S. Pytheas-Fog

Substitutions in Dynamics, Arithmetics and Combinatorics

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Preface

There are two basic ways of constructing dynamical systems. One approach is to take an already existing system from the vast reserve arising in biology, physics, geometry, or probability; such systems are typically rather complex and equipped with rigid structures (a "natural" system is generally a "smooth" system). Alternatively one can build a system by hand using some basic tools like strings of letters on a finite alphabet; the latter system will have a simple structure. Nevertheless, these "simple" systems prove to be very useful in many mathematical fields (number theory, harmonic analysis, combinatorics, ergodic theory, and so on), as well as in theoretical computer science and physics. This category also includes various "classical" systems.

To make more precise this intuitive concept of "simple" systems, we can use the combinatorial notion of complexity of a sequence of letters with values in a finite alphabet, which counts the number of factors of given length. This gives an indication of the degree of randomness of the sequence: a periodic sequence has a bounded complexity, while the g-adic expansion of a normal number in base g has an exponential complexity. There are many examples of sequences having a reasonably low complexity function, the most famous being automatic sequences and Sturmian sequences, and the "reasonably simple" dynamical systems we like to consider are those which are canonically associated with this kind of sequences.

Among them, substitutive sequences play an important role. Substitutions are very simple combinatorial objects (roughly speaking, these are rules to replace a letter by a word) which produce sequences by iteration. Let us note that substitutions will be considered here as particular cases of free group morphisms, the main simplification being that we have no problem of cancellations. Substitutive dynamical systems have a rich structure as shown by the natural interactions with combinatorics on words, ergodic theory, linear algebra, spectral theory, geometry of tilings, theoretical computer science, Diophantine approximation, transcendence, graph theory, and so on.

Notice that the notion of substitution we consider here differs from that used for self-similar tilings; in this framework, substitutions produce matching rules acting on a finite set of prototiles and determining the ways in which the tiles are allowed to fit together locally; the best known example is the

Penrose tiling. Here, we consider substitutions acting on strings which are more elementary in nature.

There exist several books in the literature on related subjects. For instance [247, 265] consider symbolic dynamics, though mainly in the positive-entropy case, while [28] deals with automatic sequences, and [271, 272] with combinatorics on words; all these works are generalist books covering rather wide areas and decribing extensively the appropriate techniques. We decided to focus on a well-delimited subject (namely, substitutive dynamical systems) but try to give as many different viewpoints as possible, with emphasis on interactions between different areas; for example, Sturmian sequences are presented in details in both [272] and our book, the former giving a complete review on their combinatorial properties and the latter a description of their links with dynamical systems. Also, our book deals with zero-entropy symbolic dynamics, a subject where few such tools are available: the most famous one at this time is [340], and our book may be seen first as its updating, and then as a sequel together with an opening on wider perspectives.

The idea for this book stemmed from the collaboration between various groups of mathematicians mainly from France, Japan and China. It is largely based on courses performed by the authors in several universities and given during various Summer Schools in the past five years.

We chose to use the pseudonym S. Pytheas-Fog for two reasons. First, our work is the work of a group which is clearly more than the sum of its individual members, and a collective identity is a good way to stress this point. Second, this group is still active, and willing to produce more mathematical publications; though we do not claim to be a new Bourbaki, we do hope that there will be at least a second book, papers in mathematical journals, and a seminar bearing the name of Pytheas-Fog, so the present volume is just a beginning, together with a motivation, for more to come.

The overall structure of this book reflects our purpose which is twofold. We first want to provide an introduction to the theory of substitutive dynamical systems by focusing on several topics including various aspects of mathematics (as for instance geometry, combinatorics, ergodic theory and spectral analysis, number theory, numeration systems, fractals and tilings) but also computer science and theoretical physics. Secondly, we want to give a state of the art on this field, spotlighting representative aspects of the theory. More precisely, we focus on the following themes:

An introduction to elementary properties of **combinatorics on words** is given in the first three chapters. In particular Chaps. 2 and 3 provide an introduction to automatic sequences, which are produced by very natural algorithms coming from theoretical computer science. Chapters 6 and 9 give an analysis of the combinatorial properties of Sturmian words: these are the sequences (or infinite words) which have the smallest complexity function among non-ultimately periodic sequences.

