mathbf{E}_1^\star([\mathbf{0}, i]^\star)$, which implies the linearity $\mathbf{E}_1^\star(\sigma)$. We also have $\mathbf{E}_1^\star(\sigma \circ \sigma') = \mathbf{E}_1^\star(\sigma') \circ \mathbf{E}_1^\star(\sigma)$ for every unimodular σ and σ' [1]. The next proposition establishes a fundamental link between discrete planes and $\mathbf{E}_1^\star$ maps.

Proposition 2.5 ([1,10]). *Let $\Gamma_{\mathbf{v}}$ be a discrete plane and σ be a unimodular substitution. We have:*

1. *$\mathbf{E}_1^\star(\sigma)(\Gamma_{\mathbf{v}})$ is the discrete plane $\Gamma_{{}^t\mathbf{M}_\sigma \mathbf{v}}$.*
2. *If $f, g \in \Gamma_{\mathbf{v}}$ are distinct, then $\mathbf{E}_1^\star(\sigma)(f) \cap \mathbf{E}_1^\star(\sigma)(g)$ does not contain any face.*

Remark 2.6. The pattern $\mathcal{U} = [\mathbf{0}, 1]^\star \cup [\mathbf{0}, 2]^\star \cup [\mathbf{0}, 3]^\star =$  is included in every discrete plane because the coordinates of the normal vector of a discrete plane are assumed to be positive.

2.2 The Brun algorithm

Let $\mathbf{v} \in \mathbb{R}^3_{>0}$ such that $\mathbf{v} = (v_1, v_2, v_3)$ and $v_1 \leqslant v_2 \leqslant v_3$. The algorithm of Brun [9] is one of the possible natural generalizations of Euclid's algorithm: subtract the second largest component of $\mathbf{v}$ to the largest, and iterate. Here we reorder the coordinates at each step, so that the condition $v_1 \leqslant v_2 \leqslant v_3$ always holds. More formally:

$$\mathbf{v} \ \mapsto \ \begin{cases} (v_1,\ v_2,\ v_3 - v_2) & \text{if } v_1 \leqslant v_2 \leqslant v_3 - v_2 \\ (v_1,\ v_3 - v_2,\ v_2) & \text{if } v_1 \leqslant v_3 - v_2 \leqslant v_2 \\ (v_3 - v_2,\ v_1,\ v_2) & \text{if } v_3 - v_2 \leqslant v_1 \leqslant v_2. \end{cases}$$

Iterating this map yields an infinite sequence of vectors $\mathbf{v}_0 = \mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \ldots$ and the algorithm can be rewritten in matrix form: $\mathbf{v}_n = \mathbf{M}_{i_n} \mathbf{v}_{n-1}$ for every $n \geqslant 1$, where

$$\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad \mathbf{M}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad \mathbf{M}_3 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

and $i_n \in \{1, 2, 3\}$. This allows us to define the *Brun expansion* of $\mathbf{v}$ as the infinite sequence $(i_n)_{n \geqslant 1}$ obtained above. It enjoys the following nice property.

Proposition 2.7 ([9]). *The Brun expansion $(i_n)_{n\geqslant 1}$ of $\mathbf{v} \in \mathbb{R}^3_{>0}$ contains infinitely many 3's if and only if $\mathbf{v}$ is totally irrational.*

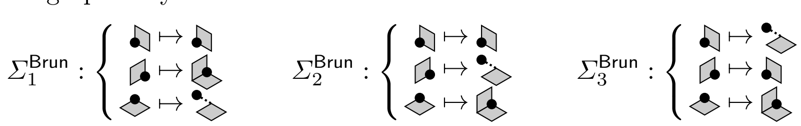
We now define some substitutions associated with the Brun algorithm.

$$\sigma_1^{\mathsf{Brun}} : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 2 \\ 3 \mapsto 32 \end{cases} \qquad \sigma_2^{\mathsf{Brun}} : \begin{cases} 1 \mapsto 1 \\ 2 \mapsto 3 \\ 3 \mapsto 23 \end{cases} \qquad \sigma_3^{\mathsf{Brun}} : \begin{cases} 1 \mapsto 2 \\ 2 \mapsto 3 \\ 3 \mapsto 13 \end{cases}$$

and $\Sigma_i^{\mathsf{Brun}} = \mathbf{E}_1^\star(\sigma_i^{\mathsf{Brun}})$. The maps $\Sigma_1^{\mathsf{Brun}}, \Sigma_2^{\mathsf{Brun}}, \Sigma_3^{\mathsf{Brun}}$ are respectively given by

