Degrees of parallelism in the continuous type hierarchy

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Abstract

A degree of parallelism is an equivalence class of Scott-continuous functions which are relatively definable by each other with respect to the language PCF (a paradigmatic sequential language). We introduce an infinite ("bi-dimensional") hierarchy of degrees. This hierarchy is inspired by representing first order continuous functions as hypergraphs. We assume some familiarity with the language PCF and with its continuous model.

Keywords: Sequentiality; Stability; Strong stability; Logical relations; Sequentiality relations

1. Introduction

A natural notion of relative definability in the continuous type hierarchy is given by the following definition:

Definition 1. Given two continuous functions $f$ and $g$, we say that $f$ is less parallel than $g$ ($f \prec_{\text{par}} g$) if there exists a closed PCF-term $M$ such that $[M]_g = f$.

A degree of parallelism is a class of the equivalence relation associated with the preorder $\leq_{\text{par}}$.

In this paper we deal with degrees of parallelism of first order boolean functions, i.e. of functions which take tuples of booleans as arguments and give booleans as results. PCF-definability for first order functions is fully characterized by the notion of sequentiality (in any of its formulations), and Sieber's sequentiality relations [5] provide a characterization of first order degrees of parallelism. Moreover this characterization is effective: given $f$ and $g$ one can decide if $f \leq_{\text{par}} g$, and recently Stoughton [6] has implemented an algorithm which solves this decision problem. Nevertheless, as far as I know, there is little knowledge of the structure of the partial order $\leq_{\text{par}}$ on first order boolean functions.

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A well known fact is that any continuous function is less parallel than the "parallel or" function (the non-strict binary disjunction) [3], and we also know that any first order stable function is less parallel than the Berry function [2, p. 334], but there is a lack of general results about the poset of degrees, whose structure turns out to be quite complicated, already at first order. Sazonov’s paper [4] may be considered as a first step towards a systematic study of the poset of degrees of parallelism.

In this paper we give a geometric account of first order degrees of parallelism, by representing first order functions as hypergraphs which highlight the structure of linearly coherent subsets in the trace of the function. Then we introduce a hierarchy of functions \( f_{(n,m)} \) which has the property that \( f_{(n,m)} \leq \text{par} f_{(n',m')} \) if and only if there exists a morphism from the hypergraph associated with \( f_{(n,m)} \) to the hypergraph associated with \( f_{(n',m')} \).

Throughout the paper PCF terms will be written in uncurried form (as n-ary functions), and some “macros” like a syntactic \( \bot \) and a sequential conjunction \( \wedge \) will be used.

2. Preliminaries

We denote by \( \mathcal{B} \) the flat domain of boolean values \( \{ \bot, \text{true}, \text{false} \} \). Tuples of boolean values are ordered componentwise. Given a continuous function \( f : \mathcal{B}^n \rightarrow \mathcal{B} \), the trace of \( f \) is defined by

\[
\text{tr}(f) = \{ (v, b) \mid v \in \mathcal{B}^n, \ b \in \mathcal{B}, \ b \neq \bot, \ f(v) = b \text{ and } \forall v' < v \ f(v') = \bot \}.
\]

A continuous function \( f : \mathcal{B}^n \rightarrow \mathcal{B} \) is stable if for all \( v_1, v_2 \in \pi_1(\text{tr}(f)), v_1 \nleq v_2 \). A subset \( A = \{ v_1, \ldots, v_k \} \) of \( \mathcal{B}^n \) is linearly coherent (or simply coherent) if

\[
\forall 1 \leq j \leq n \ ( (\forall l_1 \leq l \leq k \ v_i^l \neq \bot) \Rightarrow (\forall l_1, l_2 \ 1 \leq l_1 \leq l_2 \leq k \ v_i^{l_1} = v_i^{l_2})\) or equivalently if for any linear function \( \alpha : \mathcal{B}^n \rightarrow O, \ \alpha(\bigwedge A) = \bigwedge \alpha(A) \), where \( O \) denotes the Sierpinsky domain \( \{ \bot, \top \} \).

The set of coherent subsets of \( \mathcal{B}^n \) is denoted \( \mathcal{C}(\mathcal{B}^n) \).

**Fact 1.** If \( A \in \mathcal{C}(\mathcal{B}^n) \) and \( B \) is an Egli–Milner lower bound of \( A \) (that is if \( \forall x \in A \exists y \in B \ y \leq x \) and \( \forall y \in B \exists x \in A \ y \leq x \) then \( B \in \mathcal{C}(\mathcal{B}^n) \).

**Definition 2.** A continuous function \( f : \mathcal{B}^n \rightarrow \mathcal{B}^m \) is linearly strongly stable (or simply strongly stable) if for any \( A \in \mathcal{C}(\mathcal{B}^n) \)

1. \( f(A) \in \mathcal{C}(\mathcal{B}^m) \).
2. \( f(\bigwedge A) = \bigwedge f(A) \).