Numeration systems appear in a natural way in the study of low complexity sequences. Beatty sequences and in particular the Fibonacci numeration system are introduced in Chap. 2 and dealt with in detail in Chap. 4. Chapter 6 introduces Ostrowski's numeration system, while the numeration systems defined in Chap. 8 are linked to the geometric properties of Pisot substitutive sequences.

An interesting field of application of automatic sequences deals with the **transcendence** of formal power series. Indeed, the study of automata provides a very fruitful combinatorial transcendence criterion for formal power series with coefficients in a finite field, which can be considered as a natural translation into algebraic terms of the properties of automatic sequences. This criterion is known as Christol, Kamae, Mendès France, and Rauzy's theorem. Chapter 3 presents this criterion and surveys the most recent transcendence results obtained via finite automata theory. Note that a real number whose g-adic expansion is an automatic sequence is conjectured to be either transcendental or rational, and has been proved to be transcendental when the sequence is Sturmian. Hence this presentation emphasizes the following philosophy: algebraicity strongly depends on the (generalized) "base" in which one works. Some connected results of transcendence are studied in Chap. 4; Chapter 8 reviews some Diophantine approximation properties issued from the study of substitutive systems.

Tools from **ergodic theory and spectral analysis** are introduced in Chap. 5, via a detailed study of systems associated with sequences having a low complexity function. An elementary introduction to correlation properties, and some examples of computations of correlation measures, are given in Chap. 2. Chapter 7 surveys the latest results in the spectral study of substitutive dynamical systems. Chapter 11 gives a special account of the ergodic properties of the Perron–Frobenius transfer operator.

The question of the **geometric representation** of substitutive sequences and more generally of low complexity sequences has given birth to a great amount of work. We give an account of the development and current state of this problem in Chap. 7. The study of the Sturmian case is particularly instructive (Chap. 6); it provides a well-known and fundamental interaction between ergodic theory, number theory, and symbolic dynamics, which comes from the study of irrational rotations on the one-dimensional torus $\mathbb{T}$. With an irrational real number α we associate a geometric dynamical system, the rotation $R : x \mapsto x + \alpha \bmod 1$, an arithmetic algorithm, the usual continued fraction approximation, and a set of Sturmian sequences which are codings of trajectories under R by a canonical partition; the continued fraction algorithm arises naturally as the link between the dynamical system and the symbolic sequences, and the study of the arithmetic and symbolic objects is very useful for the study of the dynamical system. Chapters 2 and 4 allude to Beatty sequences, whereas Chap. 9 studies in detail the connections between Sturmian and invertible substitutions over a two-letter alphabet.

Chapter 8 gives a detailed introduction to the tools and techniques used in this framework, in order to generalize this interaction to further systems and sequences.

The notion of **self-similarity** is illustrated by symbolic objects, namely substitutions, and by geometric objects, the fractal sets. Chapters 7 and 8 give examples of interactions between these two notions, as the most natural geometric representations of a substitutive sequence are sets with fractal boundaries. See also Chap. 11 for the study of fractal sets associated in a natural way with piecewise linear transformations of the unit interval.

We illustrate the connections between **physics** and low complexity sequences through the study of trace maps in Chaps. 8 and 9. Indeed free monoids, groups and their morphisms occur in a natural way in physics: finite automata and substitutive sequences are very useful to model and describe certain situations in solid state physics. In particular, one important question in quasicrystal theory is to compute the traces of matrices defined inductively according to a substitutive process. Trace maps are effective algorithms for constructing the recursion relations that the traces satisfy.

We have tried to allow the reader to read the different chapters as independently as possible, and to make each chapter essentially self-contained. Furthermore, the reader is not assumed to have a detailed knowledge in each of the fields covered by this book; we rather try to provide the necessary information, allowing it to be be used by graduate students. We describe hereafter the necessary preliminary knowledge, and which chapters are required for a better understanding of the following chapters.