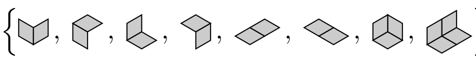
$$\begin{array}{lll} [\mathbf{0},1]^\star \mapsto [\mathbf{0},1]^\star & [\mathbf{0},1]^\star \mapsto [\mathbf{0},1]^\star & [\mathbf{0},1]^\star \mapsto [(0,1,0),3]^\star \\ [\mathbf{0},2]^\star \mapsto [\mathbf{0},2]^\star \cup [\mathbf{0},3]^\star & [\mathbf{0},2]^\star \mapsto [(0,1,0),3]^\star & [\mathbf{0},2]^\star \mapsto [\mathbf{0},1]^\star \\ [\mathbf{0},3]^\star \mapsto [(0,1,0),3]^\star & [\mathbf{0},3]^\star \mapsto [\mathbf{0},2]^\star \cup [\mathbf{0},3]^\star & [\mathbf{0},3]^\star \mapsto [\mathbf{0},2]^\star \cup [\mathbf{0},3]^\star \end{array},$$

or more graphically



2.3 Coverings and strong coverings

We call a *pattern* any finite union of faces. In the definitions below, $\mathcal{L}$ will always denote a set of patterns which is closed by translation of $\mathbb{Z}^3$, so we will define such sets by giving only one element of each translation class. The following set of patterns will be used throughout this article:

$\mathcal{L}^{\mathsf{Brun}} = \{$  $\}$.

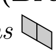
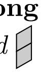
We now introduce $\mathcal{L}$-coverings and strong $\mathcal{L}$-coverings, which are the combinatorial tools we will use in order to prove the annulus property in Section 4.

Definition 2.8 ($\mathcal{L}$-covering). Let $\mathcal{L}$ be a set of patterns. A pattern P is $\mathcal{L}$-*covered* if for all faces $e, f \in P$, there exist patterns $Q_1, \ldots, Q_n \in \mathcal{L}$ such that

1. $e \in Q_1$ and $f \in Q_n$;
2. $Q_k \cap Q_{k+1}$ contains at least one face, for all $k \in \{1, \ldots, n-1\}$;
3. $Q_k \subseteq P$ for all $k \in \{1, \ldots, n\}$.

Proposition 2.9 ([15]). *Let P be an $\mathcal{L}$-covered pattern, Σ be a dual substitution and $\mathcal{L}$ be a set of patterns such that $\Sigma(Q)$ is $\mathcal{L}$-covered for every $Q \in \mathcal{L}$. Then $\Sigma(P)$ is $\mathcal{L}$-covered.*

Definition 2.10 (Strong $\mathcal{L}$-covering). A pattern P is *strongly $\mathcal{L}$-covered* if P is $\mathcal{L}$-covered and if for every pattern $X \subseteq P$ that is edge-connected and consists of two faces, there exists a pattern $Y \in \mathcal{L}$ such that $X \subseteq Y \subseteq P$.

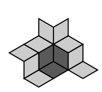
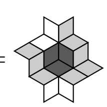
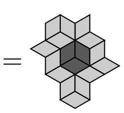
Proposition 2.11 (Brun strong covering). *Let P be an $\mathcal{L}^{\mathsf{Brun}}$-covered pattern such that the patterns* ,  *and*  *do not occur in P. Then $\Sigma_i^{\mathsf{Brun}}(P)$ is strongly $\mathcal{L}^{\mathsf{Brun}}$-covered for $i \in \{1, 2, 3\}$.*

Proof (Sketch). First, $\Sigma_i^{\mathsf{Brun}}(P)$ is $\mathcal{L}^{\mathsf{Brun}}$-covered thanks to Proposition 2.9, because $\Sigma_i^{\mathsf{Brun}}(Q)$ is $\mathcal{L}^{\mathsf{Brun}}$-covered for every $Q \in \mathcal{L}^{\mathsf{Brun}}$ (there are 24 patterns to check). To prove that $\Sigma_i^{\mathsf{Brun}}(P)$ is *strongly* $\mathcal{L}^{\mathsf{Brun}}$-covered, we can enumerate the preimages by Σ_i of all the two-face connected patterns X to check that there is always a suitable $Y \in \mathcal{L}^{\mathsf{Brun}}$ that satisfies the requirements of Definition 2.10. $\square$

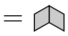
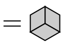
2.4 Minimal annuli

Definition 2.12 ($\mathcal{L}$-annulus). An $\mathcal{L}$-*annulus* of a pattern P is a pattern A such that A is strongly $\mathcal{L}$-covered and $P \cap \partial(P \cup A) = \varnothing$.

