1. Basic notions on substitutions

The aim of this chapter is to introduce some concepts and to fix the notation we will use throughout this book. We will first introduce some terminology in combinatorics on words. These notions have their counterpart in terms of symbolic dynamics. We shall illustrate these definitions through the example of a particular sequence, the Morse sequence, generated by an algorithmic process we shall study in details in this book, namely a substitution. After recalling some basic notions on substitutions, we shall focus on the concept of automatic sequences. We then introduce the first notions of ergodic theory and focus on the spectral description of discrete dynamical systems.

Let us start by giving a short list of basic references concerning the concepts developed here. Let us first mention the indispensable [340] on substitution dynamical systems. For detailed introductions to the symbolic dynamics, see [59, 81, 247, 265]. More generally, there are a number of excellent books on dynamical systems and ergodic theory [62, 80, 122, 145, 185, 194, 234, 241, 308, 309, 324, 326, 335, 376, 445], and topological dynamics [50, 134, 160, 161]. For references on word combinatorics, automata and formal languages, see [28, 159, 212, 271, 272, 312, 325, 368] and the references therein. See also [55, 162] for an approach to fractals, and [113, 127, 381, 383] for connections with number theory and Diophantine approximation.

We shall use the following usual notation: $\mathbb{R}$ denotes the set of *real numbers*; $\mathbb{Q}$ denotes the set of *rational numbers*; $\mathbb{Z}$ denotes the set of *integer numbers*; $\mathbb{C}$ denotes the set of *complex numbers*; $\mathbb{N}$ denotes the set of *nonnegative integers* ($0 \in \mathbb{N}$); $\mathbb{N}^+$ denotes the set of positive integers ($0 \notin \mathbb{N}^+$). By x positive we mean $x > 0$, and by x nonnegative, we mean $x \geq 0$. The *one-dimensional torus* $\mathbb{R}/\mathbb{Z}$ is denoted by $\mathbb{T}$, the *d-dimensional torus* by $\mathbb{T}^d$, and the unit circle, that is, the set of complex numbers of modulus one, is denoted by $\mathbb{U}$. The cardinality of a set S shall be denoted by Card S.

Within each section, results (theorems, propositions, lemmas, formulas, exercises, and so on) are labelled by the section and then the order of occurrence.

[1] This chapter has been written by V. Berthé and A. Siegel

1.1 Word combinatorics

Let us first introduce some terminology in word combinatorics.

1.1.1 Sequences, words, languages

Letters. Let $\mathcal{A}$ be a finite set that shall be called *alphabet*. Throughout this book the letters (also called *symbols*) of the alphabet $\mathcal{A}$ shall either be denoted as digits ($\mathcal{A} = \{0, 1, \ldots, d-1\}$) or as letters ($\mathcal{A} = \{a_1, \ldots, a_d\}$).

Monoid of finite words. A *word* or *block* is a finite string of elements in $\mathcal{A}$. The set of all finite words over $\mathcal{A}$ is denoted by $\mathcal{A}^\star$. We denote by ε the empty word.

The *concatenation* of two words $V = v_1...v_r$ and $W = w_1...w_s$ is the word $VW = v_1...v_r w_1...w_s$. This operation is associative and has a unit element, the empty word ε.

The set $\mathcal{A}^\star$ is thus endowed with the structure of a monoid, and is called the *free monoid* generated by $\mathcal{A}$. The set $\mathcal{A}^\star - \{\varepsilon\}$ is endowed with the structure of a *free semi-group*.

If $W = w_1...w_s$, s is called the *length* of W and denoted by $|W|$. We denote by $|W|_a$ the number of occurrences of a letter $a \in \mathcal{A}$ appearing in a word $W \in \mathcal{A}^\star$.

Sequences. A (one-sided) *sequence* of elements of $\mathcal{A}$, also called *(right) infinite word* on $\mathcal{A}$, shall be here an element $u = (u_n)_{n\in\mathbb{N}}$ in $\mathcal{A}^{\mathbb{N}}$.

A *two-sided sequence* in $\mathcal{A}$, also called *biinfinite word*, shall be here an element $u = (u_n)_{n\in\mathbb{Z}}$ in $\mathcal{A}^{\mathbb{Z}}$. It is also denoted by $u = \ldots u_{-2}u_{-1}.u_0u_1\ldots$.

Language. A word $v_1...v_r$ is said to *occur* at *position* (or *index*, or *rank*) m in a sequence $u = (u_n)$ or in a finite word $u_1...u_s$, if there exists a rank m such that $u_m = v_1, \ldots, u_{m+r-1} = v_r$.

We say also that the word $v_1...v_r$ is a *factor* of the sequence u.

The *language* (respectively the language of length n) of the sequence u, denoted by $\mathcal{L}(u)$ (respectively $\mathcal{L}_n(u)$), is the set of all words (respectively of length n) in $\mathcal{A}^\star$ which occur in u.

Specific properties of sequences. A sequence u is said to be *recurrent* if every factor occurs infinitely often.

A sequence u is *periodic* (respectively *ultimately periodic*) if there exists a positive integer T such that

$$\forall n,\ u_n = u_{n+T} \quad \text{(respectively } \exists n_0 \in \mathbb{N},\ \forall\, |n| \geq n_0,\ u_n = u_{n+T}).$$

We will use in both cases the terminology *shift-periodic*.

1.1.2 Complexity function

There is a classical measure of disorder for sequences taking their values in a finite alphabet, the so-called complexity function (see [16, 169] for many results linked to the notion of complexity).

Complexity. Let u be a sequence; we call *complexity function* of u, and denote by $p_u(n)$, the function which with each positive integer n associates Card $\mathcal{L}_n(u)$, that is, the number of different words of length n occurring in u.

In other words, the complexity function counts the number of distinct factors that occur through a sliding window. The complexity function is obviously non-decreasing. For any positive integer n, one has furthermore $1 \leq p_u(n) \leq d^n$, where d denotes the cardinality of the alphabet $\mathcal{A}$. Note that the complexity function can be considered as a good measure of the disorder of a sequence as it is smallest for periodic sequences. Namely, a basic result of [124] is the following.

Proposition 1.1.1. *If u is a periodic or ultimately periodic sequence, $p_u(n)$ is a bounded function. If there exists an integer n such that $p_u(n) \leq n$, u is an ultimately periodic sequence.*

Proof. The first part is trivial. In the other direction, we have $p_u(1) \geq 2$ otherwise u is constant, so $p_u(n) \leq n$ implies that $p_u(k+1) = p_u(k)$ for some k. For each word W of length k occurring in u, there exists at least one word of the form Wa occurring in u, for some letter $a \in \mathcal{A}$. As $p_u(k+1) = p_u(k)$, there can be only one such word. Hence, if $u_i...u_{i+k-1} = u_j...u_{j+k-1}$, then $u_{i+k} = u_{j+k}$. As the set $\mathcal{L}_k(u)$ is finite, there exist $j > i$ such that $u_i...u_{i+k-1} = u_j...u_{j+k-1}$, and hence $u_{i+p} = u_{j+p}$ for every $p \geq 0$, one period being $j - i$. ■

Special word. Let W be a factor of the sequence $u \in \mathcal{A}^{\mathbb{N}}$. A *right extension* (respectively *left extension*) of the factor W is a word Wx (respectively xW), where $x \in \mathcal{A}$, such that Wx (respectively xW) is also a factor of the sequence u.

A factor is said to be a *left special factor* (respectively *right special factor*) if it has more than one left (respectively right) extension.

Let W^+ (respectively W^-) denote the number of right (respectively left) extensions of W. We have

$$p_u(n+1) - p_u(n) = \sum_{W \in \mathcal{L}_n(u)} (W^+ - 1) = \sum_{W \in \mathcal{L}_n(u)} (W^- - 1).$$

This equality is a very useful tool for the computation of the complexity: see for instance Exercise 5.4.11 in Chap. 5, and also Chap. 6.

Remark. One can also introduce the notion of bispecial factors: these are the factors which are simultaneously right and left special factors (for a more detailed exposition, see [107]).

Topological entropy. There is a natural notion of entropy associated with the complexity function. The *topological entropy* of the sequence u is defined as the exponential growth rate of the complexity of u as the length increases:

$$H_{top}(u) = \lim_{n \to +\infty} \frac{\log_d(p_u(n))}{n},$$

where d denotes the cardinality of the alphabet $\mathcal{A}$. The existence of the limit follows from the subadditivity of the function $n \mapsto \log_d(p_u(n))$:

$$\forall m, n,\ \log_d(p_u(n+m)) \leq \log_d(p_u(m)) + \log_d(p_u(n)).$$

This notion of topological entropy comes from topological dynamics. One can associate in a natural way (see Sec. 1.1.3 below) a topological dynamical system with a sequence. The topological entropy of a sequence is nothing else than the topological entropy of its associated dynamical system. For more details on the entropy of dynamical systems, see for instance [393, 445].

1.1.3 Symbolic dynamical systems

Let us introduce some basic notions in symbolic dynamics. For expository books on the subject, see [59, 81, 247, 265, 340]. A more detailed exposition of this subject shall be made in Chap. 5.

The set $\mathcal{A}^{\mathbb{N}}$ shall be equipped with the product topology of the discrete topology on each copy of $\mathcal{A}$. Thus, this set is a compact space. The topology defined on $\mathcal{A}^{\mathbb{N}}$ is the topology defined by the following distance:

$$\text{for } u \neq v \in \mathcal{A}^{\mathbb{N}},\ d(u,v) = 2^{-\min\{n \in \mathbb{N};\ u_n \neq v_n\}},$$

Thus, two sequences are close to each other if their first terms coincide. Note that the space $\mathcal{A}^{\mathbb{N}}$ is complete as a metric compact space. Furthermore, it is a *Cantor set*, that is, a totally disconnected compact set without isolated points.

Let S denote the following map defined on $\mathcal{A}^{\mathbb{N}}$, called the *one-sided shift*:

$$S((u_n)_{n \in \mathbb{N}}) = (u_{n+1})_{n \in \mathbb{N}}.$$

The map S is uniformly continuous, onto but not one-to-one on $\mathcal{A}^{\mathbb{N}}$.

All these notions extend in a natural way to $\mathcal{A}^{\mathbb{Z}}$, the distance on $\mathcal{A}^{\mathbb{Z}}$ being defined as:

$$\text{for } u \neq v \in \mathcal{A}^{\mathbb{Z}},\ d(u,v) = 2^{-\min\{n \in \mathbb{N};\ u_{|n|} \neq v_{|n|}\}}.$$

Here the shift S is one-to-one.

System associated with a sequence. The *symbolic dynamical system* associated with a one-sided (respectively two-sided) sequence u with values in $\mathcal{A}$ is the system $(\overline{\mathcal{O}(u)}, S)$, where $\overline{\mathcal{O}(u)} \subset \mathcal{A}^{\mathbb{N}}$ (respectively $\mathcal{A}^{\mathbb{Z}}$) is the closure of the *orbit* of the sequence u under the action of the shift S; the orbit is the set $\{S^n u, n \in \mathbb{N}\}$ (respectively $\{S^n u, n \in \mathbb{Z}\}$). Note that $\overline{\mathcal{O}(u)}$ is finite if and only if u is shift-periodic.

The set $X_u := \overline{\mathcal{O}(u)}$ is a compact space and S is a continuous map from X_u to X_u. Indeed, the set X_u is a closed subset of the compact set $\mathcal{A}^{\mathbb{N}}$, and hence compact; $d(Sx, Sy) \leq 2d(x,y)$, therefore the continuity. This implies that if $x = \lim_{n\to+\infty} S^{k_n}u$, then $Sx = \lim_{n\to+\infty} S^{k_n+1}u$ and $Sx \in X_u$.

