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Degrees of parallelism in the continuous type hierarchy

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Abstract

A degree of parallelism is an equivalence class of Scott-continuous functions which are relatively definable by each other with respect to the language PCF (a paradigmatic sequential language). We introduce an infinite ("bi-dimensional") hierarchy of degrees. This hierarchy is inspired by representing first order continuous functions as hypergraphs. We assume some familiarity with the language PCF and with its continuous model.

Keywords: Sequentiality; Stability; Strong stability; Logical relations; Sequentiality relations

1. Introduction

A natural notion of relative definability in the continuous type hierarchy is given by the following definition:

Definition 1. Given two continuous functions f and g, we say that f is less parallel than g ($f \leqslant_{par} g$) if there exists a closed PCF-term M such that $[\![M]\!]_g = f$.

A *degree of parallelism* is a class of the equivalence relation associated with the preorder $\leqslant_{par}$.

In this paper we deal with degrees of parallelism of first order boolean functions, i.e. of functions which take tuples of booleans as arguments and give booleans as results. PCF-definability for first order functions is fully characterized by the notion of sequentiality (in any of its formulations), and Sieber's *sequentiality relations* [5] provide a characterization of first order degrees of parallelism. Moreover this characterization is effective: given f and g one can decide if $f \leqslant_{par} g$, and recently Stoughton [6] has implemented an algorithm which solves this decision problem. Nevertheless, as far as I know, there is little knowledge of the structure of the partial order $\leqslant_{par}$ on first order boolean functions.

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A well known fact is that any continuous function(al) is less parallel than the "parallel or" function (the non-strict binary disjunction) [3], and we also know that any first order stable function is less parallel than the Berry function [2, p. 334], but there is a lack of general results about the poset of degrees, whose structure turns out to be quite complicated, already at first order. Sazonov's paper [4] may be considered as a first step towards a systematic study of the poset of degrees of parallelism.

In this paper we give a geometric account of first order degrees of parallelism, by representing first order functions as *hypergraphs* which highlight the structure of *linearly coherent* [2] subsets in the trace of the function. Then we introduce a hierarchy of functions $\{f_{(n,m)}\}_{n \leqslant m \in \omega}$ which has the property that $f_{(n,m)} \leqslant_{par} f_{(n',m')}$ if and only if there exists a morphism from the hypergraph associated with $f_{(n,m)}$ to the hypergraph associated with $f_{(n',m')}$.

Throughout the paper PCF terms will be written in uncurried form (as n-ary functions), and some "macros" like a syntactic $\perp$ and a sequential conjunction $\wedge$ will be used.

2. Preliminaries

We denote by $\mathscr{B}$ the flat domain of boolean values $\{\perp, true, false\}$. Tuples of boolean values are ordered componentwise. Given a continuous function $f : \mathscr{B}^n \to \mathscr{B}$, the *trace* of f is defined by

$$tr(f) = \{(v,b) \mid v \in \mathscr{B}^n,\ b \in \mathscr{B},\ b \neq \perp,\ f(v) = b \text{ and } \forall v' < v\ f(v') = \perp\}.$$

A continuous function $f : \mathscr{B}^n \to \mathscr{B}$ is *stable* if for all $v_1, v_2 \in \pi_1(tr(f)), v_1 \not\uparrow v_2$. A subset $A = \{v_1, \ldots, v_k\}$ of $\mathscr{B}^n$ is *linearly coherent* (or simply *coherent*) if

$$\forall j\ 1 \leqslant j \leqslant n\ ((\forall l\ 1 \leqslant l \leqslant k\ v_l^j \neq \perp) \Rightarrow (\forall l_1, l_2\ 1 \leqslant l_1 \leqslant l_2 \leqslant k\ v_{l_1}^j = v_{l_2}^j))$$

or equivalently if for any linear function [3] $\alpha : \mathscr{B}^n \to O$, $\alpha(\bigwedge A) = \bigwedge \alpha(A)$, where O denotes the Sierpinsky domain $\{\perp, \top\}$.

The set of coherent subsets of $\mathscr{B}^n$ is denoted $\mathscr{C}(\mathscr{B}^n)$.

Fact 1. *If $A \in \mathscr{C}(\mathscr{B}^n)$ and B is an Egli–Milner lower bound of A (that is if $\forall x \in A \exists y \in B\ y \leqslant x$ and $\forall y \in B \exists x \in A\ y \leqslant x$) then $B \in \mathscr{C}(\mathscr{B}^n)$.*

Definition 2. A continuous function $f : \mathscr{B}^n \to \mathscr{B}^m$ is *linearly strongly stable* (or simply *strongly stable*) if for any $A \in \mathscr{C}(\mathscr{B}^n)$

- $f(A) \in \mathscr{C}(\mathscr{B}^m)$.
- $f(\bigwedge A) = \bigwedge f(A)$.