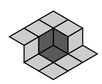
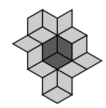
Example 2.13. Let A_1, A_2 and A_3 be defined by

$A_1 \cup \mathcal{U} =$  $A_2 \cup \mathcal{U} =$  $A_3 \cup \mathcal{U} =$ ,

where $\mathcal{U}$ is shown in dark gray and the other faces are the A_i. We have:

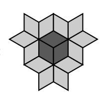
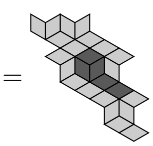
- A_1 is not an annulus of $\mathcal{U}$ because it does not satisfy $\mathcal{U} \cap \partial(\mathcal{U} \cup A_1) = \varnothing$ ($\mathcal{U}$ is not well surrounded).
- A_2 is not an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{U}$ because of the two-face pattern $X =$  depicted in white: the only pattern in $\mathcal{L}^{\mathsf{Brun}}$ that contains X is $Y =$ , but it cannot be included in A_2 so A_2 is not strongly $\mathcal{L}^{\mathsf{Brun}}$-covered.
- A_3 is an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{U}$.

Proposition 2.14 (Brun minimal annuli). *Let A be an $\mathcal{L}^{\mathsf{Brun}}$-annulus of $\mathcal{U}$ that is included in a discrete plane of normal vector $\mathbf{v}$ with $v_1 < v_2 < v_3$. Then A contains one of the following two $\mathcal{L}^{\mathsf{Brun}}$-annuli $\mathcal{A}_1^{\mathsf{Brun}}$ or $\mathcal{A}_2^{\mathsf{Brun}}$ (shown in light gray) of $\mathcal{U}$ (shown in dark gray):*

$\mathcal{A}_1^{\mathsf{Brun}} \cup \mathcal{U} =$  $\qquad \mathcal{A}_2^{\mathsf{Brun}} \cup \mathcal{U} =$  .

Proof (Sketch). This proposition can be proved by enumerating all the possible surroundings of $\mathcal{U}$ of "thickness 1", and by doing a case analysis on the problematic patterns that appear using the definition of $\mathcal{L}^{\mathsf{Brun}}$. □

Example 2.15. Let P be a pattern equal to the union of $[\mathbf{0}, 3]^\star \cup [(1, 0, -1), 2]^\star \cup [(0, 1, -1), 1]^\star$ (in dark gray) and some other faces in light gray.

$P =$  $\qquad \Sigma_1^{\mathsf{Brun}} \Sigma_2^{\mathsf{Brun}}(P) =$ 

The images of the annulus in light gray fail to be annuli. However, the annulus in P is not strongly $\mathcal{L}^{\mathsf{Brun}}$-covered, which shows the need for *strong* coverings if we want the image of an annulus to remain an annulus.

The substitutions above are chosen in such a way that $\mathbf{M}_i = {}^{\mathbf{t}}\mathbf{M}^{-1}_{\sigma_i^{\mathsf{Brun}}}$ for $i \in \{1, 2, 3\}$, which allows us to define the sequence of pattern we will use to generate the discrete plane $\Gamma_{\mathbf{v}}$, as described by the proposition below.

Proposition 2.16. *Let $\mathbf{v} \in \mathbb{R}^3_{>0}$ and $(i_n)_{n \geqslant 1}$ be its Brun expansion. We have $\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{V}) \subseteq \Gamma_{\mathbf{v}}$ for all $n \geqslant 1$, where $\mathcal{V} = \mathcal{U}$, or $\mathcal{V} = \mathcal{U} \cup \mathcal{A}_1^{\mathsf{Brun}}$ (if $i_1 \in \{1, 2\}$), or $\mathcal{V} = \mathcal{U} \cup \mathcal{A}_2^{\mathsf{Brun}}$ (if $i_1 = 3$).*

Proof. Since we have $\mathbf{v} = \mathbf{M}_{i_1}^{-1} \cdots \mathbf{M}_{i_n}^{-1} \mathbf{v}_n$, and ${}^{\mathbf{t}}\mathbf{M}_{\sigma_{i_n}^{\mathsf{Brun}}} = \mathbf{M}_{i_n}^{-1}$, and since $\mathcal{V} \subseteq \Gamma_{\mathbf{v}_n}$, it follows from Proposition 2.5 that $\Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{V}) \subseteq \Sigma_{i_1} \cdots \Sigma_{i_n}(\Gamma_{\mathbf{v}_n}) = \Gamma_{{}^{\mathbf{t}}\mathbf{M}_{\sigma_{i_1}^{\mathsf{Brun}}} \cdots {}^{\mathbf{t}}\mathbf{M}_{\sigma_{i_n}^{\mathsf{Brun}}} \mathbf{v}_n} = \Gamma_{\mathbf{v}}$. □

3 Generation graphs

We fix the following notation for this section:

- $\Sigma_1, \dots, \Sigma_k$ are dual substitutions,
- $\mathcal{X}$ and $\mathcal{Y}$ are finite sets of faces,
- $\mathcal{F}$ is an infinite family of faces.