Lemma 1.1.2. *For every sequence $w \in \mathcal{A}^{\mathbb{N}}$, the following statements are equivalent:*

1. *the sequence $w \in X_u$;*
2. *there exists a sequence $(k_n)_{n\in\mathbb{N}}$ such that $w_0...w_n = u_{k_n}...u_{k_n+n}$ for every $n \geq 0$;*
3. *$\mathcal{L}_n(w) \subset \mathcal{L}_n(u)$ for all n.*

Proof. Statement 1. is equivalent to $d(w, S^{k_n}u) < 2^{-n}$ for some sequence k_n, and that is exactly 2.; 2. implies 3. as any word occurring in w must occur in some $w_0...w_n$, and 3. implies 2. because $w_0...w_n$ is in $\mathcal{L}_{n+1}(u)$. ■

Exercise 1.1.3. Prove that a sequence u is recurrent if and only if there exists a strictly increasing sequence $(n_k)_{k\in\mathbb{N}}$ such that $u = \lim_{k\to+\infty} S^{n_k}u$ (that is, u is a cluster point of X_u). Deduce that the sequence u is recurrent if and only if S is onto on $\overline{\mathcal{O}(u)}$ (see also Lemma 5.1.11).

Cylinders. For a word $W = w_0...w_r$, the *cylinder set* $[W]$ is the set $\{v \in X_u; v_0 = w_0, ..., v_r = w_r\}$.

The cylinder sets are *clopen* (open and closed) sets and form a basis of open sets for the topology of X_u. Indeed, if the cylinder $[W]$ is nonempty and v is a point in it, $[W]$ is identified with both the open ball $\{v'; d(v,v') < 2^{-n}\}$ and the closed ball $\{v'; d(v,v') \leq 2^{-n-1}\}$.

Exercise 1.1.4. Prove that a clopen set is a finite union of cylinders.

Remark. Note that the topology extends in a natural way to $\mathcal{A}^{\mathbb{N}} \cup \mathcal{A}^{\star}$. Indeed, let $\mathcal{B}$ be a new alphabet obtained by adding a further letter to the alphabet $\mathcal{A}$; words in $\mathcal{A}^{\star}$ can be considered as sequences in $\mathcal{B}^{\mathbb{N}}$, by extending them by the new letter in $\mathcal{B}$. The set $\mathcal{A}^{\mathbb{N}} \cup \mathcal{A}^{\star}$ is thus metric and compact, as a closed subset of $\mathcal{B}^{\mathbb{N}}$. This will be needed in particular in Section 1.2.1.

The cylinders are defined over $\mathcal{A}^{\mathbb{Z}}$ as the following clopen sets, where $W_1, W_2 \in \mathcal{A}^{\star}$:

$$[W_1.W_2] = \{(v_i)_i \in X_u | v_{-|W_1|} \dots v_{-1}.v_0 \dots v_{|W_2|-1} = W_1W_2\}.$$

Depending on the chapter, we will work either with $\mathcal{A}^{\mathbb{N}}$ or with $\mathcal{A}^{\mathbb{Z}}$.

Consider now the notion of *minimal* sequence (also called *uniform recurrence*):

Minimality. A sequence $u = (u_n)$ is minimal (or uniformly recurrent) if every word occurring in u occurs in an infinite number of positions with bounded gaps, that is, if for every factor W, there exists s such that for every n, W is a factor of $u_n \dots u_{n+s-1}$.

Exercise 1.1.5. Build an example of a sequence which is recurrent but not minimal.

Let us note that the term "minimality" comes from symbolic dynamics: a sequence u is minimal if and only if the dynamical system $(\overline{\mathcal{O}(u)}, S)$ is minimal. For more details, see Sec. 1.4.1, Sec. 5.1.4, and for instance, [340].

1.1.4 Sturmian sequences

Let us illustrate the preceding section by evoking an important family of sequences, the so-called Sturmian sequences.

Sturmian sequences. A *Sturmian sequence* is defined as a (one-sided) sequence u the complexity function p_u of which satisfies:

$$\forall n \in \mathbb{N},\ p_u(n) = n + 1.$$

In particular, a Sturmian sequence is defined over a two-letter alphabet. Chaps. 6 and 9 shall study in detail the properties of Sturmian sequences. The more classical example of a Sturmian sequence is the Fibonacci sequence, defined in Sec. 1.2.1 below; a proof of the fact that the Fibonacci sequence is Sturmian is given in Exercise 5.4.11.

Exercise 1.1.6. For any $s \geq 1$, construct an example of a sequence u with complexity $p_u(n) = s + n$. *(Hint: start from a Sturmian sequence and add new letters.)*

Remark. One can define Sturmian sequences over a larger size alphabet as recurrent sequences of complexity $n + s$: see [108, 147, 202] and Chap. 4.

Arnoux-Rauzy sequences. One can also consider a generalization of Sturmian sequences over a three-letter alphabet, namely the *Arnoux-Rauzy sequences* introduced in [49]. These are recurrent sequences defined over a three-letter alphabet with complexity $2n+1$ with the following extra combinatorial property: for every n, there is exactly one right special factor and one left special factor of length n, and these special factors can be extended in three different ways. This condition is called the * condition in the seminal paper [49], and more generally in the literature on the subject. A further generalization of this notion is given in [149].

Exercise 1.1.7. Prove that the Arnoux-Rauzy sequences are uniformly recurrent. *(Hint: prove that the right special factors appear with bounded gaps.)*

One shall consider these sequences in Chap. 12.

1.2 Substitutions

Substitutions are maps defined over the set of words $\mathcal{A}^\star$, which generate in a natural way infinite sequences with low complexity function.

1.2.1 Definition

Substitution. A *substitution* σ is an application from an alphabet $\mathcal{A}$ into the set $\mathcal{A}^\star - \{\varepsilon\}$ of nonempty finite words on $\mathcal{A}$; it extends to a morphism of $\mathcal{A}^\star$ by concatenation, that is, $\sigma(WW') = \sigma(W)\sigma(W')$ and $\sigma(\varepsilon) = \varepsilon$. It also extends in a natural way to a map defined over $\mathcal{A}^{\mathbb{N}}$ or $\mathcal{A}^{\mathbb{Z}}$.

It is called *of constant length* k if $\sigma(a)$ is of length k for any $a \in A$.

Fixed point. A *fixed point* of the substitution σ is an infinite sequence u with $\sigma(u) = u$.

A *periodic point* of σ is an infinite sequence u with $\sigma^k(u) = u$ for some $k > 0$.

An n*-word* for σ is any one of the words $\sigma^n(a)$ for $a \in \mathcal{A}$.

Remark. Substitutions are very efficient tools for producing sequences. Let σ be a substitution over the alphabet $\mathcal{A}$, and a be a letter such that $\sigma(a)$ begins with a and $|\sigma(a)| \geq 2$. Then there exists a unique fixed point u of σ beginning with a. This sequence is obtained as the limit in $\mathcal{A}^\star \cup \mathcal{A}^{\mathbb{N}}$ (when n tends toward infinity) of the sequence of n-words $\sigma^n(a)$, which is easily seen to converge.

Exercise 1.2.1. Prove that every substitution σ such that $|\sigma^n(a)|$ tends to infinity with n, for every $a \in \mathcal{A}$, has at least one periodic point.

Let us consider a classical example of a substitution of constant length, namely the Morse substitution. For a nice survey of the many properties of this substitution, see [27]; see also [284]. This sequence will be studied in detail in Chaps. 2 and 5.

The Morse sequence. The *Morse sequence* u is the fixed point beginning with a of the *Morse substitution* σ_1 defined over the alphabet $\{a, b\}$ by $\sigma_1(a) = ab$ and $\sigma_1(b) = ba$.

$$u = abbabaabbaababbabaababbaabbabaabbaababbaabbabaababbabaabbaabab...$$

Another classical example, namely the Fibonacci substitution, provides the more natural example of Sturmian sequence (see Chaps. 2, 6, 8 and 9).

The Fibonacci sequence. The *Fibonacci sequence* is the fixed point v beginning with a of the the *Fibonacci substitution* σ_2 defined over the two-letter alphabet $\{a, b\}$ by $\sigma_2(a) = ab$ and $\sigma_2(b) = a$.

$$v = abaababaabaababaababaabaababaabaababaababaabaababaababaaba...$$

Invertible substitution. Let Γ_d denote the free group over d-letters. A substitution over a d-letter alphabet can be naturally extended to Γ_d by defining $\sigma(s^{-1}) = (\sigma(s))^{-1}$. It is said to be *invertible* if there exists a map $\eta : \Gamma_d \to \Gamma_d$ such that $\sigma\eta(a) = \eta\sigma(a) = a$ for every $a \in \Gamma_d$.

The Fibonacci substitution is invertible: its inverse is $a \mapsto b$, $b \mapsto b^{-1}a$. For a nice introduction to the free groups and invertible substitutions, see [69]. See also [275, 278, 416].

1.2.2 Abelianization

There is a convenient way to associate with a substitution a matrix, namely the incidence matrix of the substitution σ.

Incidence matrix. Let σ be a substitution defined over the alphabet $\mathcal{A} = \{a_1, \ldots, a_d\}$ of cardinality d. The *incidence matrix* of the substitution σ is, by definition, the $d \times d$ matrix $\mathbf{M}_\sigma$ the entry of index (i, j) of which is $|\sigma(a_j)|_{a_i}$, that is, the number of occurrences of a_i in $\sigma(a_j)$.

Let us note that for every $(i, j) \in \{1, 2, \ldots, d\}^2$ and for every $n \in \mathbb{N}$, $|\sigma^n(a_j)|_{a_i}$ is equal to the coefficient of index (i, j) of the matrix $\mathbf{M}_\sigma^n$.

Example 1.2.2. The incidence matrices of the Morse substitution σ_1 and the Fibonacci substitution σ_2 are respectively the matrices:

$$\mathbf{M}_{\sigma_1} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbf{M}_{\sigma_2} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Unimodular substitution. A substitution is said to be *unimodular* if the determinant of its incidence matrix is ± 1. The Fibonacci substitution is unimodular. The Morse substitution is not.

Canonical homomorphism. Let σ be a substitution defined over the alphabet $\mathcal{A} = \{a_1, \ldots, a_d\}$ of cardinality d. Let $\mathbf{l} : \mathcal{A}^\star \to \mathbb{Z}^d$ denote the *canonical homomorphism*, also called *homomorphism of abelianization*, defined as follows:

$$\forall\, W \in \mathcal{A}^\star, \quad \mathbf{l}(W) = (|w|_{a_i})_{1 \le i \le d} \in \mathbb{N}^d.$$

As a consequence, the incidence matrix $\mathbf{M}_\sigma$ satisfies:

$$\mathbf{M}_\sigma = (\mathbf{l}(\sigma(a_1)), \ldots, \mathbf{l}(\sigma(a_d)))\,.$$

Furthermore, we have the following commutative relation:

$$\forall\, W \in \mathcal{A}^\star,\ \mathbf{l}(\sigma(W)) = \mathbf{M}_\sigma \mathbf{l}(W).$$

This notion will be needed in Chaps. 7, 8 and 10.

1.2.3 Primitivity

Let us introduce the notion of primitivity.

Definition. A substitution σ over the alphabet $\mathcal{A}$ is *primitive* if there exists a positive integer k such that, for every a and b in $\mathcal{A}$, the letter a occurs in $\sigma^k(b)$.