Chapter 1 is needed as a prerequisite for all the other chapters. We recommend the lecture of Chaps. 2 and 3 for getting used to combinatorial manipulations on substitutions. Chapter 5 (as well as the spectral part of this book covered in some sections of Chap. 1, and in Chaps. 7 and 11) requires some basic knowledge on measure-theory and functional analysis, and is supposed to be self-contained as far as ergodic theory is concerned. This chapter will be needed for Chap. 7. We also recommend to read first Chaps. 6 and 7 before Chap. 8. Chapter 6 is essentially self-contained and is a good introduction for Chap. 9. Chapter 3 can be understood with no special algebraic knowledge except some familiarity with the notion of finite fields.

The book divides naturally into three parts, and is organized as follows.

The **introductory chapter** unifies the notation and contains the necessary background for the following chapters. Indeed we introduce in this chapter the basic introductory material that we will use throughout the book: words, languages, complexity function, substitutions, automatic sequences, substitutive dynamical systems, introduction to discrete dynamical systems and its spectral theory, group rotations, and so on.

In the **first part**, we focus on the aspects of substitutions which do not require any background on measure theory: these include combinatorial results, but also deep problems of number theory.

- The aim of **Chap. 2** is to focus on the notions of substitutions and automatic sequences by showing some typical examples of arithmetic situations in which they occur in a quite natural way. For example, the Morse sequence was first introduced to answer a question in combinatorial number theory and rediscovered by many people in various other circumstances (including geometry, group theory, logic). The Rudin-Shapiro sequence, first introduced to answer a question in harmonic analysis asked by Salem, is nowadays a basic construction in number theory and ergodic theory. The Fibonacci sequence, introduced as a natural example of a generator of a symbolic dynamical system, is deeply connected with the continued fraction algorithm and gives rise to many applications in theoretical computer science (for example to obtain good algorithms for the drawing of a straight line on a computer screen). We will consider the statistical properties of these sequences through the study of their correlation measure. The tools developed here are as simple as possible providing an elementary introduction to these classical examples. The following chapters will study them with a heavier theoretic background.
- The aim of **Chap. 3** is to investigate the connections between automatic sequences and transcendence in fields of positive characteristic, based on the following criterion due to Christol, Kamae, Mendès France, and Rauzy: a formal power series is algebraic if and only if the sequence of its coefficients is automatic, that is, it is the image by a letter-to-letter projection of a fixed point of a substitution of constant length. We also allude to the differences concerning transcendence between the real and the positive characteristic case. We then introduce some functions defined by Carlitz (exponential, logarithm, zeta) which are analogous to the corresponding real functions, and review the results of transcendence involving automata for these functions. We end this chapter by reviewing some techniques for disproving the automaticity of a sequence.
- **Chapter 4** is devoted to various partitions of the set of positive integers: Beatty sequences and connections with Sturmian sequences, partitions generated by substitutions, and similis partitions illustrated by linguistic properties of the Hungarian and Japanese languages. Special attention is devoted to non-periodic words which are shown to be fixed points of some combinatorial processes: above all the notion of log-fixed points is introduced. This study is illustrated by the Kolakoski word, which is shown to be not only a log-fixed point but also the unique fixed point of several maps: a map based on the Minkowski question-mark map, or maps defined by using the continued fraction expansion, the base-3 expansion, and so on. We then generalize these situations and present some open problems.

With any substitution, we can associate in a very natural way a dynamical system. In the **second part** we study these so-called substitutive dynamical systems, from the viewpoints of symbolic dynamics, ergodic theory and geometry.