We want to characterize the sequences $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$ and the faces $f \in \mathcal{Y}$ such that f *cannot* be reached by iterating $\Sigma_{i_1}, \dots, \Sigma_{i_n}$ starting from the "seed" $\mathcal{X}$. Our approach below is to recursively track all the possible preimages of the faces in $\mathcal{Y}$, by constructing a *generation graph* providing us with the desired characterization. The set $\mathcal{F}$ is used as a filter, in order to make the generation graph as simple as possible by eliminating some useless faces.

Definition 3.1. The *generation graph* is defined by $\mathcal{G} = \bigcup_{n\in\mathbb{N}} \mathcal{G}_n$ (an increasing union which is not always finite), where $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is the sequence of directed graphs defined by induction as follows.

1. *Initialization.* $\mathcal{G}_0$ has no edges and its set of vertices is $\mathcal{Y}$.
2. *Iteration.* Suppose that $\mathcal{G}_n$ is constructed for some $n \geqslant 0$. Start with $\mathcal{G}_{n+1}$ having the same vertices and edges as $\mathcal{G}_n$. Then, for each vertex f of $\mathcal{G}_n$, for each $i \in \{1, \dots, k\}$ and for each $g \in \mathcal{F}$ such that $f \in \Sigma_i(g)$, add the vertex g and the edge $g \xrightarrow{i} f$ to $\mathcal{G}_{n+1}$.

Proposition 3.2. *Let $\mathcal{G}$ be the graph defined in Definition 3.1, let $f_0 \in \mathcal{G}$ be a face and let $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$. Consider the following two statements.*

1. $f_0 \notin \Sigma_{i_1} \cdots \Sigma_{i_n}(\mathcal{X})$.
2. *There exists a path $f_n \xrightarrow{i_n} \cdots \xrightarrow{i_2} f_1 \xrightarrow{i_1} f_0$ in $\mathcal{G}$ with $f_n \notin \mathcal{X}$.*

We have:

- *(i) (1) $\Rightarrow$ (2) if for every $f \in \mathcal{F}$ and every $i \in \{1, \dots, k\}$, there exists $g \in \mathcal{F}$ such that $f \in \Sigma_i(g)$.*
- *(ii) (2) $\Rightarrow$ (1) if $\mathcal{X} = \mathcal{U}$ and if every face of $\mathcal{F}$ belongs to a discrete plane.*

Proof. (i). The assumption in (i) implies that a path $f_n \xrightarrow{i_n} \cdots \xrightarrow{i_2} f_1 \xrightarrow{i_1} f_0$ must exist in $\mathcal{G}$. By (1), we cannot have have $f_n \in \mathcal{X}$, which proves the first implication.

(ii). Let $P = \Sigma_{i_1} \cdots \Sigma_{i_n}(f_n)$, $g \in \mathcal{X}$ and $Q = \Sigma_{i_1} \cdots \Sigma_{i_n}(g)$. By the assumption in (ii), f_n and g must belong to a common discrete plane because $g \in \mathcal{U}$, f_n belongs to a discrete plane and $\mathcal{U}$ is included in every discrete plane. Hence, Proposition 2.5 implies the patterns P and Q do not have any face in common. It follows that $f_0 \notin \Sigma_{i_1} \cdots \Sigma_{i_n}(g)$ for every $g \in \mathcal{X}$. □

Remark 3.3. Part (i) of Proposition 3.2 will be used to obtaine "positive" results, such as proving that a given seed *always* generate a full discrete plane (see Lemma 3.5, Proposition 5.1 and Theorem 5.2 (1)). Conversely, part (ii) will be used to characterize which sequences do *not* generate a full discrete plane (see Lemma 3.4 and Theorem 5.2 (2)).

Generation graphs for Brun We now consider substitutions $\Sigma_1^{\mathsf{Brun}}, \Sigma_2^{\mathsf{Brun}}, \Sigma_3^{\mathsf{Brun}}$. We will take the filter $\mathcal{F}^{\mathsf{Brun}}$ to be the set of all the faces f that belong to a discrete plane $\Gamma_{(v_1,v_2,v_3)}$ with $0 < v_1 < v_2 < v_3$. We use Definition 3.1 to compute the following graphs.