---

2 In the sense of [1].

3 A function is linear if it is stable and it commutes with least upper bounds of finite sets.
2.1. Sequential logical relations

Definition 3 (Sieber [5]). For each \( n \geq 0 \) and each pair of sets \( A \subseteq B \subseteq \{1, \ldots, n\} \) let \( S_{n}^{A,B} \subseteq \mathcal{B}^{n} \) be defined by

\[
S_{n}^{A,B}(b_{1}, \ldots, b_{n}) \iff (\exists i \in A \ b_{i} = \bot) \lor (\forall i, j \in B \ b_{i} = b_{j})
\]

An \( n \)-ary logical relation \( R \) is called a sequentiality relation if it is an intersection of relations of the form \( S_{n}^{A,B} \).

A function \( f : \mathcal{B}^{n} \rightarrow \mathcal{B} \) is invariant under the \( m \)-ary logical relation \( R \) if for any

\[
(x_{1}, \ldots, x_{n}) \in R, (x_{2},\ldots, x_{n}) \in R, \ldots, (x_{n}, \ldots, x_{n}) \in R.
\]

one has that

\[
(f(x_{1},x_{2},\ldots,x_{n}), f(x_{2},x_{2},\ldots,x_{n}), \ldots, f(x_{n},x_{n},\ldots,x_{n})) \in R.
\]

Proposition 1. For any \( f : \mathcal{B}^{n} \rightarrow \mathcal{B} \) and \( g : \mathcal{B}^{m} \rightarrow \mathcal{B} \) continuous functions, \( f \preceq_{\text{par}} g \) if and only if for any sequentiality relation \( R \), if \( g \) is invariant under \( R \) then \( f \) is invariant too.

Actually this is a relativized version of the main theorem of [5]: a continuous function of first or second order is PCF-definable if and only if it is invariant under all sequentiality relations.

Fact 2. A set \( A = \{v_{1}, \ldots, v_{k}\} \subseteq \mathcal{B}^{n} \) is linearly coherent if and only if

\[
\forall j \in \{1, \ldots, n\}(v_{1}^{j}, v_{2}^{j}, \ldots, v_{k}^{j}) \in S_{k}^{\{1, \ldots, k\}, \{1, \ldots, k\}}
\]

The next lemma highlights the connections between strong stability and sequentiality relations.

Lemma 1. Let \( f : \mathcal{B}^{n} \rightarrow \mathcal{B} \) be continuous and \( k \geq 2 \). Then the following conditions for \( f \) are equivalent:

1. There is no coherent set \( C \subseteq \pi_{1}(\text{tr}(f)) \) with \( 2 \leq \#C \leq k \).
2. Whenever \( C \subseteq \mathcal{B}^{n} \) is coherent and \( \#C \leq k \), then
   - \( f(C) \) is coherent;
   - \( f(\bigwedge C) = \bigwedge f(C) \).
3. \( f \) is invariant under all relations of the form \( S_{m}^{A,B} \) with \( \#A \leq k \).
4. \( f \) is invariant under the relation \( S_{k+1}^{\{1, \ldots, k\}, \{1, \ldots, k+1\}} \).

Proof. (1) \( \Rightarrow \) (2): given a coherent \( C = \{v_{1}, \ldots, v_{p}\} \subseteq \mathcal{B}^{n} \) with \( p \leq k \), we prove that either there exists a \( v \in \pi_{1}(\text{tr}(f)) \) such that for any \( i \in \{1, \ldots, p\} \) \( v \leq v_{i} \) or there exists an \( i \in \{1, \ldots, p\} \) such that \( f(v_{i}) = \bot \). In both cases it is easy to see that \( f(C) \) is coherent and \( f(\bigwedge C) = \bigwedge f(C) \).
By contradiction, let us suppose that no element of $\pi_1(tr(f))$ is a lower bound of all the elements of $C$, and that any element of $C$ has a lower bound in $\pi_1(tr(f))$. For any $v_i \in C$ let $w_i \in \pi_1(tr(f))$ be such that $w_i \leq v_i$. The set

$$W = \{w \in \pi_1(tr(f)) \mid \exists i \in \{1, \ldots, p\} w = w_i\}$$

is such that $2 \leq \#W \leq p \leq k$, and it is an Egli–Milner lower bound of $C$, hence it is coherent; contradiction.

$(2) \Rightarrow (3)$: given $S_{m,n}^{A,B}$, let $\{v_1, \ldots, v_m\} \subseteq \mathcal{B}^n$ be such that

$$\forall j \in \{1, \ldots, n\} \quad (v_1^j, \ldots, v_m^j) \in S_{m,n}^{A,B}.$$

We have to prove that $(f(v_1), \ldots, f(v_m)) \in S_{m,n}^{A,B}$. We have that the set $\{v_l \mid l \in A\}$ is coherent, and that for any $k \in (B \setminus A)$ $v_k \geq \bigwedge_{l \in A} v_l$ (these two conditions are easily seen to be equivalent to $\forall j \in \{1, \ldots, n\} \quad (v_1^j, \ldots, v_m^j) \in S_{m,n}^{A,B}$). Hence by $(2)$ we get that $(f(v_1), \ldots, f(v_m)) \in S_{m,n}^{A,B}$ is coherent and that $\forall k \in (B \setminus A) \quad f(v_k) \geq \bigwedge_{l \in A} f(v_l)$, that is $(f(v_1), \ldots, f(v_m)) \in S_{m,n}^{A,B}$.