[2] In the sense of [1].

[3] A function is linear if it is stable and it commutes with least upper bounds of finite sets.

2.1. Sequential logical relations

Definition 3 (*Sieber* [5]). For each $n \geqslant 0$ and each pair of sets $A \subseteq B \subseteq \{1,\dots,n\}$ let $S_n^{A,B} \subseteq \mathscr{B}^n$ be defined by

$$S_n^{A,B}(b_1,\dots,b_n) \Leftrightarrow (\exists i \in A\ b_i = \bot) \vee (\forall i,j \in B\ b_i = b_j)$$

An n-ary logical relation R is called a *sequentiality relation* if it is an intersection of relations of the form $S_n^{A,B}$.

A function $f : \mathscr{B}^n \to \mathscr{B}$ is invariant under the m-ary logical relation R if for any

$$(x_1^1,\dots,x_1^m) \in R, (x_2^1,\dots,x_2^m) \in R,\dots,(x_n^1,\dots,x_n^m) \in R.$$

one has that

$$(f(x_1^1,x_2^1,\dots,x_n^1), f(x_1^2,x_2^2,\dots,x_n^2),\dots,f(x_1^m,x_2^m,\dots,x_n^m)) \in R.$$

Proposition 1. *For any $f : \mathscr{B}^n \to \mathscr{B}$ and $g : \mathscr{B}^m \to \mathscr{B}$ continuous functions, $f \leqslant_{par} g$ if and only if for any sequentiality relation R, if g is invariant under R then f is invariant too.*

Actually this is a relativized version of the main theorem of [5]: a continuous function of first or second order is PCF-definable if and only if it is invariant under all sequentiality relations.

Fact 2. *A set $A = \{v_1,\dots,v_k\} \subseteq \mathscr{B}^n$ is linearly coherent if and only if*

$$\forall j \in \{1,\dots,n\}(v_1^j, v_2^j,\dots,v_k^j) \in S_k^{\{1,\dots,k\},\{1,\dots,k\}}$$

The next lemma highlights the connections between strong stability and sequentiality relations.

Lemma 1. *Let $f : \mathscr{B}^n \to \mathscr{B}$ be continuous and $k \geqslant 2$. Then the following conditions for f are equivalent:*

(1) *There is no coherent set $C \subseteq \pi_1(tr(f))$ with $2 \leqslant \#C \leqslant k$.*
(2) *Whenever $C \subseteq \mathscr{B}^n$ is coherent and $\#C \leqslant k$, then*
  - *$f(C)$ is coherent;*
  - *$f(\bigwedge C) = \bigwedge f(C)$.*
(3) *f is invariant under all relations of the form $S_m^{A,B}$ with $\#A \leqslant k$.*
(4) *f is invariant under the relation $S_{k+1}^{\{1,\dots,k\},\{1,\dots,k+1\}}$*

Proof. (1)$\Rightarrow$(2): given a coherent $C = \{v_1,\dots,v_p\} \subseteq \mathscr{B}^n$ with $p \leqslant k$, we prove that either there exists a $v \in \pi_1(tr(f))$ such that for any $i \in \{1,\dots,p\}$ $v \leqslant v_i$ or there exists an $i \in \{1,\dots,p\}$ such that $f(v_i) = \bot$. In both cases it is easy to see that $f(C)$ is coherent and $f(\bigwedge C) = \bigwedge f(C)$.

By contradiction, let us suppose that no element of $\pi_1(tr(f))$ is a lower bound of all the elements of C, and that any element of C has a lower bound in $\pi_1(tr(f))$. For any $v_i \in C$ let $w_i \in \pi_1(tr(f))$ be such that $w_i \leqslant v_i$. The set

$$W = \{w \in \pi_1(tr(f)) \mid \exists i \in \{1,\dots,p\} w = w_i\}$$

is such that $2 \leqslant \#W \leqslant p \leqslant k$, and it is an Egli–Milner lower bound of C, hence it is coherent; contradiction.

(2)$\Rightarrow$(3): given $S_m^{A,B}$, let $\{v_1,\dots,v_m\} \subseteq \mathscr{B}^n$ be such that

$$\forall j \in \{1,\dots,n\} \quad (v_1^j,\dots,v_m^j\} \in S_m^{A,B}.$$

We have to prove that $(f(v_1),\dots,f(v_m)) \in S_m^{A,B}$. We have that the set $\{v_l \mid l \in A\}$ is coherent, and that for any $k \in (B \backslash A)$ $v_k \geqslant \bigwedge_{l \in A} v_l$ (these two conditions are easily seen to be equivalent to $\forall j \in \{1,\dots,n\}$ $(v_1^j,\dots,v_m^j\} \in S_m^{A,B}$). Hence by (2) we get that $\{f(v_l) \mid l \in A\}$ is coherent and that $\forall k \in (B \backslash A)$ $f(v_k) \geqslant \bigwedge_{l \in A} f(v_l)$, that is $(f(v_1),\dots, f(v_m)) \in S_m^{A,B}$.