Proposition 1.2.3. *If σ is primitive, any of its periodic points is a minimal sequence.*

Proof. Let $u = \sigma^p(u)$ be a periodic point of σ. We have $u = (\sigma^p)^k(u) = (\sigma^p)^k(u_0)(\sigma^p)^k(u_1)...$; for any $b \in \mathcal{A}$, a occurs in $(\sigma^p)^k(b)$, hence a occurs in u infinitely often with bounded gaps; but then so does every $(\sigma^p)^n(a)$ in $u = (\sigma^p)^n(u)$, hence so does any word occurring in u. ■

Remark. The following substitution is not primitive but its fixed point beginning by 0 is a minimal infinite sequence:

$$0 \mapsto 0010, \ 1 \mapsto 1.$$

This sequence is called the *Chacon sequence*. Indeed, it is not difficult to check that Chacon's sequence begins with the following sequence of words (b_n):

$$b_0 = 0, \text{ and } \forall n \in \mathbb{N}, \ b_{n+1} = b_n b_n 1 b_n.$$

One thus can deduce from this that Chacon's sequence is minimal; for more details, see Chap. 5. Notice that a rule of the form $b_{n+1} = b_n b_n 1 b_n$ is called a *catenative rule* (see [365] for introducing the notion of locally catenative rules and [390] for a generalization of this notion).

A characterization of primitivity. Let us add now the following extra assumptions on substitutions:

1. there exists a letter $a \in \mathcal{A}$ such that $\sigma(a)$ begins with a (this condition guarantees the existence of a fixed point);
2. $\lim_{n\to\infty} |\sigma^n(b)| = \infty$, for every letter $b \in \mathcal{A}$.

Let us note that these two conditions imply the existence of an (infinite) fixed point beginning with a. We will furthermore suppose:

3. all the letters in $\mathcal{A}$ actually occur in this fixed point.

Let us note that, given a primitive substitution, then there exists a power of this substitution for which these three conditions hold.

Under these assumptions, we have the following equivalence:

Proposition 1.2.4. *Let σ be a substitution satisfying the above conditions. The substitution σ is primitive if and only if the fixed point of σ beginning by a is minimal.*

The proof is left as an exercise (see also [340]).

1.2.4 The Perron–Frobenius theorem

The primitivity of a substitution can easily be expressed in terms of its incidence matrix. Indeed the substitution σ is primitive if and only if there exists an integer k such that the k-th power of its incidence matrix $\mathbf{M}_\sigma$ has positive entries. We thus say that a matrix is *primitive* if its entries are nonnegative and if there exists an integer k such that the k-th power of the matrix has positive entries. A matrix $\mathbf{M}_\sigma$ is said *irreducible* if it has no nontrivial invariant space of coordinates.

Exercise 1.2.5 ([189]).
Let $\mathbf{M}$ be an irreducible matrix (of size n).

1. Prove that $(\mathbf{I}+\mathbf{M})^{n-1} > 0$. *(Hint: prove that for any nonnegative vector* $\mathbf{y}$*, the vector* $(\mathbf{I}+\mathbf{M})\mathbf{y}$ *has a number of zero coordinates which is strictly smaller than that of* $\mathbf{y}$*.)*
2. Deduce that $\mathbf{M}$ satisfies the following: for any (i, j), there exists a positive integer k such that the entry of index (i, j) of $\mathbf{M}^k$ is positive. *(Hint: consider the binomial expansion of* $(\mathbf{I}+\mathbf{M})^{n-1}$*.)*

Primitive matrices are also called *irreducible and aperiodic matrices.* See [63, 189, 387] for more details on matrices with nonnegative coefficients.

Irreducible matrices (and hence primitive matrices) satisfy the Perron–Frobenius theorem:

Theorem 1.2.6 (Perron–Frobenius' theorem). *Let* $\mathbf{M}$ *be a nonnegative* irreducible *matrix. Then* $\mathbf{M}$ *admits a strictly positive eigenvalue* α *which dominates in modulus the other eigenvalues* λ*:* $\alpha \geq |\lambda|$*. The eigenvalue* α *is a simple eigenvalue and there exists an eigenvector with positive entries associated with* α*.*

Furthermore, if $\mathbf{M}$ *is* primitive*, then the eigenvalue* α *dominates strictly in modulus the other eigenvalues* λ*:* $\alpha > |\lambda|$*.*

This important property of primitive matrices implies the existence of frequencies for every factor of a fixed point of a primitive substitution.

Frequencies. Let u be a sequence. The *frequency* f_W of a factor W of u is defined as the limit (when n tends towards infinity), if it exists, of the number of occurrences of the factor W in $u_0u_1 \ldots u_{n-1}$ divided by n.

Theorem 1.2.7. *Let* σ *be a primitive substitution satisfying the conditions mentioned in Sec. 1.2.3. Let* u *be a fixed point of* σ*. Then every factor of* u *has a frequency. Furthermore, all the frequencies are positive. The frequencies of the letters are given by the coordinates of the positive eigenvector associated with the dominating eigenvalue, renormalized in such a way that the sum of its coordinates equals* 1*.*

For a proof of this result, see Chap. 5 or [340].

1.2.5 Substitutions of Pisot type

The case where the dominant eigenvalue is a Pisot number is of particular interest.

Pisot-Vijayaraghavan number. Let us recall that an algebraic integer $\alpha > 1$ is a *Pisot-Vijayaraghavan number* or a *Pisot number* if all its algebraic conjugates λ other than α itself satisfy $|\lambda| < 1$. This class of numbers has been intensively studied and has some special Diophantine properties (see for instance [113]).

Pisot type substitution. A substitution is *of Pisot type* if the eigenvalues of the incidence matrix satisfy the following: there exists a dominant eigenvalue α such that for every other eigenvalue λ, one gets

$$\alpha > 1 > |\lambda| > 0.$$

Proposition 1.2.8 ([103]). *Let σ be a Pisot substitution of Pisot type. Then the characteristic polynomial χ_σ of the incidence matrix $\mathbf{M}_\sigma$ is irreducible over $\mathbb{Q}$. Hence, the dominant eigenvalue α is a Pisot number and the matrix $\mathbf{M}_\sigma$ is diagonalizable (over $\mathbb{C}$), the eigenvalues being simple. Furthermore, σ cannot be of constant length.*

Proof. Suppose there are two polynomials Q and R with integer coefficients such that $QR = \chi_\sigma$. As 0 is not a root, each of the polynomials Q and R has a root in $\mathbb{C}$ with modulus larger than or equal to 1, which implies that at least two eigenvalues have a modulus larger than or equal to 1, hence the irreducibility over $\mathbb{Q}$. We deduce from this that the roots in $\mathbb{C}$ are simple. If σ is of constant length l, then l is an eigenvalue for the eigenvector $(1, 1, \cdots, 1)$, which is a contradiction for the irreducibility of the characteristic polynomial. ∎

Remark. Following the previous definition, 0 cannot be an eigenvalue of the incidence matrix of a Pisot substitution. Hence, the Morse substitution is not Pisot.

It is worth noticing the following result which is a direct consequence of the theory of irreducible matrices:

Theorem 1.2.9 ([103]). *Any Pisot type substitution is primitive.*

Proof. We deduce from the irreducibility of χ_σ, that the matrix $\mathbf{M}_\sigma$ is irreducible. The proof of the theorem is thus a direct consequence of the following result (see for instance [63]): let $\mathbf{M}$ be a matrix with nonnegative entries; $\mathbf{M}$ is primitive if and only if $\mathbf{M}$ is irreducible and the spectral radius of $\mathbf{M}$ is greater in magnitude than any other eigenvalue. ∎

We shall focus on substitutions of Pisot type in Chaps. 7, 8 and 12.

1.2.6 Substitutive dynamical systems

We deduce from Proposition 1.2.3, that if σ is primitive, then all its periodic points have the same language. In other words, all the symbolic dynamical systems associated with all its periodic points do coincide.

Substitutive system. Hence, we can associate in a natural way with a primitive substitution a symbolic dynamical system following Sec. 1.1.3. Indeed, if σ is primitive, the *symbolic dynamical system associated with* σ is that, denoted by (X_σ, S), generated by any of its periodic points. It is finite if and only if there is a periodic point for σ which is also shift-periodic. In this case, the substitution is also called *shift-periodic*. We are mainly interested in primitive and not shift-periodic substitutions.

Note that C. Holton and L. Q. Zamboni prove that a substitution of Pisot type cannot be shift-periodic:

Proposition 1.2.10 ([208]). *If σ is a primitive substitution the matrix of which has a nonzero eigenvalue of modulus less than 1, then no fixed point u of σ is shift-periodic.*

In particular, all the usual hypotheses (primitivity, non-periodicity) are satisfied by Pisot type substitutions. In Chaps. 7 and 8, we will see that Pisot type substitutive systems have very special spectral and geometrical properties.

Topological properties. The dynamical system associated with a primitive substitution can be endowed with a Borel probability measure μ. Furthermore this measure is *invariant* under the action of the shift S, that is, $\mu(S^{-1}B) = \mu(B)$, for every Borel set B. Indeed, this measure is uniquely defined by its values on the cylinders. The measure of the cylinder $[W]$ is defined as the frequency of the finite word W in any element of X_σ, which does exist following Theorem. 1.2.7.

Let us note that the system (X_σ, S) is *uniquely ergodic*: there exists a unique shift-invariant measure. For a proof, see [340], and see also Chap. 5. We will come back to this notion later in this chapter.

1.3 Automata

Among substitutions, substitutions of constant length play an important role. The aim of this section is to prove that they are connected in a fundamental way with automatic sequences.

1.3.1 Definition

Let k be an integer greater than or equal to 2. Let us introduce the notion of k-automaton: such an automaton is a finite complete deterministic automaton (also called 2*-tape automaton or transducer*). For more details, the reader is referred to Chaps. 2 and 5; see also [121, 159, 212] and the surveys [11, 22].

Definition of a k-automaton. A k-automaton is represented by a directed graph defined by:

- a finite set $\mathcal{S}$ of vertices called *states*. One of these states is called the *initial state* and is denoted by i;
- k oriented edges called *transition maps* from the set of states $\mathcal{S}$ into itself, labelled by integers between 0 and $k-1$;
- a set Y and a map φ from $\mathcal{S}$ into Y, called *output function* or *exit map*.

Automaticity. A sequence $(u(n))_{n\in\mathbb{N}}$ with values in Y is called *k-automatic* if it is generated by a k-automaton as follows: let $\sum_{i=0}^{j} n_i k^i$ $(n_j \neq 0)$ be the base k expansion of the integer n; starting from the initial state one feeds the automaton with the sequence $n_0, n_1 \ldots n_j$ (the digits being read in increasing order of powers); after doing this the automaton is in the state $a(n)$; then put $u(n) := \varphi(a(n))$. The automaton is then said to generate the k-automatic sequence in *reverse reading*.

One can similarly give another definition of k-automaticity by reading the digits in the reverse order, i.e., by starting with the most significant digit (that is, starting with n_j), but these two notions are easily seen to be equivalent (see Proposition 1.3.4 below). The automaton is then said to generate the k-automatic sequence in *direct reading*. We will use the terminology *k-automatic in direct reading* or *in reverse reading*, until Proposition 1.3.4 will be proved.

Remark. An automaton is considered here as a machine producing a sequence, and not as usual as a machine recognizing a language.

1.3.2 Cobham's theorem

There is an important connection between automatic sequences and fixed points of substitutions of constant length. The following characterization of automatic sequences is due to Cobham (see [121]).

Letter-to-letter projection. Consider a map from a finite alphabet $\mathcal{A}$ to a finite alphabet, say $\mathcal{B}$. This map extends in a natural way by concatenation to a map from $\mathcal{A}^\star \cup \mathcal{A}^{\mathbb{N}}$ to $\mathcal{B}^\star \cup \mathcal{B}^{\mathbb{N}}$. Such a map is called a *letter-to-letter projection*.

Proposition 1.3.1. *A sequence u is k-automatic in direct reading if and only if u is the image by a letter-to-letter projection of a fixed point of a substitution of constant length k.*

Proof. Let us give a constructive proof of this equivalence by showing that the image by a letter-to-letter projection of the fixed point of a substitution of constant length is generated by an automaton in direct reading with as set of states the alphabet of the substitution, with transition maps given by the substitution, and with as output function, the projection p.