$(3) \Rightarrow (4)$: trivial.

$(4) \Rightarrow (1)$: let us suppose by contradiction that there exists a coherent set $C = \{v_1, \ldots, v_p\} \subseteq \pi_1(tr(f))$ with $2 \leq p \leq k$, and consider the multiset $\{v_{e(1)}, \ldots, v_{e(k)}\}$, where $e : \{1, \ldots, k\} \to \{1, \ldots, p\}$ is defined by $e(i) = ((i-1) \mod p) + 1$. Let $w \in \mathcal{B}^n$ be defined by $w^j = \bigwedge_{v \in C} v^j$, $1 \leq j \leq n$. The $k + 1$ vectors $w_{e(1)}, \ldots, w_{e(k)}$, $w$ are such that

$$\forall j \in \{1, \ldots, n\} \quad (w_{e(1)}^j, \ldots, w_{e(k)}^j, w^j) \in S_{k+1}^{1, \ldots, k, \{1, \ldots, k+1\}}$$

since the $v_{e(i)}^j$ are coherent, and $w^j$ is their greatest lower bound. On the other hand

$$(f(w_{e(1)}), \ldots, f(w_{e(k)}), f(w)) \notin S_{k+1}^{1, \ldots, k, \{1, \ldots, k+1\}}$$

since for any $i \in \{1, \ldots, k\}$ $f(w_{e(i)}) > \bot$ and $f(w) = \bot$ (this last condition being assured by the fact that $k > p \geq 2$). Hence $f$ is not invariant under $S_{k+1}^{1, \ldots, k, \{1, \ldots, k+1\}}$, contradiction.

**Fact 3.** A function $f : \mathcal{B}^n \to \mathcal{B}$ is stable if and only if it satisfies one of the conditions of Lemma 1 for $k = 2$. It is strongly stable if and only if it satisfies one of the conditions of Lemma 1 for all $k \geq 2$.

The following proposition states that strong stability captures the notion of sequential definability, at least at first order.

**Proposition 2.** $f : \mathcal{B}^n \to \mathcal{B}$ is strongly stable, if and only if it is PCF-definable.

This follows from condition (3) of Lemma 1, because for a first order function $f$ it is invariant under the relations $S_{n+1}^{A,B}$ implies invariance under all sequentiality relations.
In the next example we show a typical use of sequentiality relations to explore the poset of degrees of parallelism:

Example 1. We show that there exists a degree of parallelism strictly in between the degree of the weak parallel or function \( \text{Por}^- : \mathcal{B}^2 \rightarrow \mathcal{B} \) and the one of the parallel or function \( \text{Por} : \mathcal{B}^2 \rightarrow \mathcal{B} \), where

\[
\text{tr}(\text{Por}^-) = \{((\text{true}, \bot), \text{true}), ((\bot, \text{true}), \text{true})\}
\]

and

\[
\text{tr}(\text{Por}) = \{((\text{true}, \bot, \text{true}), \text{true}), ((\bot, \text{true}, \text{true}), \text{true}), ((\text{false}, \text{false}, \bot), \text{false})\}.
\]

We are looking for a (first order) function \( f \) such that \( \text{Por}^- <_{\text{par}} f <_{\text{par}} \text{Por} \). Let \( f : \mathcal{B}^4 \rightarrow \mathcal{B} \) be defined by

\[
\text{tr}(f) = \{((\text{true}, \bot, \text{true}, \text{true}), \text{true}), ((\bot, \text{true}, \text{true}, \text{true}), \text{true}), ((\text{false}, \text{false}, \bot, \text{false}), \text{false})\}.
\]

A general method for showing that, for two given first order functions \( g \) and \( g' \), \( g <_{\text{par}} g' \) is to define a PCF-term \( M \) such that \([M]_{g'} = g\), showing that \( g <_{\text{par}} g' \), and a sequentiality relation \( R \) such that \( g \) is invariant under \( R \) and \( g' \) is not, which entails \( g' \leq_{\text{par}} g \) by Proposition 1.

Let us use this method for \( \text{Por}^- <_{\text{par}} f \). Firstly

\[
\text{Por}^- = [\lambda y \lambda x_1 x_2 y(x_1, x_2, \text{true}, \text{true})] f
\]

Then consider the sequentiality relation \( S_4^{\{1, \ldots, 4\}, \{1, \ldots, 4\}} \). It is easy to see that \( \text{Por}^- \) is invariant under it and \( f \) is not.