(3)$\Rightarrow$(4): trivial.

(4)$\Rightarrow$(1): let us suppose by contradiction that there exists a coherent set $C = \{v_1,\dots,v_p\} \subseteq \pi_1(tr(f))$ with $2 \leqslant p \leqslant k$, and consider the multiset $\{v_{e(1)},\dots,v_{e(k)}\}$, where $e : \{1,\dots,k\} \to \{1,\dots,p\}$ is defined by $e(i) = ((i-1) \operatorname{MOD} p) + 1$. Let $w \in \mathscr{B}^n$ be defined by $w^j = \bigwedge_{v \in C} v^j$, $1 \leqslant j \leqslant n$. The $k+1$ vectors $v_{e(1)},\dots,v_{e(k)}, w$ are such that

$$\forall j \in \{1,\dots,n\} \quad (v_{e(1)}^j,\dots,v_{e(k)}^j, w^j) \in S_{k+1}^{\{1,\dots,k\},\{1,\dots,k+1\}}$$

since the $v_{e(i)}^j$ are coherent, and w^j is their greatest lower bound. On the other hand

$$(f(v_{e(1)}),\dots,f(v_{e(k)}), f(w)) \notin S_{k+1}^{\{1,\dots,k\},\{1,\dots,k+1\}}$$

since for any $i \in \{1,\dots,k\}$ $f(v_{e(i)}) > \bot$ and $f(w) = \bot$ (this last condition being assured by the fact that $k \geqslant p \geqslant 2$). Hence f is not invariant under $S_{k+1}^{\{1,\dots,k\},\{1,\dots,k+1\}}$, contradiction. $\square$

Fact 3. *A function $f : \mathscr{B}^n \to \mathscr{B}$ is stable if and only if it satisfies one of the conditions of Lemma* 1 *for $k = 2$. It is strongly stable if and only if it satisfies one of the conditions of Lemma* 1 *for all $k \geqslant 2$.*

The following proposition states that strong stability captures the notion of sequential definability, at least at first order.

Proposition 2. *$f : \mathscr{B}^n \to \mathscr{B}$ is strongly stable, if and only if it is PCF-definable.*

This follows from condition (3) of Lemma 1, because for a first order function invariance under the relations $S_n^{A,B}$ implies invariance under all sequentiality relations.

In the next example we show a typical use of sequentiality relations to explore the poset of degrees of parallelism:

Example 1. We show that there exists a degree of parallelism strictly in between the degree of the *weak parallel or* function $Por^- : \mathscr{B}^2 \to \mathscr{B}$ and the one of the *parallel or* function $Por : \mathscr{B}^2 \to \mathscr{B}$, where

$$tr(Por^-) = \{((true, \bot), true), ((\bot, true), true)\}$$

and

$$tr(Por) = \{((true, \bot), true), ((\bot, true), true), ((false, false), false)\}.$$

We are looking for a (first order) function f such that $Por^- <_{par} f <_{par} Por$. Let $f : \mathscr{B}^4 \to \mathscr{B}$ be defined by

$$tr(f) = \{((true, \bot, true, true), true),\ ((\bot, true, true, true), true),$$
$$((false, false, false, \bot), false),\ ((false, false, \bot, false), false)\}.$$

A general method for showing that, for two given first order functions g and $g', g <_{par} g'$ is to define a PCF-term M such that $[\![M]\!]_{g'} = g$, showing that $g \leqslant_{par} g'$, and a sequentiality relation R such that g is invariant under R and g' is not, which entails $g' \not\leqslant_{par} g$ by Proposition 1.

Let us use this method for $Por^- <_{par} f$. Firstly

$$Por^- = [\![\lambda y \lambda x_1 x_2 y(x_1, x_2, true, true)]\!] f$$

Then consider the sequentiality relation $S_4^{\{1,\dots,4\},\{1,\dots,4\}}$. It is easy to see that Por^- is invariant under it and f is not.