- Let us first start with the fixed point $u = (u(n))_{n\in\mathbb{N}}$ of the substitution σ of constant length k defined over the alphabet $\mathcal{A}$. Define $\sigma_i : \mathcal{A} \to \mathcal{A}$, which sends the letter a to the $(i+1)$-th letter of $\sigma(a)$. We then have $\sigma(a) = \sigma_0(a)\cdots\sigma_{k-1}(a)$. Furthermore, for any integer n,
$$\sigma(u_n) = u_{kn}u_{kn+1}\ldots u_{kn+k-1} = \sigma_0(u_n)\cdots\sigma_{k-1}(u_n),$$
that is, $\sigma_i(u(n)) = u(kn+i)$, for any integer n and for any $i \in [0, k-1]$.
Let us construct a k-automaton A in direct reading which recognizes the sequence u: consider as set of states the alphabet $\mathcal{A}$, and as set of edges the maps σ_i. There is an edge from a to b if b occurs in $\sigma(a)$, labelled by i if b is the $i+1$-th letter of $\sigma(a)$. Put as initial state u_0 the first letter of u. Define as output function the identity.
It is easily seen that the k-automaton A generates in direct reading the sequence u. Indeed, write $n = \sum_{i=0}^{j} n_i k^i$: we start from $u(0)$, then go to the $n_t + 1$-th letter of $\sigma(u(0))$, denoted by a_1, then go to the $n_{t-1} + 1$-th letter of $\sigma(a_1)$, which is also the $kn_t + n_{t-1} + 1$-th letter of $\sigma^2(u(0))$, and, after n steps, we arrive to the $n+1$-th letter of $\sigma^{t+1}(u(0))$, which is $u(n)$.
If now, v is the image by a letter-to-letter projection $p : \mathcal{A} \to \mathcal{B}$ of the fixed point $u = (u(n))_{n\in\mathbb{N}} \in \mathcal{A}^{\mathbb{N}}$ of a substitution σ of constant length k defined on the alphabet $\mathcal{A}$, then v is generated by the automaton previously defined for u, but with output function the projection p.
- Conversely, let u be a sequence generated by a k-automaton A in direct reading. Let $\mathcal{S}$ be the set of states of the automaton A and let $f_0, \cdots, f_{k-1}$ be the transition maps. Define the substitution of constant length $\sigma = f_0 \cdots f_{k-1}$ over $\mathcal{S}$. Let v be the fixed point of σ beginning with the initial state i (note that the edge f_0 maps i onto itself by definition, hence the existence of a fixed point). It is easily checked that the sequence u is the image by the output function φ of the fixed point v. ■

Remark. Following Prop. 1.3.1, k-automatic sequences are also called *k-uniform tag sequences* in [121].

Example 1.3.2. One deduces from the proposition above that the 2-automaton (Fig. 1.1) with initial state a and exit map $Id_{\{a,b\}}$ generates the Morse sequence in direct reading.

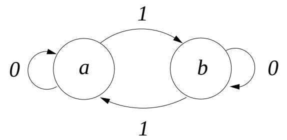

Fig. 1.1. Automaton associated with the Morse sequence.

1.3.3 The kernel of a sequence

The aim of this section is to give a further characterization of automaticity in terms of a family of subsequences.

Proposition 1.3.3. *A sequence $u \in \mathcal{A}^{\mathbb{N}}$ is k-automatic in reverse reading if and only if the k*-kernel *$N_k(u)$ of the sequence u is finite, where $N_k(u)$ is the set of subsequences of the sequence $(u(n))_{n\in\mathbb{N}}$ defined by*

$$N_k(u) = \{(u(k^l n + r))_{n\in\mathbb{N}};\ l \geq 0;\ 0 \leq r \leq k^l - 1\}.$$

Proof.
Let us first suppose that the k-kernel $N_k(u)$ of the sequence u is finite. We denote by $u = u_1, u_2, \cdots, u_d$ the sequences of $N_k(u)$. Consider a finite set of d states $\mathcal{S} = \{a_1, \cdots, a_d\}$ in bijection with $N_k(u)$ (a_j denotes the state in bijection with the sequence u_j). Define, for any r in $[0, k-1]$, the map $r : \mathcal{S} \to \mathcal{S}$ which associates with any state a_i the state $r.a_i$ in bijection with the sequence of the k-kernel $(u_i(kn+r))_{n\in\mathbb{N}}$. Let $n = \sum_{i=0}^{j} n_i k^i$ be the base k expansion of the integer n, with $n_i \in\ [0, k-1]$ and $n_j \neq 0$. We define the map n from $\mathcal{S}$ into itself by $n(a_i) = n_j(n_{j-1}(\cdots(n_0(a_i))))$, if $n \neq 0$, otherwise the map 0 equals the identity. It is easily seen that $n.a_1$ is the state in bijection with $(u(k^{j+1}l + n))_{l\in\mathbb{N}}$. Hence if $n.a_1 = m.a_1$, then $u(n) = u(m)$, by considering the first term of the two corresponding subsequences. We thus can define as output function the map φ (which is well defined) which associates with a state a_i the value $u(n)$ for any integer such that $n.a_1 = a_i$ (such an integer always exists by construction). The sequence u is hence generated by the automaton in reverse reading with states $\mathcal{S}$, initial state a_1, transition maps $0, 1, \cdots, k-1$ and output function φ.

Let us show that a k-automatic sequence u in reverse reading has a finite k-kernel. Let A be a finite automaton which generates the sequence u of initial state i. The subsequence $(u(k^l n + r))_{n\geq 0}$, where $l \geq 0$ and $0 \leq r \leq k^l - 1$, is generated by A with the initial state $\overline{r}.i$, where $\overline{r}$ is the word of k letters obtained by concatenating in front of the base q expansion of r as many zeros as necessary. As the automaton A has a finite number of states, we hence obtain a finite number of subsequences. ∎

Proposition 1.3.4. *A sequence is k-automatic in direct reading if and only if it is k-automatic in reverse reading.*

Proof. Following Proposition 1.3.1 and Proposition 1.3.3, it is sufficient to prove the equivalence between being the image by a letter-to-letter projection of the fixed point of a substitution of constant length k and having a finite k-kernel.

Let us prove that the image by a letter-to-letter projection of the fixed point of a substitution of constant length k has a finite k-kernel. It suffices to prove this result for the fixed point $(u(n))_{n\in\mathbb{N}}$ of a substitution σ of constant length k defined over the alphabet $\mathcal{A}$. For $0 \leq i \leq k-1$, define as in the preceding proof $\sigma_i : \mathcal{A} \to \mathcal{A}$ which associates with a letter x the $(i+1)$-th letter in its image under σ. We have $\sigma_i(u(n)) = u(kn+i)$, for any integer n and for any $i \in [0, k-1]$. Let $l \geq 0$, $0 \leq r \leq k^l - 1$. Write $r = \sum_{i=0}^{l-1} r_i k^i$, where $0 \leq r_i \leq k-1$. We thus have $u(k^l n + r) = \sigma_{r_0} \circ \sigma_{r_1} \circ \cdots \circ \sigma_{r_{l-1}}(u(n))$. As there are at most $(\mathrm{Card}(\mathcal{A}))^{\mathrm{Card}(\mathcal{A})}$ such maps, this concludes the proof.

Conversely, suppose that the k-kernel $N_k(u)$ of the sequence u is finite. We denote by $u = u_1, u_2, \cdots, u_d$ the sequences of $N_k(u)$. Let $U = (U(n))_{n\in\mathbb{N}}$ be the sequence with values in $\mathcal{A}^d$ defined by $U(n) = (u_1(n), \cdots, u_d(n))$. Let us construct a substitution σ of constant length k defined on $\mathcal{A}^d$ with U as a fixed point. As $N_k(u)$ is stable by the maps $\mathcal{A}_r$, where $\mathcal{A}_r(v(n))_{n\in\mathbb{N}} = v(kn+r))_{n\in\mathbb{N}}$, then, for any $0 \leq r \leq k-1$, $U(kn+r) = U(km+r)$, if $U(n) = U(m)$. Let us define, for $0 \leq r \leq k-1$, σ_r: $\mathcal{A}^d \to \mathcal{A}^d$, $U(n) \mapsto U(kn+r)$ if there exists n such that $(a_1, \cdots, a_r) = U(n)$, and otherwise $(a_1, \cdots, a_r) \mapsto (0, \cdots, 0)$. The sequence U is thus the fixed point of the substitution of constant length $\sigma : x \mapsto \sigma_0(x) \cdots \sigma_{k-1}(x)$ and the sequence u is the image of U by the projection on the first coordinate from $\mathcal{A}^d$ into $\mathcal{A}$.■

Remark. These proofs are constructive, i.e., given the k-kernel of a sequence, then one can compute a k-automaton in reverse reading, generating the sequence u, the states of which are in bijection with the subsequences of the k-kernel. Furthermore a k-automaton in direct reading can be derived from a substitution of constant length generating u from the proof of Proposition 1.3.1. We give other characterizations of automaticity in Chap. 3, known as the Christol, Kamae, Mendès France and Rauzy theorem (see [118] and also [119]).

Exercise 1.3.5. Give a description of the 2-kernel of the Thue-Morse sequence. Prove that the automaton in Fig. 1.1 also generates the Thue-Morse sequence in reverse reading.

1.4 An introduction to topological dynamics and measure-theoretic dynamical systems

We have introduced the notion of a symbolic dynamical system associated with an infinite sequence. Such a discrete system belongs to the larger class of topological dynamical systems, which have been intensively studied (for instance, see [122]).

The aim of this section is to provide a brief introduction to basic notions in topological and measure-theoretic dynamics, that may be useful for a better understanding of Part 2. Note that most of the notions introduced here will be studied in more details through the study of representative examples in Chap. 5.

1.4.1 Topological dynamical systems: basic examples

Definition. A *topological dynamical system* is defined as a compact metric space X together with a continuous map T defined onto the set X.

Remark. Note that topological systems may also be defined without assuming that the map T is onto. In all the examples we consider here, the maps are onto.

Minimality. A topological dynamical system (X, T) is *minimal* if for all x in X, the orbit of x (that is, the set $\{T^n x;\ n \in \mathbb{N}\}$) is dense in X.

Exercise 1.4.1. 1. Prove that minimality is equivalent to each of the following assertions:
   a) X does not contain a nontrivial closed subset E such that $T(E) \subset E$;
   b) for every non-empty subset U of X, then $\bigcup_{n \geq 0} T^{-n}(U) = X$.
2. A Borel function f on X is said T*-invariant* if $\forall x \in X,\ f(Tx) = f(x)$. Suppose that (X, T) is minimal. Prove that if $f : X \to X$ is continuous, then f is T-invariant implies that f is constant.

Existence of an invariant measure. Let X be a compact metrizable space. Recall that the set $\mathcal{M}_X$ of Borel probability measures on X is identified with a convex subset of the unit ball of the dual space of the set $\mathcal{C}(X)$ of the continuous complex-valued functions on X for the weak-star topology. The set $\mathcal{M}_X$ is metrizable, and compact. In the weak-star topology, one gets

$$\mu_n \to \mu \Longleftrightarrow \forall f \in \mathcal{C}(X),\ \int f d\mu_n \to \int f d\mu.$$

Recall that a Borel measure μ defined over X is said T-invariant if $\mu(T^{-1}(B)) = \mu(B)$, for every Borel set B. This is equivalent to the fact that for any continuous function $f \in \mathcal{C}(X)$, then $\int f(Tx) d\mu(x) = \int f(x) d\mu(x)$.