As for \( f <_{\text{par}} \text{Por} \), it is enough to show that \( \text{Por} \leq_{\text{par}} f \), since any function is less parallel than \( \text{Por} \) [3]. Let us consider the sequentiality relation \( R = S_3^{\{1, 2, 3\}, \{1, 2, 3\}} : \text{Por} \) is not invariant under \( R \) since

\[
(\text{true}, \bot, \text{false}), (\bot, \text{true}, \text{false}) \in R
\]

and

\[
(\text{Por(\text{true}, \bot), Por(\bot, \text{true}), Por(\text{false}, \text{false})) = (\text{true}, \text{true}, \text{false}) \notin R.}
\]

On the other hand \( f \) is invariant under \( R \): let \( v_1, v_2, v_3 \in \mathcal{B}^4 \) be such that \( \forall j \in \{1, \ldots, 4\} \ (v_1^j, v_2^j, v_3^j) \in R \). Then, by Fact 2, \( \{v_1, v_2, v_3\} \in \mathcal{E}(\mathcal{B}^4) \). If \( \exists j \in \{1, 2, 3\} \ f(v_j) = \bot \) then \( (f(v_1), f(v_2), f(v_3)) \in R \), otherwise there exists an Egli–Milner lower bound \( E \) of \( \{v_1, v_2, v_3\} \) in \( \pi_1(\text{tr}(f)) \) such that \( \#E \leq 3 \). It is easy to see that this implies \( f(v_1) = f(v_2) = f(v_3) \), and hence we conclude that \( (f(v_1), f(v_2), f(v_3)) \in R \). □
3. Hypergraphs for boolean functions

We consider a category whose objects are (colored) hypergraphs and whose morphisms are arcs-preserving and coloring-preserving maps:

**Definition 4.** A colored hypergraph \( h = (V_h, A_h, C_h) \) is given by a set \( V_h \) of vertices, a set \( A_h \subseteq \{ A \subseteq V_h \mid \# A \geq 2 \} \) of (hyper)arcs and a coloring function \( C_h : V_h \rightarrow \{ \text{black, white} \} \). A morphism from a hypergraph \( h \) to a hypergraph \( h' \) is a function \( m : V_h \rightarrow V_{h'} \) such that

- for all \( A \subseteq V_h \), if \( A \in A_h \) then \( m(A) \in A_{h'} \).
- for all \( x, x' \in V_h \), \( C_h(x) = C_h(x') \) if and only if \( C_{h'}(m(x)) = C_{h'}(m(x')) \).

**Definition 5.** Let \( f : \mathcal{B}^n \rightarrow \mathcal{B} \) be the \( n \)-ary function defined by \( \text{tr}(f) = \{(v_1, b_1), \ldots, (v_k, b_k)\} \). The hypergraph \( H(f) \) is defined by

- \( V_{H(f)} = \{1, 2, \ldots, k\} \).
- \( A_{H(f)} = \{(i, i_2, \ldots, i_l) \subseteq V_{H(f)} \mid l \geq 2 \text{ and } \{v_{i_1}, v_{i_2}, \ldots, v_{i_l}\} \in \mathcal{C}(\mathcal{B}^n)\} \).
- \( C_{H(f)}(i) = \text{if } b_i \text{ then white else black} \).

**Example 2.** Consider the function \( G : \mathcal{B}^3 \rightarrow \mathcal{B} \) defined by

\[
\text{tr}(G) = \{((\bot, \text{true, } \text{false}), \text{true}), ((\text{false, } \bot, \text{true}), \text{true}), ((\text{true, } \text{false, } \bot), \text{false})\}
\]

and the function \( \text{Por} : \mathcal{B}^2 \rightarrow \mathcal{B} \) defined in Example 1. We have

\[
H(G) = (\{1, 2, 3\}, \{\{1, 2, 3\}\}, C_{H(G)}(1) = C_{H(G)}(2) = \text{white}, C_{H(G)}(3) = \text{black})
\]

\[
H(\text{Por}) = (\{1, 2, 3\}, \{\{1, 2\}, \{1, 2, 3\}\},
C_{H(\text{Por})}(1) = C_{H(\text{Por})}(2) = \text{white}, C_{H(\text{Por})}(3) = \text{black}).
\]

The map \( m : H(G) \rightarrow H(\text{Por}) \) defined by \( m(i) = i \), for \( i = 1, 2, 3 \), is a morphism. A term \( M \) such that \( [M]_{\text{Por}} = G \) is

\[
M = \lambda f \lambda x_1 x_2 x_3 f(t_1, t_2),
\]

where

\[
t_1 = \text{if } x_1 \text{ then } (\text{if } x_2 \text{ then } \bot \text{ else } \text{false} ) \text{ else if } x_3 \text{ then true else } \bot
\]

\[
t_2 = \text{if } x_2 \text{ then } (\text{if } x_3 \text{ then } \bot \text{ else } \text{true} ) \text{ else if } x_1 \text{ then false else } \bot.
\]