As for $f <_{par} Por$, it is enough to show that $Por \not\leqslant_{par} f$, since any function is less parallel than Por [3]. Let us consider the sequentiality relation $R = S_3^{\{1,2,3\},\{1,2,3\}}$: Por is not invariant under R since

$$(true, \bot, false), (\bot, true, false) \in R$$

and

$$(Por(true, \bot), Por(\bot, true), Por(false, false)) = (true, true, false) \notin R.$$

On the other hand f is invariant under R: let $v_1, v_2, v_3 \in \mathscr{B}^4$ be such that $\forall j \in \{1, \dots, 4\}$ $(v_1^j, v_2^j, v_3^j) \in R$. Then, by Fact 2, $\{v_1, v_2, v_3\} \in \mathscr{C}(\mathscr{B}^4)$. If $\exists j \in \{1,2,3\}$ $f(v_j) = \bot$ then $(f(v_1), f(v_2), f(v_3)) \in R$, otherwise there exists an Egli–Milner lower bound E of $\{v_1, v_2, v_3\}$ in $\pi_1(tr(f))$ such that $\#E \leqslant 3$. It is easy to see that this implies $f(v_1) = f(v_2) = f(v_3)$, and hence we conclude that $(f(v_1), f(v_2), f(v_3)) \in R$. $\square$

3. Hypergraphs for boolean functions

We consider a category whose objects are (colored) hypergraphs and whose morphisms are arcs-preserving and coloring-preserving maps:

Definition 4. A *colored hypergraph* $h=(V_h,A_h,C_h)$ is given by a set V_h of *vertices*, a set $A_h \subseteq \{A \subseteq V_h \mid \#A \geqslant 2\}$ of *(hyper)arcs* and a *coloring function* $C_h : V_h \to \{black, white\}$. A morphism from a hypergraph h to a hypergraph h' is a function $m: V_h \to V_{h'}$ such that

- for all $A \subseteq V_h$, if $A \in A_h$ then $m(A) \in A_{h'}$.
- for all $x,x' \in V_h$, $C_h(x)=C_h(x')$ if and only if $C_{h'}(m(x))=C_{h'}(m(x'))$.

Definition 5. Let $f:\mathscr{B}^n \to \mathscr{B}$ be the n-ary function defined by $tr(f)=\{(v_1,b_1),\ldots,(v_k,b_k)\}$. The hypergraph $H(f)$ is defined by

- $V_{H(f)}=\{1,2,\ldots,k\}$.
- $A_{H(f)}=\{\{i_1,i_2,\ldots,i_l\} \subseteq V_{H(f)} \mid l \geqslant 2 \text{ and } \{v_{i_1},v_{i_2},\ldots,v_{i_l}\} \in \mathscr{C}(\mathscr{B}^n)\}$.
- $C_{H(f)}(i)=$ if b_i then white else black.

Example 2. Consider the function $G:\mathscr{B}^3 \to \mathscr{B}$ defined by

$$tr(G)=\{((\bot, true, false), true), ((false, \bot, true), true), ((true, false, \bot), false)\}$$

and the function $Por:\mathscr{B}^2 \to \mathscr{B}$ defined in Example 1. We have

$$H(G)=(\{1,2,3\},\{\{1,2,3\}\},C_{H(G)}(1)=C_{H(G)}(2)=white, C_{H(G)}(3)=black)$$

$$H(Por)=(\{1,2,3\},\{\{1,2\},\{1,2,3\}\},$$
$$C_{H(Por)}(1)=C_{H(Por)}(2)=white, C_{H(Por)}(3)=black).$$

The map $m:H(G)\to H(Por)$ defined by $m(i)=i$, for $i=1,2,3$, is a morphism. A term M such that $[\![M]\!]Por=G$ is

$$M=\lambda f \lambda x_1x_2x_3 f(t_1,t_2),$$

where

$$t_1 = if\ x_1\ then\ (if\ x_2\ then\ \bot\ else\ false)\ else\ if\ x_3\ then\ true\ else\ \bot$$

$$t_2 = if\ x_2\ then\ (if\ x_3\ then\ \bot\ else\ true)\ else\ if\ x_1\ then\ false\ else\ \bot. \qquad \square$$

Example 3. Let $3Por:\mathscr{B}^3 \to \mathscr{B}$ be defined by

$$tr(3Por)=\{((true,\bot,\bot), true), ((\bot, true, \bot), true), ((\bot,\bot, true), true)\}.$$

The associated hypergraph is

$$H(3Por)=(\{1,2,3\},\{\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\},$$
$$C(1)=C(2)=C(3)=white).$$

It is easy to see that there exists no morphism $m : H(3Por) \to H(Por)$. Nevertheless, $3Por \leqslant_{par} Por$, since for instance

$$3Por = [\![M]\!]Por,$$

where

$$M = \lambda f \lambda x_1 x_2 x_3 \quad \textit{if } f(f(x_1,x_2),x_3) \textit{ then true else } \bot. \qquad \square$$

The following are simple properties relating hypergraphs and degrees of parallelism:

Fact 4. *Let $f : \mathscr{B}^n \to \mathscr{B}$ be a continuous function: f is stable if and only if $H(f)$ has no 2-arc. It is strongly stable if and only $H(f)$ has no arc.*

This follows from Fact 3 and condition (1) of Lemma 1.