Lemma 1.4.2. *A topological system (X, T) always has an invariant probability measure.*

Proof. Indeed, given a point $x \in X$, any cluster point for the weak-star topology of the sequence of probability measures:

$$\frac{1}{N} \sum_{n<N} \delta_{T^n x}$$

is a T-invariant probability measure, where δ_y denotes the Dirac measure at point y. ■

Since the space $\mathcal{C}(X)$ is metrizable, it is equivalent to say that there exists an increasing sequence of integers $(N_k)_{k \in \mathbb{N}}$ such that

$$\forall f \in \mathcal{C}(X),\ \frac{1}{N_k} \sum_{n<N_k} f(T^n x) \to \int f d\mu.$$

Unique ergodicity. The case where there exists only one T-invariant measure is of particular interest. A topological dynamical system (X,T) is *uniquely ergodic* if there exists one and only one T-invariant Borel probability measure over X.

Conjugacy. A natural question is to try to "compare" two dynamical systems. Two dynamical systems (X,S) and (Y,T) are said to be *topologically conjugate* (or *topologically isomorphic*) if there exists an homeomorphism f from X onto Y which conjugates S and T, that is:

$$f \circ S = T \circ f.$$

Semi-conjugacy. Two dynamical systems (X,S) and (Y,T) are said to be *semi-topologically conjugate* if there exist two sets X_1 and Y_1 which are at most countable such that $B_1 = X \setminus X_1$, $B_2 = Y \setminus Y_1$, and f is a bicontinuous bijection from B_1 onto B_2 which conjugates S and T. The map f is said to be a *semi-topological conjugacy.* For an example of semi-topological conjugacy, see Sec. 5.2.3.

Topological factor. A topological dynamical system (Y,T) is a *topological factor* of (X,S) if there exists a continuous map from X onto Y which conjugates the maps S and T. The system (X,T) is then called an *extension* of (Y,S).

Exercise 1.4.3. Let u,v be two sequences with values in finite alphabets, and let X_u, X_v denote respectively the associated symbolic dynamical systems.

1. Suppose that (X_v,S) is a topological factor of (X_u,S), where S denotes the shift. Let ϕ denote the conjugation map from X_u onto X_v. Prove that the map ϕ satisfies the following: there exists a positive integer q such that for every i, the coordinate of index i of $\phi(x)$ depends only on $(x_i, \dots x_{i+q})$. *(Hint: let $\phi_0 : X_u \to \mathcal{B}$, $x = (x_n) \mapsto |\phi(x)|_0$, where $\mathcal{B}$ denotes the alphabet of v; prove that $\phi_0^{-1}([a])$ is a clopen set, for every letter a; see also Lemma 5.1.14.)*
2. Deduce that if (X_u,S) and (X_v,S) are topologically conjugate, then they have the same topological entropy (see Sec. 1.1.2), and if (X_v,S) is a topological factor of (X_u,S), then $H_{top}(v) \leq H_{top}(u)$.

Rotations. The simplest example of a topological dynamical system consists of *toral translations.* Toral translations belong to a larger class of dynamical systems, namely, the *translations over a compact group* G, the invariant probability measure being the Haar measure. These are usually called *rotations* of the group G. This class contains in particular all the toral rotations, the additions over a finite group, translations over p-adic integer groups or translations over p-adic solenoids (see Sec. 1.6.2).

Exercise 1.4.4. Let G be a compact metric group. Let $T : G \to G$, $x \mapsto ax$ be a rotation of G. Prove that (G,T) is minimal if and only if $\{a^n,\ n \in \mathbb{N}\}$ is dense in X.

Following Kronecker's theorem, the minimality of toral rotations can be expressed as follows (for a proof, see for instance [234]):

Proposition 1.4.5. *The rotation by the vector* $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ *on the d-dimensional torus* $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ *is minimal if and only if* $\alpha_1, \ldots, \alpha_d$ *and 1 are rationally independent.*

Remark. Note that the minimality of a rotation (G, T) (G being a compact metric group) is equivalent to its unique ergodicity. For a proof, see for instance [445]. In particular, a rotation with irrational angle on the one-dimensional torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is minimal and uniquely ergodic, the invariant measure being the Haar measure.

1.4.2 Measure-theoretic dynamical systems

We have considered here the notion of dynamical system, that is, a map acting on a given set, in a topological context. This notion can be extended to measurable spaces: we thus get measure-theoretic dynamical systems. For more details about all of the notions defined in this section, one can refer to [122, 194, 445].

Definition. A *measure-theoretic dynamical system* is defined as a system $(X, T, \mu, \mathcal{B})$, where μ is a probability measure defined on the σ-algebra $\mathcal{B}$ of subsets of X, and $T : X \to X$ is a measurable map which preserves the measure μ.

Remark. Despite unique ergodicity is a concept from topological dynamics, one should note that any uniquely ergodic topological dynamical system (X, T) is a measure-theoretic dynamical system with respect to the unique probability measure which is invariant under T on the σ-algebra of Borel sets of X.

Example 1.4.6. A uniquely ergodic rotation on a compact metric abelian group or a primitive substitutive dynamical system are measure-theoretic dynamical systems.

Ergodicity. A measure-theoretic dynamical system (X, T, μ) is *ergodic* if every Borel subset B of X such that $T^{-1}(B) = B$ has zero measure or full measure.

Exercise 1.4.7. Prove that ergodicity is equivalent to each of the following assertions:

- every Borel set B such that $\mu(T^{-1}B \Delta B) = 0$ has zero or full measure;
- every Borel set B with $\mu(B) > 0$ satisfies $\mu(\cup_{n=0}^{\infty} T^{-n}B) = 1$.

Let us recall that a Borel function f on X is said T-invariant if $f \circ T = f$. If the system (X, T, μ) is ergodic, then every Borel function which is T-invariant is almost everywhere constant. Otherwise, if f is not constant almost everywhere, then one can cut its image into two disjoint sets, whose inverse images have a nontrivial measure and are invariant sets.

Remark. The T-invariant measure of a uniquely ergodic system (X, T) is ergodic. Otherwise, if E satisfies $T^{-1}(E) = E$ and $\mu(E) > 0$, then the measure defined over the Borel sets by: $\mu(B) = \frac{\mu(B \cap E)}{\mu(E)}$ is another T-invariant probability measure.

Ergodic transformations have the following interesting property, which is a measure-theoretical counter-part of the property stated in Exercise 1.4.1 (for a proof, see for instance [445], Sec. 1.5):

Theorem 1.4.8. *Let $(X, T, \mu, \mathcal{B})$ be a measure-theoretic dynamical system. The map T is ergodic if and only if every T-invariant measurable function $f : X \to X$ is constant almost everywhere.*

Exercise 1.4.9. Prove that if $f : X \to \mathbb{R}$ is measurable and $f(Tx) \geq f(x)$ for almost every $x \in X$, then f is almost everywhere constant. *(Hint: introduce sets of the form $\{x \in X,\ f(x) \geq c\}$, with $c \in \mathbb{R}$.)*

Group rotations. In the case of rotations on compact metric groups, one has furthermore the following equivalence (for a proof, see for instance [445]):

Theorem 1.4.10. *Let G be a compact metric group, and let $T : G \to G$, $x \mapsto ax$ be a rotation of G. The following properties are equivalent:*

1. *T is minimal;*
2. *T is ergodic;*
3. *T is uniquely ergodic;*
4. *$\{a^n,\ n \in \mathbb{N}\}$ is dense in G.*

Remark. We thus deduce by using the density of $\{a^n,\ n \in \mathbb{N}\}$ that the ergodicity of T implies that G is abelian.

Isomorphism. Let us introduce an equivalent of the notion of topological conjugacy for measure-theoretic dynamical systems. The idea here is to remove sets of measure zero in order to conjugate the spaces via an invertible measurable transformation. For a nice exposition of connected notions of isomorphism, see [445].

Two measure-theoretic dynamical systems $(X_1, T_1, \mu_1, \mathcal{B}_1)$ and $(X_2, T_2, \mu_2, \mathcal{B}_2)$ are said to be *measure-theoretically isomorphic* if there exist two sets of full measure $B_1 \in \mathcal{B}_1$, $B_2 \in \mathcal{B}_2$, a measurable map $f : B_1 \to B_2$ called *conjugacy map* such that

- the map f is one-to-one,
- the reciprocal map of f is measurable,
- f conjugates T_1 and T_2 over B_1 and B_2,
- μ_2 is the image $f_*\mu_1$ of the measure μ_1 with respect to f, that is,

$$\forall B \in \mathcal{B}_2,\ \mu_1(f^{-1}(B)) = \mu_2(B).$$

If the map is f is only onto, then $(X_2, T_2, \mu_2, \mathcal{B}_2)$ is said to be a *measure-theoretic factor* of $(X_1, T_1, \mu_1, \mathcal{B}_1)$.

A nice way to obtain explicit factors is to study the spectral properties of the system. The aim of the next section is to introduce this spectral aspect.

1.5 Spectral theory

With any measure-theoretic dynamical system (X, T, μ), one can associate a Hilbert space: the space $\mathcal{L}^2(X, \mu)$. The aim of this section is to introduce a unitary operator acting on this Hilbert space and show how to use *spectral theory* to give us insight into the dynamics of (X, T, μ). Good references on the subject are [122, 194, 309, 340, 445]. See also for more details, Sec. 5.2.2 in Chap. 5.

1.5.1 First properties

Unitary operator. Let (X, T, μ) be a measure-theoretic dynamical system, where T is *invertible* (that is, T^{-1} is also measurable and measure-preserving). One can associate with it in a natural way an *operator* U acting on the Hilbert space $\mathcal{L}^2(X, \mu)$ defined as the following map:

$$\begin{aligned} U : \mathcal{L}^2(X, \mu) &\to \mathcal{L}^2(X, \mu) \\ f &\mapsto f \circ T. \end{aligned} \tag{1.1}$$

Since T preserves the measure, the operator U is easily seen to be a *unitary operator*. Note that the surjectivity of the operator U comes from the invertibility of the map T.

The *eigenvalues* of (X, T, μ) are defined as being those of the map U. By abuse of language, we will call *spectrum* the set of eigenvalues of the operator U. It is a subgroup of the unit circle. The *eigenfunctions* of (X, T, μ) are defined to be the eigenvectors of U. Let us note that the map U always has 1 as an eigenvalue and any non-zero constant function is a corresponding eigenfunction.

The spectrum is said to be *irrational* (respectively *rational*) if it is included in $\exp(2i\pi\mathbb{R} \setminus \mathbb{Q})$ (respectively in $\exp(2i\pi\mathbb{Q})$).

Eigenvalues and ergodicity. One can deduce ergodic information from the spectral study of the operator U.

In particular, T is ergodic if and only if the eigenvalue one is a simple eigenvalue, that is, if all eigenfunctions associated with 1 are constant almost everywhere. Indeed, this follows from Theorem 1.4.8 and from the following remark: if a Borel set E of nontrivial measure satisfies $T^{-1}E = E$, then 1_E is a nonconstant eigenfunction associated with one.

Furthermore, if T is ergodic, every eigenfunction is simple and every eigenfunction is of constant modulus. Indeed, if f is an eigenfunction for the eigenvalue β, $|f|$ is an eigenfunction for the eigenvalue $|\beta| = 1$ and hence is a constant. If f_1 and f_2 are eigenfunctions for β, $|f_2|$ is a nonzero constant, and f_1/f_2 is an eigenfunction for 1 and hence a constant (see also Chap. 5, Sec. 5.2.2).

Structure of the spectrum. The spectrum is said to be *discrete* if $\mathcal{L}^2(X,\mu)$ admits an Hilbert basis of eigenfunctions, that is, if the eigenfunctions span $\mathcal{L}^2(X,\mu)$. Hence, if $\mathcal{L}^2(X,\mu)$ is separable (this is the case for instance if X is a compact metric set), then there are at most a countable number of eigenvalues.