**Example 3.** Let \( 3\text{Por} : \mathcal{B}^3 \rightarrow \mathcal{B} \) be defined by

\[
\text{tr}(3\text{Por}) = \{((\text{true, } \bot, \bot), \text{true}), ((\text{false, } \bot, \bot), \text{true}), ((\bot, \bot, \text{true}), \text{true})\}.
\]

The associated hypergraph is

\[
H(3\text{Por}) = (\{1, 2, 3\}, \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\},
C(1) = C(2) = C(3) = \text{white}).
\]
It is easy to see that there exists no morphism \( m : H(3\text{Por}) \to H(\text{Por}) \). Nevertheless, \( 3\text{Por} \preceq_{\text{par}} \text{Por} \), since for instance

\[
3\text{Por} = \lfloor M \rfloor \text{Por},
\]

where

\[
M = \lambda f \lambda x_1 x_2 x_3 \quad \text{if } f( f(x_1, x_2), x_3) \text{ then true else } \bot.
\]

The following are simple properties relating hypergraphs and degrees of parallelism:

**Fact 4.** Let \( f : \mathbb{B}^n \to \mathbb{B} \) be a continuous function: \( f \) is stable if and only if \( H(f) \) has no 2-arc. It is strongly stable if and only \( H(f) \) has no arc.

This follows from Fact 3 and condition (1) of Lemma 1.

**Proposition 3.** Let \( f : \mathbb{B}^n \to \mathbb{B} \) and \( g : \mathbb{B}^m \to \mathbb{B} \) be such that

\[
\min\{\#A | A \in A_{H(f)}\} < \min\{\#A | A \in A_{H(g)}\}.
\]

Then \( f \not\preceq_{\text{par}} g \).

**Proof.** Let \( k = \min\{\#A | A \in A_{H(f)}\} \). By Lemma 1 (conditions (1) and (4)) we know that \( g \) is invariant under the sequentiality relation \( S_{k+1}^{[1, \ldots, k+1]} \), and that \( f \) is not. By Proposition 1 we are done. \( \square \)

4. A hierarchy of degrees

**Definition 6.** Given two natural numbers \( m \geq n \geq 3 \), let \( h_{(n,m)} \) be the hypergraph defined by

\[
h_{(n,m)} = (\{1, 2, \ldots, m\}, \{A \subseteq \{1, 2, \ldots, m\} | \#A \geq n\}, \text{ for all } i \text{ } C(i) = \text{white}).
\]

Given \( h_{(n,m)} \) and \( h_{(n',m')} \), we are interested in determining the conditions under which there exists a morphism \( f : h_{(n,m)} \to h_{(n',m')} \). Since the \( h_{(i,j)} \)'s are monocolored, the only condition to be satisfied for a function \( f : \{1, \ldots, m\} \to \{1, \ldots, m'\} \) to be a morphism is the preservation of arcs. It is easy to see that \( f \) is a morphism if and only if

\[
\max\{\#f^{-1}(B) | B \subseteq \{1, \ldots, m'\} \text{ and } \#B = n' - 1\} < n
\]

since only in that case every arc of \( h_{(n,m)} \) is mapped by \( f \) to an arc of \( h_{(n',m')} \). Hence there exists a morphism from \( h_{(m,n)} \) to \( h_{(m',n')} \) if and only if

\[
n > \min_{f : \{1, \ldots, m\} \to \{1, \ldots, m'\}} \max\{\#f^{-1}(B) | B \subseteq \{1, \ldots, m'\} \text{ and } \#B = n' - 1\}.
\]
It is quite easy to see that one of the functions $f : \{1, \ldots, m\} \to \{1, \ldots, m'\}$ which realize the minimum above is $f_0(i) = ((i - 1) \mod m') + 1$, and that

$$
\max\{\#f_0^{-1}(B) \mid B \subseteq \{1, \ldots, m'\} \text{ and } \#B = n' - 1\}
$$

$$
= \left(\min\{n' - 1, m \mod m'\} \times \left \lfloor \frac{m}{m'} \right \rfloor \right)
$$

$$
+ \left(\max\{0, (n' - 1) - (m \mod m')\} \times \left \lfloor \frac{m}{m'} \right \rfloor \right),
$$

where $[x]$ and $\lfloor x \rfloor$ denote the integer parts of $x + 1$ and $x$, respectively. If we denote this natural number by $C_{m,n',m'}$, we have that there exists a morphism from $h_{(n,m)}$ to $h_{(n',m')}$ if and only if

$$
n > C_{m,n',m'}.
$$

We define now a set of boolean functions $\{f_{(n,m)}\}$ such that for all $n,m$ (with $3 \leq n \leq m$), $H(f_{(n,m)}) = h_{(n,m)}$, and we show that for all $n,m,n',m'$ $f_{(n,m)} \leq_{par} f_{(n',m')}$ if and only if $n > C_{m,n',m'}$. We start by showing how to construct, for any given $h_{(n,m)}$, a boolean function $f$ such that $H(f) = h_{(n,m)}$. The trace of $f$ has to contain $m$ elements, its second projection has to be the singleton \{true\} and for any subset $A$ of the first projection of the trace, $A$ has to be coherent if and only if $\#A \geq n$. Before describing the general method for constructing such a function $f$, let us consider an example:

**Example 4.** The function $f$ described by the following trace (which we represent as a matrix), is such that $H(f) = h_{(3,4)}$ (Table 1):

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Actually a subset of the first projection of this trace is coherent if and only if its cardinality is at least 3, since for any binary subset \{i,j\} of rows there exists a column $l$ such that the elements $(i,l)$ and $(j,l)$ are defined and different. □

For constructing a function $f_{(n,m)}$ whose associated hypergraph is $h_{(n,m)}$ we have just to generalize the idea above: for any subset of less than $n$ rows (and of at least two rows), there must exist a column which makes that subset incoherent. The arity of the function is $\sum_{i=2}^{n-1} \binom{m}{i}$, and in the $j$th column, only elements corresponding to rows in the $j$th subset (with respect to any enumeration whatsoever) will be defined, say by \textit{true} for the first row in that subset and by \textit{false} for the other rows.
Example 5. The following matrix represents $\pi_1(tr(f_{4,4}))$:

\[
\begin{array}{cccccccc}
v_1 &=& \text{true} & \text{true} & \text{true} & \bot & \bot & \bot & \text{true} & \text{true} & \text{true} & \bot \\
v_2 &=& \text{false} & \bot & \bot & \text{true} & \text{true} & \bot & \text{false} & \bot & \text{false} & \text{true} \\
v_3 &=& \bot & \text{false} & \bot & \text{false} & \bot & \text{true} & \text{false} & \bot & \text{false} & \bot \\
v_4 &=& \bot & \bot & \bot & \text{false} & \bot & \text{false} & \bot & \text{false} & \bot & \text{false} & \bot & \text{false} & \text{false} & \text{false} & \text{false} \end{array}
\]

and the following one represents $\pi_1(tr(f_{3,3}))$:

\[
\begin{array}{cccc}
w_1 &=& \text{true} & \text{true} & \bot \\
w_2 &=& \text{false} & \bot & \text{true} \\
w_3 &=& \bot & \text{false} & \text{false} \\
\end{array}
\]

Proposition 4. If $n,m,n',m' \in \omega$ are such that $3 \leq n \leq m$, $3 \leq n' \leq m'$ and $n > C_{m,n',m'}$, then $f_{(n,m)} \leq_{\text{par}} f_{(n',m')}$. 

Proof. Let $k = \sum_{i=2}^{n-1} \binom{m}{i}$ and $k' = \sum_{i=2}^{n'-1} \binom{m'}{i}$, and let $A = \pi_1(tr(f_{(n,m)})) = \{v_1, \ldots, v_m\}$ and $B = \pi_1(tr(f_{(n',m')})) = \{w_1, \ldots, w_{m'}\}$. By hypothesis there exists a function $f : \{1, \ldots, m\} \to \{1, \ldots, m'\}$ which maps every non-singleton coherent subset of $A$ to a non-singleton coherent subset of $B$. Let us consider, for $1 \leq j \leq k'$, the function $g_j : B^k \to B$ defined by

\[
tr(g^j) = \{(v_i, w^j_{f(i)}) | 1 \leq i \leq m \land w^j_{f(i)} \neq \bot \}.
\]

We prove that $g^j$ is strongly stable, for $j \in \{1, \ldots, k'\}$, by using the condition (1) of Lemma 1: let $C = \{v_{i_1}, \ldots, v_{i_l}\} \subseteq \pi_1(tr(g^j))$ be such that $2 \leq l$. Let $H = f\{i_1, \ldots, i_l\}$. By definition of $g^j$ we have that for any $h \in H$, $w^j_h \neq \bot$, hence by construction of $f_{(n',m')}$, we get $\#H \leq n' - 1$, hence $\#C \leq n - 1$ and $C$ is not coherent.

The $g^j$'s are strongly stable, first order functions, hence by Proposition 2, for all $i \leq k'$ there exists a PCF term $t^i(x_1, \ldots, x_k)$ which defines $g^j$. Consider the term

\[
M = \lambda y \lambda x_1 x_2 \ldots x_k \lambda t^1(x_1, \ldots, x_k), t^2(x_1, \ldots, x_k), \ldots, t^{k'}(x_1, \ldots, x_k)).
\]

In order to prove that $[M] f_{(n',m')} = f_{(n,m)}$, we just remark that, by construction,

$$\forall v \in B^k ((\exists j \leq m'(g^j(v), \ldots, g^{k'}(v))) \geq w_j) \iff (\exists i \leq m v \geq v_i \text{ and } f(i) = j).$$

Example 6. Let us apply the construction above to show that $f_{(4,4)} \leq_{\text{par}} f_{(3,3)}$ (remark that $C_{4,3,3} = 3$) (we refer to Example 5).