Proposition 3. *Let $f : \mathscr{B}^n \to \mathscr{B}$ and $g : \mathscr{B}^m \to \mathscr{B}$ be such that*

$$\min\{\#A \mid A \in A_{H(f)}\} < \min\{\#A \mid A \in A_{H(g)}\}.$$

Then $f \nleqslant_{par} g$.

Proof. Let $k = \min\{\#A \mid A \in A_{H(f)}\}$. By Lemma 1 (conditions (1) and (4)) we know that g is invariant under the sequentiality relation $S_{k+1}^{\{1,2,\dots,k\},\{1,2,\dots,k+1\}}$, and that f is not. By Proposition 1 we are done. $\square$

4. A hierarchy of degrees

Definition 6. Given two natural numbers $m \geqslant n \geqslant 3$, let $h_{(n,m)}$ be the hypergraph defined by

$$h_{(n,m)} = (\{1,2,\dots,m\}, \{A \subseteq \{1,2,\dots,m\} \mid \#A \geqslant n\}, \text{ for all } i \ C(i) = \textit{white}).$$

Given $h_{(n,m)}$ and $h_{(n',m')}$, we are interested in determining the conditions under which there exists a morphism $f : h_{(n,m)} \to h_{(n',m')}$. Since the $h_{(i,j)}$'s are monocolored, the only condition to be satisfied for a function $f : \{1,\dots,m\} \to \{1,\dots,m'\}$ to be a morphism is the preservation of arcs. It is easy to see that f is a morphism if and only if

$$\max\{\#f^{-1}(B) \mid B \subseteq \{1,\dots,m'\} \text{ and } \#B = n' - 1\} < n$$

since only in that case every arc of $h_{(n,m)}$ is mapped by f to an arc of $h_{(n',m')}$. Hence there exists a morphism from $h_{(m,n)}$ to $h_{(m',n')}$ if and only if

$$n > \min_{f:\{1,\dots,m\} \to \{1,\dots,m'\}} \max\{\#f^{-1}(B) \mid B \subseteq \{1,\dots,m'\} \text{ and } \#B = n' - 1\}.$$

It is quite easy to see that one of the functions $f:\{1,\dots,m\}\to\{1,\dots,m'\}$ which realize the minimum above is $f_0(i)=((i-1)\,\mathrm{MOD}\,m')+1$, and that

$$\max\{\#f_0^{-1}(B)\mid B\subseteq\{1,\dots,m'\}\text{ and }\#B=n'-1\}$$
$$=\left(\min\{n'-1,m\,\mathrm{MOD}\,m'\}*\left\lceil\frac{m}{m'}\right\rceil\right)$$
$$+\left(\max\{0,(n'-1)-(m\,\mathrm{MOD}\,m')\}*\left\lfloor\frac{m}{m'}\right\rfloor\right),$$

where $\lceil x\rceil$ and $\lfloor x\rfloor$ denote the integer parts of $x+1$ and x, respectively. If we denote this natural number by $C^{m,n',m'}$, we have that there exists a morphism from $h_{(n,m)}$ to $h_{(n',m')}$ if and only if $n > C^{m,n',m'}$.

We define now a set of boolean functions $\{f_{(n,m)}\}$ such that for all n,m (with $3\leqslant n\leqslant m$), $H(f_{(n,m)})=h_{(n,m)}$, and we show that for all n,m,n',m' $f_{(n,m)}\leqslant_{par}f_{(n',m')}$ if and only if $n > C^{m,n',m'}$. We start by showing how to construct, for any given $h_{(n,m)}$, a boolean function f such that $H(f)=h_{(n,m)}$. The trace of f has to contain m elements, its second projection has to be the singleton $\{true\}$ and for any subset A of the first projection of the trace, A has to be coherent if and only if $\#A\geqslant n$. Before describing the general method for constructing such a function f, let us consider an example:

Example 4. The function f described by the following trace (which we represent as a matrix), is such that $H(f)=h_{(3,4)}$ (Table 1):

Table 1

true	true	true	⊥	⊥	⊥	true
false	⊥	⊥	true	true	⊥	true
⊥	false	⊥	false	⊥	true	true
⊥	⊥	false	⊥	false	false	true

Actually a subset of the first projection of this trace is coherent if and only if its cardinality is at least 3, since for any binary subset $\{i,j\}$ of rows there exists a column l such that the elements (i,l) and (j,l) are defined and different. □

For constructing a function $f_{(n,m)}$ whose associated hypergraph is $h_{(n,m)}$ we have just to generalize the idea above: for any subset of less than n rows (and of at least two rows), there must exist a column which makes that subset incoherent. The arity of the function is $\sum_{i=2}^{n-1}\binom{m}{i}$, and in the jth column, only elements corresponding to rows in the jth subset (with respect to any enumeration whatsoever) will be defined, say by *true* for the first row in that subset and by *false* for the other rows.