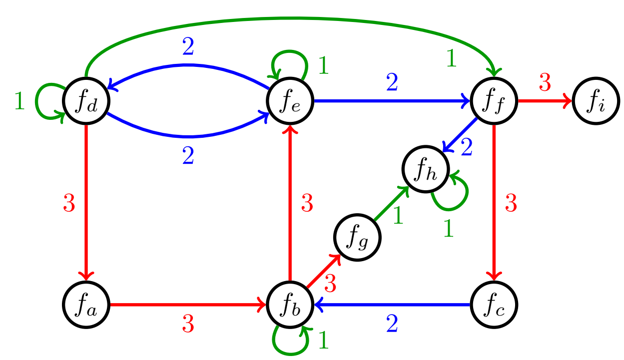
- The graph $\mathcal{G}^{\mathsf{Brun}}$ is obtained by starting with $\mathcal{Y} = \mathcal{A}_1^{\mathsf{Brun}} \cup \mathcal{A}_2^{\mathsf{Brun}}$. Its computation stops after two iterations of the algorithm. It has 19 vertices and 47 edges. We will use it below with $\mathcal{X} = \mathcal{U}$.
- The graph $\mathcal{H}^{\mathsf{Brun}}$ is obtained by starting with $\mathcal{Y}$ equal to the set of faces of all the possible minimal $\mathcal{L}^{\mathsf{Brun}}$-annuli of $\mathcal{A}_1^{\mathsf{Brun}}$ and $\mathcal{A}_2^{\mathsf{Brun}}$ (a total of 60 faces). Its computation stops after six iterations of the algorithm. It has 101 vertices and 240 edges. We will use it below with $\mathcal{X} = \mathcal{U} \cup \mathcal{A}_1^{\mathsf{Brun}}$ or $\mathcal{U} \cup \mathcal{A}_2^{\mathsf{Brun}}$.

Lemma 3.4. *The graph $\mathcal{G}^{\mathsf{Brun}}$ verifies (i) and (ii) of Proposition 3.2 with $\mathcal{X} = \mathcal{U}$.*

Lemma 3.5. *The graph $\mathcal{H}^{\mathsf{Brun}}$ verifies item (i) of Proposition 3.2, both with $\mathcal{X} = \mathcal{U} \cup \mathcal{A}_1^{\mathsf{Brun}}$ and $\mathcal{X} = \mathcal{U} \cup \mathcal{A}_2^{\mathsf{Brun}}$.*

Proof. For both lemmas we have to check that the assumption in (i) is satisfied. Let $f \in \mathcal{F}^{\mathsf{Brun}}$ and let $i \in \{1,2,3\}$. Because $f \in \mathcal{F}^{\mathsf{Brun}}$, there exists $\mathbf{v} = (v_1, v_2, v_3)$ such that $0 < v_1 < v_2 < v_3$ and $f \in \Gamma_{\mathbf{v}}$. By Proposition 2.5 and by definition of the Brun algorithm (Section 2.2), we have $\Gamma_{\mathbf{v}} = \Sigma_i^{\mathsf{Brun}}(\Gamma_{\mathbf{w}})$, where $\mathbf{w} = {}^{\mathbf{t}}\mathbf{M}_{\sigma_i^{\mathsf{Brun}}}^{-1}\mathbf{v}$. We have $0 < w_1 < w_2 < w_3$, so all the faces of $\Gamma_{\mathbf{w}}$ belong to $\mathcal{F}^{\mathsf{Brun}}$, so there exists a face $g \in \mathcal{F}^{\mathsf{Brun}}$ such that $f \in \Sigma_i(g)$ because $f \in \Gamma_{\mathbf{v}} = \Sigma_i(\Gamma_{\mathbf{w}})$. Finally (for Lemma 3.5 only), the assumptions required in (ii) trivially hold. □

In Section 5 we will need to consider only the infinite paths in $\mathcal{G}^{\mathsf{Brun}}$ and $\mathcal{H}^{\mathsf{Brun}}$ that contain infinitely many edges labelled by 3, and that avoid $\mathcal{X}$. In the case of $\mathcal{H}^{\mathsf{Brun}}$, there turns out to be no such infinite path, which is the key point to prove Proposition 5.1. However $\mathcal{G}^{\mathsf{Brun}}$ is more interesting, and removing all the vertices which are not contained in such a path yields the following graph.