If the spectrum contains only the eigenvalue 1, with multiplicity 1, the system is said to be *weakly mixing* or to have a *continuous spectrum*. This implies in particular that T is ergodic. We will see in Sec. 1.5.2 where the terminology "continuous spectrum" comes from.

We say that the system has a *partially continuous spectrum* if the system neither has discrete spectrum, nor is weakly mixing.

Spectrum of group rotations. Consider a rotation $T : x \mapsto ax$ on a compact abelian group G equipped with the Haar measure μ. Let $\hat{G}$ denote the group of *characters* of the group G, that is, the continuous group morphisms $\gamma : G \to \mathbb{C}^*$. Note that a character γ takes its values in a compact set of the unit circle $\mathbb{U} := \{z \in \mathbb{C},\ |z| = 1\}$ of the complex plane.

Example 1.5.1. We have $\hat{\mathbb{U}} = \{z \mapsto z^n,\ n \in \mathbb{Z}\}$ and $\hat{\mathbb{U}}$ is isomorphic to $\mathbb{Z}$. The group $\hat{\mathbb{T}}^d = \{(\alpha_1, \ldots, \alpha_d) \mapsto \sum_{j=1}^{d} k_j\alpha_j,\ (k_1, \ldots, k_d) \in \mathbb{Z}^d\}$ and $\hat{\mathbb{T}}^d$ is isomorphic to $\mathbb{Z}^d$. For a proof, see [445].

Note that if G is a compact metric space (this is the situation we will be in throughout this book), then $\mathcal{L}^2(G,\mu)$ is separable, and $\hat{G}$ is at most countable. One thus can prove that the set of characters form an Hilbert basis of $\mathcal{L}^2(G,\mu)$. We deduce that λ is an eigenvalue for U if and only if there exists $\gamma \in \hat{G}$ such that $\lambda = \gamma(a)$ (recall that a denotes the translation vector). Hence, any group rotation (G, T, μ) has discrete spectrum.

Exercise 1.5.2. Show that (see also Chap. 5 Prop. 5.2.18):

1. the spectrum of the rotation R_α on the one-dimensional torus $\mathbb{T}$ with irrational angle α is $\exp(2i\pi\mathbb{Z}\alpha) = \{e^{2i\pi\, k\alpha};\ k \in \mathbb{Z}\}$;
2. the spectrum of the addition of 1 on the group $\mathbb{Z}/r\mathbb{Z}$ $(r \geq 1)$, is $\exp(2i\pi\frac{\mathbb{Z}}{r}) = \{e^{2i\pi\, k/r};\ k \in \mathbb{Z}\}$;
3. the spectrum of the rotation of angle $(\alpha_1, \ldots \alpha_d)$ on the d-dimensional torus $\mathbb{T}^d$ is $\exp(2i\pi \sum_j \mathbb{Z}\alpha_j) = \{e^{2i\pi\, \sum_j k_j\alpha_j};\ k_j \in \mathbb{Z}\}$.

An invariant for measure-theoretical isomorphism. One of the main questions in ergodic theory is to decide when two dynamical systems are isomorphic. A first answer to this problem can be provided by looking at some natural conjugacy invariants. A property of a topological (respectively measure-theoretic) dynamical system is a topological (respectively measure-theoretic) *invariant* if given (X_1, T_1) which has this property and (X_2, T_2) which is topologically conjugate (respectively measure-theoretically isomorphic) to (X_1, T_1), then (X_2, T_2) has this property. An invariant is said *complete* if (X_1, T_1) and (X_2, T_2) are conjugate if and only if they have this same invariant.

Exercise 1.5.3. Which of the following properties is a topological invariant or a measure-theoretic invariant: unique ergodicity, minimality, weak-mixing, ergodicity?

It is usually difficult to find complete invariants. Nevertheless, Von Neumann proved that, restricting to invertible ergodic system with discrete spectrum, the spectrum is a complete measure-theoretic invariant (see a proof in [194]).

Theorem 1.5.4 (Von Neumann). *Two invertible and ergodic transformations with identical discrete spectrum are measure-theoretically isomorphic.*

Any invertible and ergodic system with discrete spectrum is measure theoretically isomorphic to a rotation on a compact abelian group, equiped with the Haar measure.

The proof of the second assertion of the theorem is based on the following idea: consider the group Λ of eigenvalues of U endowed with the discrete topology; the group of the rotation will be the character group of Λ, which is compact and abelian.

Topological discrete spectrum. This theorem has its counterpart in topological terms. Let (X,T) be a topological dynamical system, where T is an homeomorphism. A nonzero complex-valued continuous in $\mathcal{C}(X)$ is an *eigenfunction* for T if there exists $\lambda \in \mathbb{C}$ such that $\forall x \in X,\ f(Tx) = \lambda f(x)$. The set of the eigenvalues corresponding to those eigenfunctions is called the *topological spectrum* of the operator U. If two systems are topologically conjugate they have the same group of eigenvalues. The operator U is said to have *topological discrete spectrum* if the eigenfunctions span $\mathcal{C}(X)$.

Example 1.5.5. An ergodic rotation on a compact metric abelian group has topological discrete spectrum.

The theorem of Von Neumann becomes (see for instance [445]):

Theorem 1.5.6. *Two minimal topological dynamical systems (X_1, T_1) and (X_2, T_2) with discrete spectrum, where T_1 and T_2 are homeomorphisms, are topologically conjugate if and only if they have the same eigenvalues.*

Any invertible and minimal topological dynamical system with topological discrete spectrum is topologically conjugate to a minimal rotation on a compact abelian metric group.

1.5.2 Spectral type

For systems which do not have a discrete spectrum, more precise invariants solve the problem of isomorphism: the most important of them are the spectral type introduced below and the spectral multiplicity. These notions are illustrated in Sec. 5.2.2.

Fourier coefficients. Let μ be a probability measure defined over $\mathbb{T}$. The *Fourier coefficients* $(\hat{\mu}(n))_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ of μ are defined by:

$$\forall n \in \mathbb{Z},\ \hat{\mu}(n) = \int_{\mathbb{T}} exp(2i\pi nt)d\mu(t).$$

Positive definite sequence. A sequence $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is *positive definite* if for any complex sequence $(z_i)_{i\geq 1}$, we have:

$$\forall n \geq 1,\ \sum_{1\leq i,j\leq n} z_i\overline{z_j}a_{i-j} \geq 0.$$

In particular, the sequence $(\hat{\mu}(n))_{n\in\mathbb{Z}}$ is positive definite. Conversely, one can associate with any positive definite sequence a positive Borel measure on the torus (see for instance [235]).

Theorem 1.5.7 (Bochner-Herglotz theorem). *Any positive definite sequence $(a_n)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ is the Fourier transform of a finite positive Borel measure on $\mathbb{T}$.*

Consider now a measure-theoretical dynamical system (X, T, μ) and let U be the unitary operator associated with it.

Exercise 1.5.8. Let $f \in \mathcal{L}^2(X, \mu)$. Prove that the sequence $(U^n f, f)_{n\in\mathbb{Z}}$ is positive definite (where $(U^n f, f) = \int_X U^n f.\overline{f}d\mu$).

Spectral type. Let $f \in \mathcal{L}^2(X, \mu)$. The *spectral type* ϱ_f of f is the finite positive Borel measure ϱ_f on the torus $\mathbb{T}$ defined by

$$\forall n \in \mathbb{Z},\ \hat{\varrho_f}(n) = (U^n f, f) = \int_X f \circ T^n.\overline{f}d\mu.$$

Its total mass is $\hat{\varrho_f}(0) = ||f||_2$. For examples of methods of computation of spectral types, see Chap. 5.

Exercise 1.5.9. Let f be an eigenfunction with norm 1 associated with the eigenvalue $e^{2i\pi\lambda}$. Prove that $\varrho_f = \delta_\lambda$.

Remark. Let us recall that the spectrum is said continuous if 1 is the only eigenvalue of T and the only eigenfunctions are the (almost everywhere) constants. In this case if $f \in \mathcal{L}^2(X, \mu)$ is orthogonal to the constant function 1, then ϱ_f is continuous.

Maximal spectral type. Let μ, ν be two Borel measures. Recall that μ is said *absolutely continuous* with respect to ν ($\mu << \nu$) if $\nu(E) = 0$ implies $\mu(E) = 0$, for any Borel set E. Two measures μ and ν are said to be *equivalent* if $\mu << \nu$ and $\nu << \mu$.

Theorem 1.5.10. *Let (X,T,μ) be a measure-theoretic dynamical system. There exists a probability measure ϱ defined over $\mathbb{T}$ characterized by the following property:*

$$\forall f,\ \varrho_f << \varrho.$$

Let ν be a positive measure on $\mathbb{T}$, if $\nu << \varrho$, then there exists f such that $\nu = \varrho_f$.

By definition, this measure ϱ is unique up to equivalence. The measure ϱ is called the maximal spectral type *of U.*

Remark. We thus have that $e^{2i\pi\lambda}$ is an eigenvalue of U if and only if $\varrho\{\lambda\} \neq 0$. For more details, see [194, 340, 445]. Note that $\varrho\{1\} \neq 0$.

The above theorem comes from a more general theorem stating that the Hilbert space $\mathcal{L}^2(X,\mu)$ can be uniquely decomposed as an orthogonal sum of cyclic spaces H_i, where the spaces H_i are stable under the action of U. A *cyclic space* is the closure of a set $\{U^n(f), n \in \mathbb{Z}\}$, for an element f in $\mathcal{L}^2(X,\mu)$. We say that U has a spectrum of *multiplicity* at most k if $\mathcal{L}^2(X,\mu)$ is the direct sum of k cyclic spaces; if $k=1$, we say that the spectrum is *simple.* For more details, see [340] or Sec. 5.2.2. In particular, Proposition 5.2.21 deals with the spectrum of the Morse system which is proved to be nondiscrete and simple. An example of system having a spectrum with multiplicity at most four, and functions having a spectral type equivalent to the Lebesgue measure is given in Sec. 5.3.2.

1.5.3 Correlation measures of a sequence

There is an interesting way to associate with an infinite sequence u with values in a finite alphabet, a family of finite positive measures on $\mathbb{T}$. We will see that such measures can be considered as spectral types of the associated dynamical system (X_u, S).

Correlation measure. Let u be a sequence with values in the finite alphabet $\mathcal{A} \subset \mathbb{C}$. Let us note that the following definitions and results also hold for bounded complex sequences.

A *correlation sequence* of u is any cluster point of the sequence $(\gamma_N)_{N\geq 1}$ with values in $\mathbb{C}^{\mathbb{N}}$, where γ_N is defined as follows for a fixed N:

$$\gamma_N : \mathbb{N} \to \mathbb{C},\ k \mapsto \frac{1}{N}\sum_{n<N} u_{n+k}\overline{u_n}.$$

By compacity, if γ is a correlation sequence of u, there exists a sequence $(N_j)_{j\geq 1}$ such that

$$\forall k \in \mathbb{N},\ \gamma_k = \lim_{j\to+\infty} \frac{1}{N_j}\sum_{n<N_j} u_{n+k}\overline{u_n}.$$

Let us extend the definition of γ to the negative numbers by introducing $\tilde{\gamma}$ defined over $\mathbb{Z}$ as follows: $\forall k \in \mathbb{N},\ \tilde{\gamma}_{-k} = \overline{\gamma_k}$. The sequence γ is positive definite.

A *correlation measure* of u associated with the correlation sequence γ is the finite positive measure defined on $\mathbb{T}$, the Fourier transform of which is given by the positive definite sequence γ.

Let us note that a correlation measure μ is not necessarily a probability measure. Its total mass is equal to $\lim_{j\to+\infty} \frac{1}{N_j} \sum_{n<N_j} |u_n|^2$.