Example 5. The following matrix represents $\pi_1(tr(f_{(4,4)}))$:

$v_1 =$	true	true	true	$\perp$	$\perp$	$\perp$	true	true	true	$\perp$
$v_2 =$	false	$\perp$	$\perp$	true	true	$\perp$	false	$\perp$	false	true
$v_3 =$	$\perp$	false	$\perp$	false	$\perp$	true	false	false	$\perp$	false
$v_4 =$	$\perp$	$\perp$	false	$\perp$	false	false	$\perp$	false	false	false

and the following one represents $\pi_1(tr(f_{(3,3)}))$:

$w_1 =$	true	true	$\perp$
$w_2 =$	false	$\perp$	true
$w_3 =$	$\perp$	false	false

Proposition 4. *If $n, m, n', m' \in \omega$ are such that $3 \leqslant n \leqslant m$, $3 \leqslant n' \leqslant m'$ and $n > C^{m,n',m'}$, then $f_{(n,m)} \leqslant_{par} f_{(n',m')}$.*

Proof. Let $k = \sum_{i=2}^{n-1} \binom{m}{i}$ and $k' = \sum_{i=2}^{n'-1} \binom{m'}{i}$, and let $A = \pi_1(tr(f_{(n,m)})) = \{v_1, \ldots, v_m\}$ and $B = \pi_1(tr(f_{(n',m')})) = \{w_1, \ldots, w_{m'}\}$. By hypothesis there exists a function $f : \{1, \ldots, m\} \to \{1, \ldots, m'\}$ which maps every non-singleton coherent subset of A to a non-singleton coherent subset of B. Let us consider, for $1 \leqslant j \leqslant k'$, the function $g^j : \mathscr{B}^k \to \mathscr{B}$ defined by

$$tr(g^j) = \{(v_i, w^j_{f(i)}) \mid 1 \leqslant i \leqslant m \wedge w^j_{f(i)} \neq \perp\}.$$

We prove that g^j is strongly stable, for $j \in \{1, \ldots, k'\}$, by using the condition (1) of Lemma 1: let $C = \{v_{i_1}, \ldots, v_{i_l}\} \subseteq \pi_1(tr(g^j))$ be such that $2 \leqslant l$. Let $H = f\{i_1, \ldots, i_l\}$. By definition of g^j we have that for any $h \in H$ $w^j_h \neq \perp$, hence by construction of $f_{(n',m')}$ we get $\#H \leqslant n' - 1$, hence $\#C \leqslant n - 1$ and C is not coherent.

The g^j's are strongly stable, first order functions, hence by Proposition 2, for all $i \leqslant k'$ there exists a PCF term $t^j(x_1, \ldots, x_k)$ which defines g^j. Consider the term

$$M = \lambda y\, \lambda x_1 x_2 \ldots x_k\, y(t^1(x_1, \ldots, x_k), t^2(x_1, \ldots, x_k), \ldots, t^{k'}(x_1, \ldots, x_k)).$$

In order to prove that $[\![M]\!] f_{(n',m')} = f_{(n,m)}$ we just remark that, by construction,

$$\forall v \in B^k((\exists j \leqslant m'(g^1(v), \ldots, g^{k'}(v)) \geqslant w_j) \Leftrightarrow (\exists i \leqslant m v \geqslant v_i \text{ and } f(i) = j)). \quad \square$$

Example 6. Let us apply the construction above to show that $f_{(4,4)} \leqslant_{par} f_{(3,3)}$ (remark that $C^{4,3,3} = 3$) (we refer to Example 5).

Any surjective function $f : \{1, 2, 3, 4\} \to \{1, 2, 3\}$ satisfies the condition of being a morphism from $h_{(4,4)}$ to $h_{(3,3)}$; let us choose for instance

$$f(1) = f(4) = 1, \quad f(2) = 2, \quad f(3) = 3.$$

The corresponding g^j are defined by

$$tr(g^1) = \{(v_1, true), (v_2, false), (v_4, true)\},$$

$$tr(g^2) = \{(v_1, true), (v_3, false), (v_4, true)\},$$

$$tr(g^3) = \{(v_2, true), (v_3, false)\}.$$

The terms t^j are essentially sequences of conditional statements: for instance

$$t^3 = \lambda x_1 \ldots x_{10} \quad \text{if } x_4 \quad \text{then (if } (\neg x_1 \wedge x_5 \wedge \neg x_7 \wedge \neg x_9 \wedge x_{10}) \text{ then true else } \bot)$$
$$\text{else (if } (\neg x_2 \wedge x_6 \wedge \neg x_7 \wedge \neg x_8 \wedge \neg x_{10}) \text{ then true else } \bot).$$