The faces corresponding to the vertices of the graph are

$$
\begin{array}{lll}
f_a = [(1,1,-1),1]^\star & f_d = [(-1,1,0),2]^\star & f_g = [(-1,0,1),2]^\star \\
f_b = [(1,-1,1),3]^\star & f_e = [(-1,0,1),3]^\star & f_h = [(-1,-1,1),3]^\star \\
f_c = [(1,1,-1),2]^\star & f_f = [(-1,1,0),3]^\star & f_i = [(1,1,-1),3]^\star.
\end{array}
$$

4 The annulus property

Definition 4.1 (Property A). Let Σ be a dual substitution and let $\mathcal{L}$ be a set of edge-connected patterns. *Property A* holds for Σ with $\mathcal{L}$ if for all faces f, g, f_0, g_0 such that $f \in \Sigma(f_0)$, $g \in \Sigma(g_0)$, $f \cup g$ is connected and $f_0 \cup g_0$ is disconnected, there cannot exist a pattern P and an $\mathcal{L}$-annulus A of P which are included in a common discrete plane Γ such that $f_0 \in P$, $g_0 \notin A \cup P$, and $f_0 \cup g_0 \subseteq \Gamma$ and $f \cup g \subseteq \Sigma(\Gamma)$.

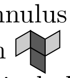
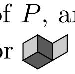
Proposition 4.2. *Let Σ be a dual substitution and $\mathcal{L}$ be a set of edge-connected patterns such that Property A holds for Σ with $\mathcal{L}$, and such that the image by Σ of every strongly $\mathcal{L}$-covered pattern is strongly $\mathcal{L}$-covered. Let P be a pattern and A be an $\mathcal{L}$-annulus of P, both included in a common discrete plane. Then $\Sigma(A)$ is an $\mathcal{L}$-annulus of $\Sigma(P)$.*

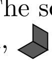
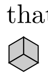
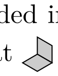
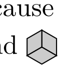
Proof. The pattern A is strongly $\mathcal{L}$-covered because it is an $\mathcal{L}$-annulus, so $\Sigma(A)$ is also strongly $\mathcal{L}$-covered, by assumption. It remains to show that $\Sigma(P) \cap \partial(\Sigma(P) \cup \Sigma(A)) = \varnothing$. Suppose the contrary. This means that there exist faces f, g, f_0, g_0 such that $f \in \Sigma(f_0)$, $g \in \Sigma(g_0)$, $f \cup g$ is connected, and $f_0 \cup g_0$ is disconnected (because $f_0 \in P$ and $g_0 \notin A \cup P$). These are precisely the conditions stated in Property A, so such a situation cannot occur and the proposition holds. □

Proposition 4.3 (Property A for Brun). *Property A holds for Brun substitutions with $\mathcal{L}^{\mathsf{Brun}}$, when restricted to planes $\Gamma_{(v_1,v_2,v_3)}$ with $v_1 \leqslant v_2 \leqslant v_3$.*

Proof. There are finitely many two-face connected patterns $f \cup g$, so we can enumerate all the faces f, g, f_0, g_0 that satisfy the three conditions of Definition 4.1, for Σ_1^{Brun}, Σ_2^{Brun} and Σ_3^{Brun}. It turns out that there are 9 such possibilities, where the corresponding values for $f_0 \cup g_0$ are shown in the table below.

Σ_1^{Brun}	Σ_2^{Brun}	Σ_3^{Brun}
$[\mathbf{0}, 2]^\star \cup [(0,1,0), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(1,0,-1), 3]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,1,-1), 3]^\star$
$[\mathbf{0}, 2]^\star \cup [(1,-1,0), 2]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,1,1), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,0,1), 2]^\star$
$[\mathbf{0}, 2]^\star \cup [(0,1,1), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(0,0,1), 1]^\star$	$[\mathbf{0}, 3]^\star \cup [(1,0,1), 2]^\star$

Let us treat the case $f_0 \cup g_0 = [\mathbf{0}, 2]^\star \cup [(1,-1,0), 2]^\star$. Suppose that there exists a pattern P and an $\mathcal{L}^{\mathsf{Brun}}$-annulus A of P that is included in a discrete plane such that $f_0 \in P$ and $g_0 \in A$. Because A is an annulus of P, any extension of $f_0 \cup g_0$ within a discrete plane must be of the form  or , where $f_0 \cup g_0$ is shown in light gray and the dark gray faces are included in A.