- In **Chap. 5**, we introduce the fundamental notions of ergodic theory, also through the study of a few examples of symbolic dynamical systems, both in the topological and measure-theoretic framework. These include substitutions, which are maps on a finite alphabet, and which define naturally a class of infinite sequences, and the shift on this set. We study four examples of substitutions (Morse, Rudin-Shapiro, Fibonacci, already introduced in Chap. 2, and Chacon) and use them, together with symbolic notions (language, frequencies, complexity) to define basic ergodic notions (minimality, ergodicity), and begin the study of elementary spectral properties, geometric representation of symbolic systems, and the vast problem of joinings between systems.
- The main idea of **Chap. 6** is to show how it is possible to recover all the classical properties of rotations and continued fractions in a purely combinatorial way; we only use the combinatorial definition of Sturmian sequences, and obtain the existence of a combinatorial continued fraction acting on the set of Sturmian sequences. Some proofs, and also some geometric interpretations, become simpler and more natural in this setting; it also suggests non-trivial ways to generalize the usual continued fraction.
- **Chapter 7** presents an overview of the general spectral theory of substitutive dynamical systems. After recalling the tools and concepts required (including subshifts of finite type and adic systems, the notions of recognizability, and Markov partitions), a complete description of the related literature is given, including very recent work and some important conjectures in this subject.
- The problem of the geometric representation of substitutive dynamical systems is studied in detail in **Chap. 8**. Some basic tools and notions are introduced: stepped surface, generation by generalized substitutions, renormalization, study of the fractal boundary, and so on. Special attention is devoted to some important applications, as the quasi-periodicity of the tiling related to the stepped surface, the existence of Markov partitions of group automorphisms on $\mathbb{T}^3$ or Diophantine approximation properties in connection with the modified Jacobi-Perron algorithm.

In the **third part**, we extend the notion of substitution in two directions, the automorphisms of the free group and the piecewise linear maps. We also state a few of the many open problems related to substitutions.

- The purpose of **Chap. 9** is to study the properties of factors of sequences generated by invertible substitutions over a two-letter alphabet; this study is based on the following important result: invertible substitutions over a two-letter alphabet (that is, substitutions which are automorphisms of free groups) are shown to be exactly the Sturmian substitutions. We first discuss the local isomorphism between two invertible substitutions, reducing the problem to the study of some special classes of invertible substitutions. We then study some elementary properties of factors (including palindrome properties, powers of factors), which generalize some classical results for the Fibonacci sequence. By introducing the notion of singular factors, we establish a decomposition of the fixed points of invertible substitutions and we discuss the factor properties associated with this decomposition.

- **Chapter 10** investigates more deeply trace maps. Trace maps are dynamical systems attached to endomorphisms of free groups, which occur in a wide range of physical situations. By exploiting polynomial identities in rings of matrices, recursion formulas are produced between the traces of matrices defined by an induction using substitutions. We then study from an algebraic point of view endomorphisms of free monoids and free groups. Such endomorphisms are shown to induce a map of the ring of Fricke characters into itself. Particular emphasis is given to the group structure of trace maps and the Fricke-Vogt invariant.
- **Chapter 11** deals with Cantor sets generated by expanding piecewise linear maps. The main tool is the α-Fredholm matrix. This is the extension of the Fredholm matrix which is introduced to study the spectral properties of the Perron–Frobenius operator associated with one-dimensional maps. Using this α-Fredholm matrix, the Hausdorff dimension of the Cantor sets we consider is studied, as well as the ergodic properties of the dynamical system on it.
- This book ends with a survey in **Chap. 12** of some open problems in the subject. The first is the S-adic conjecture, about the equivalence for a minimal sequence of having sub-linear complexity and being generated by a finite number of substitutions. Then we look at possible generalizations of the interaction between rotations and Sturmian sequences through the usual continued fraction algorithm; these involve Arnoux-Rauzy sequences, interval exchanges and codings of the $\mathbb{Z}^2$-action of two rotations on the one-dimensional torus. This approach induces various open questions about geometric representations of substitutions, arithmetics in $SL(3,\mathbb{Z})$ and $SL(3,\mathbb{N})$, and about the definition in the two-dimensional case of some fundamental combinatorial objects as the complexity function or the notion of substitution. In **Appendix A**, J. Rivat states that infinitely many prime matrices exist in $SL(3,\mathbb{Z})$, contrary to what happens in the two-dimensional case.

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