The rest of this section is devoted to a proof that the condition $n > C^{m,n',m'}$ is indeed necessary for having $f_{(n,m)} \leqslant_{par} f_{(n',m')}$:

Proposition 5. *If n, m, n', m' are such that $3 \leqslant n \leqslant m$, $3 \leqslant n' \leqslant m'$ and $n \leqslant C^{m,n',m'}$, then $f_{(n,m)} \not\leqslant_{par} f_{(n',m')}$.*

Proof. By Proposition 1 it is sufficient to define a sequential logical relation R such that $f_{(n',m')}$ is invariant with respect to R and $f_{(n,m)}$ is not.

The first projection of $tr(f_{(n,m)})$ is

$$\pi_1(tr(f_{(n,m)})) = \{(x_1^1, \ldots x_1^{\sum_{i=2}^{n-1}\binom{m}{i}}), \ldots, (x_m^1, \ldots, x_m^{\sum_{i=2}^{n-1}\binom{m}{i}})\}.$$

Remark that, by definition, any "column" of the first projection of $tr(f_{(n,m)})$, i.e. any tuple

$$\{(x_1^i, x_2^i, \ldots, x_m^i)\}_{1 \leqslant i \leqslant \sum_{i=2}^{n-1}\binom{m}{i}}$$

contains at most $n-1$ components different from $\bot$. Hence it is easy to see that $f_{(n,m)}$ is not invariant with respect to the $(m+1)$-ary sequential logical relation

$$R = \left(\bigcap_{A \subseteq \{1,2,\ldots,m\},\ \#A \geqslant n} S_{m+1}^{A,A} \right) \cap (S_{m+1}^{\{1,\ldots,m\},\{1,\ldots,m+1\}})$$

since the tuples

$$\{(x_1^i, x_2^i, \ldots, x_m^i, \bot)\}_{1 \leqslant i \leqslant \sum_{i=2}^{n-1}\binom{m}{i}}$$

are in R, and the application of $\underbrace{(f_{(n,m)}, \ldots, f_{(n,m)})}_{m+1}$ to those tuples yields the tuple $(\underbrace{true, true, \ldots, true}_{m}, \bot)$ which is not in R.

If we prove that $f_{(n',m')}$ is invariant with respect to R we are done. By *reductio ad absurdum*, let us suppose that $f_{(n',m')}$ is not invariant. Then there exist $m+1$ tuples

$$v_1 = (y_1^1, \dots, y_1^{\sum_{i=2}^{n'-1}\binom{m'}{i}})$$

$$v_2 = (y_2^1, \dots, y_2^{\sum_{i=2}^{n'-1}\binom{m'}{i}})$$

$$\vdots$$

$$v_{m+1} = (y_{m+1}^1, \dots, y_{m+1}^{\sum_{i=2}^{n'-1}\binom{m'}{i}})$$

such that

$$\forall j \quad 1 \leqslant j \leqslant \sum_{i=2}^{n'-1} \binom{m'}{i} (y_1^j, \dots, y_{m+1}^j) \in R$$

and

$$(f_{(n',m')}(v_1), \dots, f_{(n',m')}(v_{m+1})) \notin R.$$

It is easy to see that this is the case if and only if

$$f_{(n',m')}(v_1) = f_{(n',m')}(v_2) = \cdots = f_{(n',m')}(v_m) = true \quad \text{and} \quad f_{(n',m')}(v_{m+1}) = \bot.$$

Hence for any $1 \leqslant i \leqslant m$ there exists an element $w_{f(i)}$ of the first projection of $tr(f_{(n',m')})$ such that $v_i \geqslant w_{f(i)}$,[4] for some function $f : \{1, 2, \dots, m\} \to \{1, 2, \dots, m'\}$. Since $n \leqslant C^{m,n',m'}$, there exists a set $A \subseteq \{1, \dots, m\}$ such that $\#A \geqslant n$ and $\#f(A) \leqslant n' - 1$. Let A be a maximal set with that property.

If $f(A) = \{l\}$ is a singleton, then $\#f(A) < n' - 1$, hence $A = \{1, \dots, m\}$ because A is maximal. This means $v_i \geqslant w_l$ for $i = 1, \dots, m$ and hence also $v_{m+1} \geqslant w_l$ since $(y_1^j, \dots, y_{m+1}^j) \in R \subseteq S_{m+1}^{\{1,\dots,m\},\{1,\dots,m+1\}}$. This implies $f_{(n',m')}(v_{m+1}) \geqslant f_{(n',m')}(w_l) = true$, contradiction.