The first case cannot happen because it contains an occurrence of , which is forbidden since we are restricted to discrete planes with normal vector $\mathbf{v}$ satisfying $v_1 < v_2 < v_3$ (see Remark 2.2). The second case also cannot happen, because A is strongly $\mathcal{L}^{\mathsf{Brun}}$-covered. Indeed,  $\subseteq A$, so there must exist a translation of a pattern of $\mathcal{L}^{\mathsf{Brun}}$ that is included in A and that contains . The only such pattern in $\mathcal{L}^{\mathsf{Brun}}$ is  (note that  $\notin \mathcal{L}^{\mathsf{Brun}}$). This is impossible because then  and $f_0 \cup g_0$ overlap, which is a contradiction because $f_0, g_0 \notin A$ and  $\in A$. The same reasoning applies to the eight other cases. □

5 Main results

Let P be a pattern that contains $\mathcal{U}$. The *combinatorial radius* of P is the length of the shortest path of faces $f_1, \ldots, f_n$ in P such that: $f_1 \in \mathcal{U}$, the f_i and f_{i+1} are adjacent, and f_n shares an edge with the boundary of P.

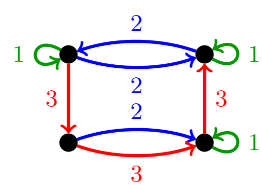
Proposition 5.1. *Let $(u_n = u_{n,1} \cdots u_{n,k_n})_{n \geqslant 1}$ be an infinite sequence of words in $\{1,2,3\}^\star$ such that the number of 3's in the u_n is strictly increasing. Let $\mathcal{V}$ be equal either $\mathcal{U} \cup \mathcal{A}_1^{\mathsf{Brun}}$ or to $\mathcal{U} \cup \mathcal{A}_2^{\mathsf{Brun}}$. Then the pattern $\Sigma_{u_{n,1}}^{\mathsf{Brun}} \cdots \Sigma_{u_{n,k_n}}^{\mathsf{Brun}}(\mathcal{V})$ has arbitrarily large combinatorial radius when $n \to \infty$.*

Proof. Let R be a positive integer (an arbitrary radius that we want to bound above). We can algorithmically check that, in the graph $\mathcal{H}^{\mathsf{Brun}}$ described in Section 3, there are no infinite paths containing infinitely many 3's that avoid $\mathcal{A}_1^{\mathsf{Brun}} \cup \mathcal{A}_2^{\mathsf{Brun}} \cup \mathcal{U}$. Hence by Lemma 3.5 there exists an integer N such that R annuli are generated from $\mathcal{V}$ by $\Sigma_{u_{N,1}}^{\mathsf{Brun}} \cdots \Sigma_{u_{N,k_N}}^{\mathsf{Brun}}$. By Propositions 4.2 and 4.3, these annuli remain annuli, so the combinatorial radius cannot be less that R. ☐

Theorem 5.2. *Let $\mathbf{v} \in \mathbb{R}^3_{>0}$ be an ordered totally irrational vector and let $(i_n) \in \{1,2,3\}^{\mathbb{N}}$ be its Brun expansion. We have:*

1. $\bigcup_{n \geqslant 1} \Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{V}) = \Gamma_{\mathbf{v}}$, *where* $\mathcal{V} = \begin{cases} \mathcal{U} \cup \mathcal{A}_1^{\mathsf{Brun}} & \text{if } i_1 \in \{1,2\}, \\ \mathcal{U} \cup \mathcal{A}_2^{\mathsf{Brun}} & \text{if } i_1 = 3. \end{cases}$

2. $\bigcup_{n \geqslant 1} \Sigma_{i_1}^{\mathsf{Brun}} \cdots \Sigma_{i_n}^{\mathsf{Brun}}(\mathcal{U}) \subsetneq \Gamma_{\mathbf{v}}$ *if and only if there exists $N \geqslant 0$ such that $\bullet \xleftarrow{i_N} \bullet \xleftarrow{i_{N+1}} \cdots$ is an infinite path in the following graph:*



Proof. Assertion (1) follows from Propositions 2.16 and 5.1, and (2) follows directly from (1), Lemma 3.4 and the description of $\mathcal{G}^{\mathsf{Brun}}$ given in Section 3. ☐

Some applications Theorem 5.2 implies the following for *finite* products of Brun substitutions: for every $\sigma = \sigma_{i_1}^{\mathsf{Brun}} \cdots \sigma_{i_n}^{\mathsf{Brun}}$ such that at least one $i_n = 3$, we have $\bigcup_{n \geqslant 1} \mathbf{E}_1^\star(\sigma)(\mathcal{U}) = \Gamma_{\mathbf{v}}$ if and only if there is no infinite periodic path labelled by $(i_1 \cdots i_n)^\infty$ in the graph above. This has several consequences, as mentioned in Section 1.1. Note that such substitutions σ are always Pisot irreducible [2].

Another application (also mentioned in Section 1.1) is Corollary 5.3 below: the convergence of the Brun algorithm. Indeed, similarly as in the proof of Proposition 2.16, the approximated discrete planes $\Gamma_{\mathbf{w}_n}$ contain patterns of arbitrarily large radius. These patterns are also included in $\Gamma_{\mathbf{v}}$, so the approximated vectors $\mathbf{w}_n$ are constrained and their direction must tend to that of $\mathbf{v}$.