Remark. If $\forall k \in \mathbb{N},\ \lim_{N\to+\infty} \frac{1}{N} \sum_{n<N} u_n \overline{u_{n+k}}$ exists, then the sequence u admits a unique correlation measure.

Exercise 1.5.11. Consider the dynamical system (X_u, S) associated with the sequence u. Let μ be an S-invariant measure which is a cluster point for the weak-star topology of the sequence of probability $(\frac{1}{N} \sum_{n<N} \delta_{S^n u})_{N\geq 1}$. Consider, for a fixed $m \in \mathbb{N}$, the map $\pi_m : X_u \to \mathbb{C},\ x \mapsto x_m$. Prove that the spectral type of π_m is a correlation measure, and that it does not depend on m.

For examples of correlation measures, see Chap. 2, where the correlation measure for the Morse (Sec. 2.1.3) and the Rudin-Shapiro (Sec. 2.2.2) sequence are explicitly computed. See also [79, 340, 348] and the nice surveys [22, 342].

1.6 Factors of substitutive dynamical systems

In this section, we illustrate the important connection between the eigenvalues of a measure theoretical dynamical system and its translation factors. We will then apply these results to substitutive dynamical systems, by alluding to the fact that the spectrum of a dynamical system appears to be an efficient way to determine the largest rotation component. These results will be needed in particular in Chap. 7.

1.6.1 Translation factors on a torus or on a finite group

First, let us present the relationship between the eigenvalues of a measure-theoretical dynamical system and its translation factors on a torus or a finite group. We will next study in the following section examples of factor rotations on inverse limit groups, such as the group of p-adic numbers or the 2-adic solenoid.

Lemma 1.6.1. *The spectrum of a measure-theoretic dynamical system contains the spectrum of any of its measure-theoretic factors.*

Proof. Let (X_1, T_1, μ_1) be a factor of (X, T, μ). Let f be the conjugacy map. Let g_1 be an eigenfunction of (X_1, T_1, μ_1) for the eigenvalue λ. We have $g_1 \circ T_1 = \lambda\, g_1$. Let $g = g_1 \circ f$. Then $g \circ T = g_1 \circ f \circ T = g_1 \circ T_1 \circ f = \lambda\, g_1 \circ f = \lambda\, g$. Thus, g is an eigenfunction of (X, T, μ) for λ. ∎

In the other direction, the following lemmas state that the knowledge on the existence of some eigenvalues of a dynamical system allows the determination of some rotation factors:

Lemma 1.6.2. *A rotation R_α of irrational angle α on the one-dimensional torus $\mathbb{T}$, that we denote $(\mathbb{T}, R_\alpha)$, is a measure-theoretic factor of an ergodic dynamical system (X, T, μ) if and only if $e^{2i\pi\,\alpha}$ is an eigenvalue of (X, T, μ).*

Proof. The necessary condition is a consequence of Lemma 1.6.1.

Let g be an eigenfunction of (X, T, μ) for the eigenvalue $e^{2i\pi\,\alpha}$. We will prove that the rotation $T_\alpha : \mathbb{U} \to \mathbb{U}$, $x \mapsto e^{2i\pi\,\alpha} x$ is a measure-theoretic factor of (X, T, μ). This is equivalent (since $(\mathbb{U}, T_\alpha)$ and $(\mathbb{T}, R_\alpha)$ are conjugate) to the fact that $(\mathbb{T}, R_\alpha)$ is a factor of (X, T, μ).

By ergodicity, g is of constant modulus, which can be chosen equal to 1. We thus have $g : X \to \mathbb{U}$, with $g \circ T = e^{2i\pi\,\alpha} g = T_\alpha \circ g$. It remains to prove that g is onto.

We have $\mu(g^{-1}\mathbb{U}) = \mu(X) = 1 \neq 0$, and the measure $\mu_* g$ on $\mathbb{U}$ is nonzero and invariant under T_α. By unique ergodicity of T_α, this measure is nothing else than the Haar measure. Since $g(X)$ is invariant under T_α and of nonzero measure, we get $g(X) = \mathbb{U}$, by ergodicity of T_α, and $(\mathbb{U}, T_\alpha)$ is a measure-theoretic factor of (X, T, μ). ∎

Exercise 1.6.3. Prove that a minimal rotation R_α on the torus $\mathbb{T}^d$, with $\alpha = (\alpha_1, \ldots, \alpha_d)$, is a measure-theoretic factor of an ergodic dynamical system (X, T, μ) if and only if for every $1 \le i \le d$, $e^{2i\pi\alpha_i}$ is an eigenvalue of (X, T, μ).

Similar results are obtained for the addition of 1 on the finite group $\mathbb{Z}/r\mathbb{Z}$, where r is a positive integer, this map being uniquely ergodic.

Lemma 1.6.4. *The addition of 1 on the finite group $\mathbb{Z}/r\mathbb{Z}$, that we denote $(\mathbb{Z}/r\mathbb{Z}, 1)$, is a measure-theoretic factor of the ergodic dynamical system (X, T, μ) if and only if $e^{2i\pi/r}$ is an eigenvalue of (X, T, μ).*

Proof. The necessary condition comes from Lemma 1.6.1.

Suppose that g is an eigenfunction of (X, T, μ) for $\exp(2i\pi/r)$, normalized so that it is of modulus 1. For all $0 \le \varepsilon \le 1/p^n$, the set $X_\varepsilon = g^{-1}(\cup_{k\in\mathbb{Z}} \exp\{2i\pi(k/r + [0, \varepsilon[)\})$ is measurable and invariant through the action of T. Consequently, the measure of this set is either 0 or 1. Moreover, the sequence $(X_\varepsilon)_{0\le\varepsilon\le r}$ is an increasing sequence of sets, X_0 has measure 0, and $X_{1/r}$ has measure 1. Thus, there exists a lowerbound for the set of the reals ε such that X_ε has measure 1. This real is denoted by ε_0.

Then $\cup_{\varepsilon<\varepsilon_0} X_\varepsilon = g^{-1}(\cup_{k\in\mathbb{Z}} \exp\{2i\pi(k/r + [0,\varepsilon_0[)\})$ has measure 0, whereas $\cup_{\varepsilon>\varepsilon_0} X_\varepsilon = g^{-1}(\cup_{k\in\mathbb{Z}} \exp\{2i\pi(k/r + [0,\varepsilon_0])\})$ has measure 1. Let $z_0 = e^{2i\pi\varepsilon_0}$. Then $g^{-1}\{z_0 e^{2i\pi(k/r)}, k \in \mathbb{Z}\}$ has measure 1.

This implies that there exists a unique function $f : X \to \mathbb{Z}/r\mathbb{Z}$ such that $g = z_0 \exp(2i\pi/r\, f)$ in $\mathcal{L}^2(X,\mu)$. This map satisfies $f \circ T = 1 + f$. By unique ergodicity of $\mathbb{Z}/p^n\mathbb{Z}$, we prove as in the preceding lemma that f preserves the measure and is onto. ∎

Remark. Let (X,T,μ) be an ergodic dynamical system, and let β be an eigenvalue of the operator U. According to the fact that the argument of β is rational or not, we deduce that the system (X,T,μ) admits as a factor a rotation on a finite group or on the torus $\mathbb{T}$. Let us consider in the next section, the case of a p-adic or of a solenoidal factor.

1.6.2 Solenoidal and p-adic translation factors

The aim of this section is to introduce two classes of translations over a compact abelian group which totally differ from that of the translations studied in the previous section, that is, translations over p-adic groups and p-adic solenoids. Such objects appear in a natural way in the study of the spectral properties of substitutive dynamical systems: for instance, for a precise study of the Morse system, see Chap. 5; we also refer to p-adic rotations and geometric representation of substitutive dynamical systems in Chap. 7. Nevertheless, the reader may consider the following as a digression more than a fundamental notion on substitutions.

p-adic translations. Let us first consider additions on p-adic groups. We will suppose that the reader is already familiar with the notion of p-adic numbers. One can refer to [34, 83, 192, 360] for more details on this notion. See also Chaps. 3 and 5.

Let us recall that the *ring of p-adic integers* $\mathbb{Z}_p$ can be realized as the *inverse limit* of the finite groups $\mathbb{Z}/p^n\mathbb{Z}$. More precisely, for $i \geq 1$, let p^i_{i+1} denote the canonical projection from $\mathbb{Z}/p^{i+1}\mathbb{Z}$ onto $\mathbb{Z}/p^i\mathbb{Z}$. Then we have

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z} = \{(a_i)_{i\geq 1};\ a_i \in \mathbb{Z}/p^i\mathbb{Z},\ \forall i \geq 1,\, p^i_{i+1}(a_{i+1}) = a_i\}.$$

We get an expansion of the elements of $\mathbb{Z}_p$ with respect to the reference sequence $(p^i)_{i\in\mathbb{Z}}$, which tends to zero in $\mathbb{Q}_p$. This expansion, similar to the decimal expansion in $\mathbb{R}$, is called *Hensel expansion*:

$$\forall\, z \in \mathbb{Z}_p,\ z \neq 0,\ \exists\,(b_i)_{i\geq 0} \in \{0,\ldots,p-1\},\ \ z = \sum_{i\geq 1} b_i\, p^i.$$

The *field of p-adic numbers* $\mathbb{Q}_p$, that is, the fraction field $\mathbb{Q}_p$ of $\mathbb{Z}_p$ is the completion of $\mathbb{Q}$ for the *p-adic absolute value* defined on $\mathbb{Q}$ as $|z|_p = p^{-v_p(z)}$, where $v_p(z)$ is the largest power of p which appears in the decomposition of p in prime factors. This absolute value is *non-Archimedean*, that is, for any two elements x, y with $|x| \neq |y|$, then $|x + y| = \max(|x|, |y|)$.

Symbolic representation as an odometer. From a symbolic point of view, the Hensel expansion means that the addition of 1 on $\mathbb{Z}_p$ is topologically conjugate to the following symbolic system called *p-odometer* (see Chap. 5 and [193] for a definition of a generalized odometer associated with a numeration system).

Definition 1.6.5. *The p-odometer is the topological dynamical system* $(\{0, \ldots, p-1\}^{\mathbb{N}}, \chi)$ *where* χ *is defined by:*

$$\begin{aligned} \chi : \{0, \ldots, p-1\}^{\mathbb{N}} &\to \{0, \ldots, p-1\}^{\mathbb{N}} \\ w = w_0 w_1 \ldots &\mapsto v = v_0 v_1 \ldots \end{aligned}$$

with

$$\begin{aligned} &\textit{if } \ \forall\, i \geq 0,\, w_i = p-1, \quad \textit{then } \ \forall\, i \geq 0, v_i = 0; \\ &\textit{if } \ \begin{cases} \forall\, i < i_0,\, w_i = p-1 \\ w_{i_0} \neq p-1, \end{cases} \quad \textit{then } \ \begin{cases} \forall\, i < i_0,\, v_i = 0 \\ v_{i_0} = w_{i_0} + 1, \\ \forall\, i > i_0,\, v_i = w_i. \end{cases} \end{aligned}$$

By Theorem 1.4.10, we obtain that the addition of 1 on the group $\mathbb{Z}_p$, that we denote $(\mathbb{Z}_p, 1)$ is a minimal and uniquely ergodic topological dynamical system.