Any surjective function $f : \{1,2,3,4\} \to \{1,2,3\}$ satisfies the condition of being a morphism from $h_{(4,4)}$ to $h_{(3,3)}$; let us choose for instance

$$f(1) = f(4) = 1, \quad f(2) = 2, \quad f(3) = 3.$$
The corresponding \( g^j \) are defined by

\[
tr(g^1) = \{(v_1, true), (v_2, false), (v_4, true)\},
\]

\[
tr(g^2) = \{(v_1, true), (v_3, false), (v_4, true)\},
\]

\[
tr(g^3) = \{(v_2, true), (v_3, false)\}.
\]

The terms \( t^j \) are essentially sequences of conditional statements: for instance

\[
t_3 = \lambda x_1 \ldots x_{10} \text{ if } x_4 \text{ then (if } x_1 \Lambda x_5 \Lambda \neg x_7 \Lambda \neg x_9 \Lambda x_{10} \text{ then true else } \bot) \]

\[
\text{else (if } \neg x_2 \Lambda x_6 \Lambda \neg x_7 \Lambda \neg x_8 \Lambda \neg x_{10} \text{ then true else } \bot).
\]

The rest of this section is devoted to a proof that the condition \( n > C_{m,n',m'} \) is indeed necessary for having \( f_{(n,m)} \leq_{par} f_{(n',m')} \):

**Proposition 5.** If \( n, m, n', m' \) are such that \( 3 < n < m, 3 < n' < m' \) and \( n < C_{m,n',m'} \), then \( f_{(n,m)} \not\leq_{par} f_{(n',m')} \).

**Proof.** By Proposition 1 it is sufficient to define a sequential logical relation \( R \) such that \( f_{(n,m)} \) is invariant with respect to \( R \) and \( f_{(n,m)} \) is not.

The first projection of \( tr(f_{(n,m)}) \) is

\[
\pi_1(tr(f_{(n,m)})) = \{(x_1, \ldots, x_1^{\sum_{i=2}^{n-1}(\gamma)}, \ldots, (x_m, \ldots, x_m^{\sum_{i=2}^{n-1}(\gamma)})\}.
\]

Remark that, by definition, any "column" of the first projection of \( tr(f_{(n,m)}) \), i.e. any tuple

\[
\{(x_1^i, x_2^i, \ldots, x_m^i)\}_{1 \leq i \leq \sum_{i=2}^{n-1}(\gamma)}
\]

contains at most \( n - 1 \) components different from \( \bot \). Hence it is easy to see that \( f_{(n,m)} \) is not invariant with respect to the \((m + 1)\)-ary sequential logical relation

\[
R = \bigcap_{A \subseteq \{1, 2, \ldots, m\}, |A| \geq n} S_{m+1}^{m,A} \cap (S_{m+1}^{1,\ldots,m \cup \{m+1\}}, m)
\]

since the tuples

\[
\{(x_1^i, x_2^i, \ldots, x_m^i, \bot)\}_{1 \leq i \leq \sum_{i=2}^{n-1}(\gamma)}
\]

are in \( R \), and the application of \( \underbrace{f_{(n,m)}, \ldots, f_{(n,m)}}_{m+1} \) to those tuples yields the tuple

\[
(\text{true, true, \ldots, true, } \bot)
\]

which is not in \( R \).
If we prove that \( f_{(n',m')} \) is invariant with respect to \( R \) we are done. By *reductio ad absurdum*, let us suppose that \( f_{(n',m')} \) is not invariant. Then there exist \( m+1 \) tuples
\[
\begin{align*}
v_1 &= (y_1^1, \ldots, y_1^{m'}) \\
v_2 &= (y_2^1, \ldots, y_2^{m'}) \\
&\vdots \\
v_{m+1} &= (y_{m+1}^1, \ldots, y_{m+1}^{m'})
\end{align*}
\]
such that
\[
\forall j \quad 1 \leq j \leq m' \quad \sum_{i=2}^{m'} (\binom{m'}{i}) (y_1^j, \ldots, y_{m+1}^j) \in R
\]
and
\[
(f_{(n',m')}(v_1), \ldots, f_{(n',m')}(v_{m+1})) \not\in R.
\]
It is easy to see that this is the case if and only if
\[
f_{(n',m')}(v_1) = f_{(n',m')}(v_2) = \cdots = f_{(n',m')}(v_m) = \text{true} \quad \text{and} \quad f_{(n',m')}(v_{m+1}) = \perp.
\]
Hence for any \( 1 \leq i \leq m \) there exists an element \( w_f(i) \) of the first projection of \( \text{tr}(f_{(n',m')}) \) such that \( v_i \geq w_f(i) \),\(^4\) for some function \( f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, m'\} \). Since \( n \leq C^{m.n',m'} \), there exists a set \( A \subseteq \{1, \ldots, m\} \) such that \( \#A \geq n \) and \( \#f(A) \leq n' - 1 \). Let \( A \) be a maximal set with that property.