Corollary 5.3. *Let $\mathbf{v} \in \mathbb{R}^3_{>0}$ be an ordered totally irrational vector and let $(i_n) \in \{1,2,3\}^{\mathbb{N}}$ be its Brun expansion. Let $\mathbf{w}_n = \mathbf{M}_{i_1}^{-1} \cdots \mathbf{M}_{i_n}^{-1} \cdot (1,1,1)$, where the $\mathbf{M}_i$ are the Brun matrices given in Section 2.2. Then, the sequence $(\mathbf{w}_n / \|\mathbf{w}_n\|)_{n \geqslant 1}$ converges to $\mathbf{v}/\|\mathbf{v}\|$ as $n \to \infty$.*

Lastly, note that the above results do not directly imply that iterating substitutions from $\mathcal{U}$ generates patterns containing translations of patterns with arbitrarily large radius. This requires another proof (to be published in a forthcoming article), and is linked with the Pisot conjecture (see Section 1.1).

References

1. Arnoux, P., Ito, S.: Pisot substitutions and Rauzy fractals. Bull. Belg. Math. Soc. Simon Stevin 8(2), 181–207 (2001)
2. Avila, A., Delecroix, V.: Pisot property for the Brun and fully subtractive algorithms (2013), preprint
3. Berthé, V., Jolivet, T., Siegel, A.: Substitutive arnoux-rauzy sequences have pure discrete spectrum. Unif. Distrib. Theory 7(1), 173–197 (2012)
4. Berthé, V., Jolivet, T., Siegel, A.: Connectedness of Rauzy fractals associated with Arnoux-Rauzy substitutions (2013), preprint
5. Berthé, V., Siegel, A.: Tilings associated with beta-numeration and substitutions. Integers 5(3), A2, 46 pp. (2005)
6. Berthé, V., Siegel, A., Thuswaldner, J.M.: Substitutions, Rauzy fractals, and tilings. In: Combinatorics, Automata and Number Theory, Encyclopedia of Mathematics and its Applications, vol. 135. Cambridge University Press (2010)
7. Berthé, V., Steiner, W., Thuswaldner, J.: Tilings with S-adic Rauzy fractals (2013), preprint
8. Berthé, V., Jamet, D., Jolivet, T., Provençal, X.: Critical connectedness of thin arithmetical discrete planes. In: Discrete Geometry for Computer Imagery, Lecture Notes in Computer Science, vol. 7749, pp. 107–118 (2013)
9. Brun, V.: Algorithmes euclidiens pour trois et quatre nombres. In: Treizième congrès des mathèmaticiens scandinaves, tenu à Helsinki 18-23 août 1957, pp. 45–64
10. Fernique, T.: Multidimensional Sturmian sequences and generalized substitutions. Internat. J. Found. Comput. Sci. 17(3), 575–599 (2006)
11. Fernique, T.: Generation and recognition of digital planes using multi-dimensional continued fractions. Pattern Recognition 42(10), 2229–2238 (2009)
12. Fogg, N.P.: Substitutions in dynamics, arithmetics and combinatorics, Lecture Notes in Mathematics, vol. 1794. Springer-Verlag, Berlin (2002)
13. Frougny, C., Solomyak, B.: Finite beta-expansions. Ergodic Theory Dynam. Systems 12(4), 713–723 (1992)
14. Furukado, M., Ito, S., Yasutomi, S.I.: The condition for the generation of the stepped surfaces in terms of the modified Jacobi-Perron algorithm (2013), preprint
15. Ito, S., Ohtsuki, M.: Parallelogram tilings and Jacobi-Perron algorithm. Tokyo J. Math. 17(1), 33–58 (1994)
16. Ito, S., Rao, H.: Atomic surfaces, tilings and coincidence. I. Irreducible case. Israel J. Math. 153, 129–155 (2006)
17. Paysant-Le Roux, R., Dubois, E.: Une application des nombres de Pisot à l'algorithme de Jacobi-Perron. Monatsh. Math. 98(2), 145–155 (1984)
18. Rauzy, G.: Nombres algébriques et substitutions. Bull. Soc. Math. France 110(2), 147–178 (1982)
19. Schweiger, F.: Multidimensional continued fractions. Oxford Science Publications, Oxford University Press (2000)
20. Siegel, A., Thuswaldner, J.M.: Topological properties of Rauzy fractals. Mém. Soc. Math. Fr. (N.S.) (118), 140 (2009)