A basis for the topology of $\mathbb{Q}_p$ consists in the clopen balls $z + p^k\mathbb{Z}_p$, where $k \in \mathbb{N}$ and $z \in \mathbb{Q}_p$ ($\mathbb{Z}_p$ is the closed unit ball of $\mathbb{Q}_p$). In particular, one can note that the Hensel map sends the clopen sets $z + p^k\mathbb{Z}_p$ in $\mathbb{Z}_p$ onto the cylinders in $\{0, \ldots, p-1\}^{\mathbb{N}}$. On this space, the Bernoulli measure is invariant under the action of the odometer. A consequence is that the Haar measure μ_p on $\mathbb{Z}_p$ is explicit; it is defined on the clopen sets by:

$$\forall\, z \in \mathbb{Z}_p, \quad \forall\, k \geq 0, \qquad \mu_p(z + k^n\, \mathbb{Z}_p) = 1/p^k.$$

Spectral theory. The determination of the spectrum of the addition of 1 on $\mathbb{Z}_p$ can be realized as a generalization of that of the eigenvalues of the addition of 1 on a finite group. We left it as an exercise:

Exercise 1.6.6. The spectrum of the addition of 1 on the p-adic group $\mathbb{Z}_p$ is the group generated by

$$\bigcup_{n \geq 1} \exp(2i\pi \frac{\mathbb{Z}}{p^n}) = \{e^{2i\pi\, k/p^n};\, n \geq 1,\, k \in \mathbb{Z}\}.$$

Conversely, the following lemma states that having the elements $\{e^{2i\pi\, k/p^n};\, n \geq 1,\, k \in \mathbb{Z}\}$ as eigenvalues is equivalent with having $\mathbb{Z}_p$ as a factor. See also the proof of Proposition 5.2.18.

Lemma 1.6.7. *The addition of 1 on the p-adic group* $\mathbb{Z}_p$ *is a measure-theoretic factor of the ergodic dynamical system* (X, T, μ) *if and only if, for every* $n \geq 1$, $e^{2i\pi/p^n}$ *is an eigenvalue of* (X, T, μ).

Proof. The condition is necessary from Lemma 1.6.1.

From Lemma 1.6.4, if $e^{2i\pi/p^n}$ is an eigenvalue of (X, T, μ), there exists an eigenfunction $g_n : X \to \mathbb{C}$ (of constant modulus 1) for the eigenvalue $e^{2i\pi/p^n}$, and a constant $z_n \in \mathbb{C}$ such that $g_n = z_n \exp(2i\pi/p^n\, f_n)$, with $f_n : X \to \mathbb{Z}/p^n\mathbb{Z}$ and $f_n \circ T = 1 + f_n$.

Since g_{n+1} is an eigenfunction for $e^{2i\pi/p^{n+1}}$, the map $(g_{n+1})^p$ is also an eigenfunction for $e^{2i\pi/p^n}$. By ergodicity of (X, T, μ) , the eigenvalues are simple, and $(g_{n+1})^p$ and g_n are proportional. So that we get $z_n \exp(2i\pi/p^n\, f_n) = {z_{n+1}}^p \exp(2i\pi/p^n\, f_{n+1})$. In particular, $f_n - f_{n+1}$ is constant modulo p^n.

It will be recalled that p^i_{i+1} is the canonical projection of $\mathbb{Z}/p^{i+1}\mathbb{Z}$ onto $\mathbb{Z}/p^i\mathbb{Z}$. Modifying the map f_{n+1} by subtracting the constant (we thus get $f_n = f_{n+1}$ modulo p^n), we obtain a sequence of maps $f_n : X \to \mathbb{Z}/p^n\mathbb{Z}$ such that $f_n \circ T = 1 + f_n$ and $p^n_{n+1} \circ f_{n+1} = f_n$, for every positive integer n.

Let $f : X \to \prod_{n\geq 1} \mathbb{Z}/p^n\mathbb{Z}$ be the map defined by $f(x) = (f_n(x))_{n\geq 1}$. According to what precedes, we have for all $i \geq 1$, $p^i_{i+1}(f_{i+1}(x)) = f_i(x)$. Thus, $f(X)$ is included in the inverse limit of $\mathbb{Z}/p^n\mathbb{Z}$, that is, $\mathbb{Z}_p$. Moreover, we have $f \circ T = 1 + f$. By unique ergodicity of the addition of 1 on $\mathbb{Z}_p$, we conclude just as in Lemma 1.6.2 that $\mathbb{Z}_p$ is a measure-theoretic factor of (X, T, μ). ■

The 2-adic solenoid. Solenoids usually appear in dynamical systems as examples of attractors (see for instance the Smale attractor in [234], or [145]). See also [247] for a purely symbolic approach in connection with the notion of dimension group. We will study some properties of the 2-adic solenoid solenoid through an exercise. Everything extends in a natural way to the p-adic solenoid.

The 2-*adic solenoid* $\mathcal{S}_2$ can be realized as the following inverse limit space:

$$\mathcal{S}_2 = \{\mathbf{x} = (x_k)_{k\geq 1} |\ \forall k \geq 1,\quad x_k \in \mathbb{T},\quad x_k = 2x_{k+1} \bmod 1\}\,.$$

Exercise 1.6.8. Let us first prove some basic properties of the 2-adic solenoid:

1. Prove that $\mathcal{S}_2$ is nonempty by giving one of its elements.
2. Give two elements $\mathbf{a}$ and $\mathbf{b}$ in $\mathcal{S}_2$ such that $a_2 = b_2$ and $a_3 \neq b_3$.
3. Prove that the projections $\pi_k : \mathcal{S}_2 \to \mathbb{T}$, $\mathbf{x} \mapsto x_k$ are onto.
4. Prove that $\mathcal{S}_2$ is a compact abelian group.

Let us study now the minimality of translations on $\mathcal{S}_2$. Let $\mathbf{a} \in \mathcal{S}_2$ such that the addition of a_1 on $\mathbb{T}$ is minimal, that is, any real number representing a_1 is irrational.

1. Prove that the addition of any coordinate a_k on $\mathbb{T}$ is also minimal.
2. Prove that the addition of $\mathbf{a}$ on $\mathcal{S}_2$ is onto.
3. Prove that the addition of $\mathbf{a}$ on $\mathcal{S}_2$ is minimal.

Let us finally study some spectral properties of the minimal addition of $\mathbf{a}$ on the 2-adic solenoid.

1. Prove that the spectrum of the addition of $\mathbf{a}$ on $\mathcal{S}_2$ contains all the numbers $e^{2i\pi a_k}$, $k \in \geq 1$, and that it is exactly equal to the group generated by
$$\cup_{k\geq 1} \exp(2i\pi\mathbb{Z}[\alpha/2^k]) = \{e^{2i\pi \frac{m\alpha}{2^k}},\ k \geq 1, m \in \mathbb{Z}\}.$$
(Hint: a description of the characters of $\mathcal{S}_2$ is given in [247].)
2. Let α be an irrational number. Let $\mathbf{a} = (\alpha, \alpha/2, \dots, \alpha/2^k, \dots)$. Prove that the addition of $\mathbf{a}$ on $\mathcal{S}_2$, that we denote $(\mathcal{S}_2, R_{\mathbf{a}})$, is a measure-theoretic factor of the ergodic dynamical system (X, T, μ) if and only if for every $k \geq 1$, $e^{2i\pi \frac{\alpha}{2^k}}$ is an eigenvalue of (X, T, μ).

Example. The spectrum of the addition of $(\sqrt{2}/2^k)_{k\geq 1}$ on $\mathcal{S}_2$ is the group generated by $\cup_{k\geq 1} \exp(2i\pi\mathbb{Z}[\sqrt{2}/2^k])$.

1.6.3 Application to the spectrum.

Let (X, T, μ) be an invertible ergodic dynamical system with discrete spectrum. From Theorem 1.5.4, this system is a compact group rotation. Let us collect the results we have obtained on the existence of translation factors in order to describe this rotation:

- if $\exp(2i\pi\alpha)$, with α irrational, is an eigenvalue of (X, T, μ), then the system $(\mathbb{T}, R_\alpha)$ is a factor of (X, T, μ); its spectrum contains $\exp(2i\pi\mathbb{Z}\alpha)$;
- let R_α be a minimal rotation on the torus $\mathbb{T}^d$, with $\alpha = (\alpha_1, \dots, \alpha_d)$; if for every $1 \leq i \leq d$, $\exp(2i\pi\alpha_i)$ is an eigenvalue of (X, T, μ), then the system $(\mathbb{T}^d, R_\alpha)$ is a factor of (X, T, μ); its spectrum contains $\exp(2i\pi \sum \mathbb{Z}\alpha_i)$;
- if $\exp(2i\pi/r)$, with $r \in \mathbb{N}^+$ is an eigenvalue of (X, T, μ), then the system $(\mathbb{Z}/r\mathbb{Z}, 1)$ is a factor of (X, T, μ); its spectrum contains $\exp(2i\pi\frac{\mathbb{Z}}{r})$;
- if all the elements of $\cup_{k\geq 1} \exp(2i\pi\mathbb{Z}[\alpha/p^k])$, with α irrational and p prime, are eigenvalues of (X, T, μ), then the system $(\mathcal{S}_p, R_{\mathbf{a}})$, where $\mathbf{a} = (\alpha, \alpha/p, \dots, \alpha/p^k, \dots) \in \mathcal{S}_p$, is a factor of (X, T, μ);
- if all the elements of $\cup_{k\geq 1} \exp(2i\pi\frac{\mathbb{Z}}{p^k})$, with p prime, are eigenvalues of (X, T, μ), then the system $(\mathbb{Z}_p, 1)$ is a factor of (X, T, μ).

Let us partition now the spectrum into disjoint components which are rationally independent; note that we have lost no information in restricting ourselves in the above enumeration to prime integers, since we require rational independence between the components. By using the results stated in this enumeration, one can construct a minimal compact group rotation which admits exactly the same spectrum: this group will be built as a direct product of tori, finite groups, p-adic groups or solenoids.

1.6.4 Maximal equicontinuous factor

Theorem 1.5.6 has numerous consequences in topological dynamics. Indeed, for any topological dynamical system, one can look at a specific set of topological factors, which contains all the rotation factors, that is, the equicontinuous factors.

Equicontinuity. A topological dynamical system (X, T) is said to be *equicontinuous* if the set of maps $\{T^n,\, n \in \mathbb{Z}\}$ is uniformly equicontinuous, that is:

$$(\forall\, \varepsilon > 0)\,(\exists\, \alpha > 0)\,(\forall\, x, y \in X)\ (\, d(x,y) < \alpha \implies \\ (\forall\, n \in \mathbb{Z}),\, d(T^n(x), T^n(y)) < \varepsilon\,).$$

In particular, any rotation on a compact abelian metric group is equicontinuous. For more details on this notion, see [50, 160, 161, 134].

One interest of this notion is that equicontinuous factors can be compared:

Maximal equicontinuous factor. Every topological dynamical system (X, T) admits a largest equicontinuous factor, (Y, T_1), in the sense that any topological equicontinuous factor of (X, T) is a topological factor of (Y, T_1). This largest equicontinuous factor is called *maximal equicontinuous factor* [161].

Application to the spectrum of substitutive dynamical systems. One interesting point in the spectral study of substitutive dynamical systems is that we do not need to distinguish between topological and measure-theoretic eigenfunctions, and thus between topological and measure-theoretic factors (see also Exercise 7.3.12 in Chap. 7).

Theorem 1.6.9 (B. Host [215]). *Let σ be a primitive and not shift-periodic substitution, and let (X_σ, S) denote the uniquely ergodic topological dynamical system associated with it. Then, any class (in $\mathcal{L}^2$) of eigenfunctions contains a continuous eigenfunction.*

An invertible and uniquely ergodic equicontinuous topological dynamical system has a discrete spectrum [183]. Furthermore every rotation factor of (X_σ, S) is a factor of its maximal equicontinuous factor. As a consequence, the maximal equicontinuous factor of the dynamical system (X_σ, S) is nothing else but a rotation on a compact abelian group, determined by the discrete part of its spectrum.

An important literature is devoted to the determination of the maximal equicontinuous factor of a substitutive dynamical system, with a special emphasis on the discrete spectrum case. A review of these results is given in Chap. 7. The description of the spectrum of some representative examples of substitutions is detailed in Chap. 5.

Part I

Arithmetics and combinatorics of substitutions