If \( f(A) = \{1\} \) is a singleton, then \( \#f(A) < n' - 1 \), hence \( A = \{1, \ldots, m\} \) because \( A \) is maximal. This means \( v_i \geq w_I \) for \( i = 1, \ldots, m \) and hence also \( v_{m+1} \geq w_I \) since \( (y_1^j, \ldots, y_{m+1}^j) \in R \subseteq S_{m+1}^{(1, \ldots, m'), (1, \ldots, m')} \). This implies \( f_{(n',m')}(v_{m+1}) \geq f_{(n',m')}(v_I) = \text{true} \), contradiction.

If \( \#f(A) \geq 2 \), then \( C = \text{def} \{ w_f(i) \mid i \in A \} \subseteq \pi_1(\text{tr}(f_{(n',m')})) \) cannot be coherent because \( 2 \leq \#C \leq n' - 1 \). Hence also \( C' = \text{def} \{ v_i \mid i \in A \} \) cannot be coherent because \( C \) is an Egli-Milner lower bound of \( C' \). This contradicts \( (y_1^j, \ldots, y_{m+1}^j) \in S_{m+1}^{(1, \ldots, m'), (1, \ldots, m')} \) for all \( f \).

Hence \( f_{(n,m)} \leq_{\text{par}} f_{(n',m')} \) if and only if \( n > C^{m.n',m'} \), It is easy to see that this hierarchy is non-trivial:

**Proposition 6.** If \( f_{(n,m)} \leq_{\text{par}} f_{(n',m')} \) and \( f_{(n',m')} \leq_{\text{par}} f_{(n,m)} \) then \( n = n' \) and \( m = m' \).

**Proof.** By Proposition 3 we get that \( n = n' \). Let us suppose, without loss of generality, that \( m > m' \). Thus any morphism from \( h_{(n,m)} \) to \( h_{(n',m')} \) is non-injective and, since \( n = n' \), it is easy to see that there is no morphism at all. \( \Box \)

\(^4\) This element is unique since \( f_{(n',m')} \) is stable.
In order to draw a picture of (a part of) this hierarchy of degrees, let us compute some typical values of $C_{i,j}^{k,l}$:

\[ C_{n+1,n,n} = 2 + (n - 2) - n \Rightarrow \forall n \geq 3 \ f(n+1,n+1) \leq_{\text{par}} f(n,n) \]

\[ C_{n,n-1,n+1} = n - 2 \Rightarrow \forall n \geq 4 \ f(n-1,n) \leq_{\text{par}} f(n-1,n+1) \]

\[ C_{n+1,n-1,n} = 2 + (n - 3) = n - 1 \Rightarrow \forall n \geq 4 \ f(n,n+1) \leq_{\text{par}} f(n-1,n) \]

We can prove that the inequalities above are strict by using the same method: for the first one we have for instance

\[ C_{n,n+1,n+1} = n \Rightarrow \forall n f(n,n) \not=_{\text{par}} f(n+1,n+1) \]

The following picture shows some degrees in the hierarchy (arrows denote $\leq_{\text{par}}$ relations, $(n,m)$ denotes $f(n,m)$):

```
(3,5)  
|    |
(4,6)  
|    |
(3,4)  
|    |
(5,7)  
|    |
(4,5)  
|    |
(3,3)  
|    |
(5,6)  
|    |
(4,4)  
|    |
(5,5)  
```

This hierarchy contains infinite ascending chains, infinite descending chains and infinite antichains (i.e. denumerable sets of pairwise unrelated elements). The diagram could be a bit misleading since the regular pattern it shows is not complete: there exist arrows that cannot be extrapolated by the finite portion of the hierarchy in the picture. For instance, it turns out that $f(5,8) <_{\text{par}} f(3,4)$, even if they lie on the same row in the diagram. On the other hand the elements of the row containing $(3,3)$ (or any $(n,n)$) form an antichain.
5. Conclusions

The hypergraph that we associate with a function \( f \) yields some information about the degree of parallelism of \( f \).

Actually, as shown by Example 3, the existence of a morphism from \( H(f) \) to \( H(g) \) is not a necessary condition for \( f \leq_{\text{par}} g \), but some of the results we obtained (like Proposition 3, or the existence of the hierarchy \( f_{(n,m)} \)), support our feeling that the study of the combinatory of hypergraphs can result in a better understanding of the poset of degrees of parallelism.

A complete characterization of first order degrees of parallelism can be considered as preliminary to the study of the decidability problem for \( \leq_{\text{par}} \) at higher order, which is open.

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