If $\#f(A) \geqslant 2$, then $C =_{def} \{w_{f(i)} \mid i \in A\} \subseteq \pi_1(tr(f_{(n',m')}))$ cannot be coherent because $2 \leqslant \#C \leqslant n' - 1$. Hence also $C' =_{def} \{v_i \mid i \in A\}$ cannot be coherent because C is an Egli–Milner lower bound of C'. This contradicts $(y_1^j, \dots, y_{m+1}^j) \in S_{m+1}^{A,A}$ for all j. □

Hence $f_{(n,m)} \leqslant_{par} f_{(n',m')}$ if and only if $n > C^{m,n',m'}$, It is easy to see that this hierarchy is non-trivial:

Proposition 6. *If $f_{(n,m)} \leqslant_{par} f_{(n',m')}$ and $f_{(n',m')} \leqslant_{par} f_{(n,m)}$ then $n = n'$ and $m = m'$.*

Proof. By Proposition 3 we get that $n = n'$. Let us suppose, without loss of generality, that $m > m'$. Thus any morphism from $h_{(n,m)}$ to $h_{(n',m')}$ is non-injective and, since $n = n'$, it is easy to see that there is no morphism at all. □

[4] This element is unique since $f_{(n',m')}$ is stable.

In order to draw a picture of (a part of) this hierarchy of degrees, let us compute some typical values of $C^{i,j,l}$:

$$C^{n+1,n,n} = 2 + (n-2) = n \;\Rightarrow\; \forall n \geqslant 3\;\; f_{(n+1,n+1)} \leqslant_{par} f_{(n,n)}$$

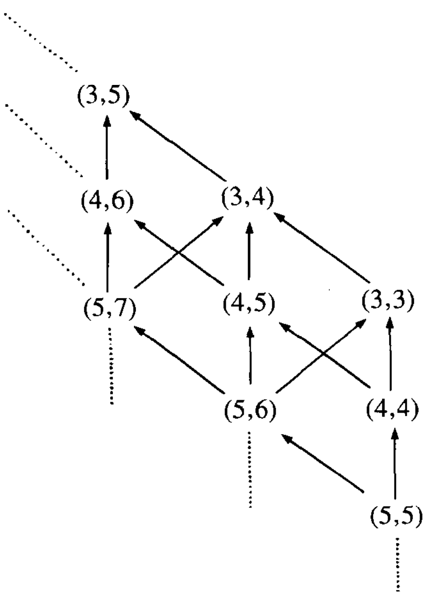
$$C^{n,n-1,n+1} = n - 2 \;\Rightarrow\; \forall n \geqslant 4\;\; f_{(n-1,n)} \leqslant_{par} f_{(n-1,n+1)}$$

$$C^{n+1,n-1,n} = 2 + (n-3) = n - 1 \;\Rightarrow\; \forall n \geqslant 4\;\; f_{(n,n+1)} \leqslant_{par} f_{(n-1,n)}.$$

We can prove that the inequalities above are strict by using the same method: for the first one we have for instance

$$C^{n,n+1,n+1} = n \;\Rightarrow\; \forall n f_{(n,n)} \not\leqslant_{par} f_{(n+1,n+1)}.$$

The following picture shows some degrees in the hierarchy (arrows denote $\leqslant_{par}$-relations, (n,m) denotes $f_{(n,m)}$):



This hierarchy contains infinite ascending chains, infinite descending chains and infinite antichains (i.e. denumerable sets of pairwise unrelated elements). The diagram could be a bit misleading since the regular pattern it shows is not complete: there exist arrows that cannot be extrapolated by the finite portion of the hierarchy in the picture. For instance, it turns out that $f_{(5,8)} <_{par} f_{(3,4)}$, even if they lie on the same row in the diagram. On the other hand the elements of the row containing $(3,3)$ (or any (n,n)) form an antichain.

5. Conclusions

The hypergraph that we associate with a function f yields some information about the degree of parallelism of f.

Actually, as shown by Example 3, the existence of a morphism from $H(f)$ to $H(g)$ is not a necessary condition for $f \leqslant_{par} g$, but some of the results we obtained (like Proposition 3, or the existence of the hierarchy $f_{(n,m)}$), support our feeling that the study of the combinatory of hypergraphs can result in a better understanding of the poset of degrees of parallelism.

A complete characterization of first order degrees of parallelism can be considered as preliminary to the study of the decidability problem for $\leqslant_{par}$ at higher order, which is open.

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