Describing an $n \log n$ algorithm for minimizing states in deterministic finite automaton

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Abstract

There are several well known algorithms to minimize deterministic finite automata. In this paper, an algorithm is given for minimizing the number of states in a finite automaton or for determining if two finite automata are equivalent. The asymptotic running time of the algorithm is bounded by $kn \log n$ where $k$ is some constant depends linearly on the size of the input alphabet and $n$ is the number of states.

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1 Introduction

The problem of writing efficient algorithms to find the minimal deterministic finite automaton equivalent to a given automaton can be traced back to 1950’s with the works of Huffman [Huf55] and Moore [Moo58]. Over the years several alternative algorithms were proposed. However, a worst case analysis of these algorithms indicate that they are $n^2$ processes where $n$ is the number of states. Hopcroft [Hop71] introduced in 1971 an $O(n \log n)$ algorithm for minimizing a finite deterministic automaton of $n$ states. The constant of proportionality depends linearly on the number of input symbols. Clearly the same algorithm can be used to determine if two finite automata are equivalent. Recent researches [BC04, BBC08] show this bound is tight.

The text is organized as follows. In Section 2 we present some definitions and notation used throughout the paper. In Section 3 we describe Hopcroft’s algorithm, explain how it works.
2 Preliminaries

2.1 DFA, word and language

Definition 2.1. A deterministic finite automaton (DFA) \( D \) is a tuple \( (Q, \Sigma, \delta, q_0, F) \) where:

- \( Q \) is a finite set of states;
- \( \Sigma \) is the input alphabet;
- \( \delta : Q \times \Sigma \rightarrow Q \) is the transition function;
- \( q_0 \) is the initial state;
- \( F \subseteq Q \) is the set of final states.

When the transition function is total, the automaton \( D \) is said to be complete.

Definition 2.2. Any finite sequence of alphabet symbols \( a \in \Sigma \) is a word.

Let \( \epsilon \) denote the empty word. We define the extended transition function \( \hat{\delta} : Q \times \Sigma^* \rightarrow Q \) in the following way:

\[
\begin{align*}
\hat{\delta}(q, \epsilon) &= q \\
\hat{\delta}(q, xa) &= \delta(\hat{\delta}(q, x), a)
\end{align*}
\]

Definition 2.3. The language accepted by \( D \), \( L(D) \), is the set of all words \( w \in \Sigma^* \) such that \( \hat{\delta}(q_0, w) \in F \).

2.2 Equivalent states and Minimal DFA

Given a DFA \( D = (Q, \Sigma, \delta, q_0, F) \), two states \( q_1, q_2 \in Q \) are said to be equivalent, denoted \( q_1 \approx q_2 \), if for every \( w \in \Sigma^* \), \( \delta(q_1, w) \in F \Leftrightarrow \delta(q_2, w) \in F \). Two states that are not equivalent are called distinguishable. The equivalent minimal automaton \( D/\approx \) is called quotient automaton, and its states correspond to the equivalence class of \( \approx \). It is proved to be unique up to isomorphism.

Definition 2.4. Two DFAs \( D \) and \( \tilde{D} \) are equivalent if and only if \( L(D) = L(\tilde{D}) \).

Definition 2.5. A DFA is called minimal if there is no other equivalent DFA with fewer states.

3 Minimization

3.1 The classical algorithm

Definition 3.1. A state \( q \in Q \) of a DFA \( D = (Q, \Sigma, \delta, q_0, F) \) is called accessible if \( \delta(q_0, w) = q \) for some \( w \in \Sigma^* \).

Definition 3.2. If all states in \( Q \) are accessible, a complete DFA \( D \) is called initially-connected (ICDFA).

Lemma 3.1. Suppose that \( D \) is an ICDFA. Then \( D \) is minimal if and only if all pairs of states are distinguishable.
Proof. (⇒) This implication is easy, for if \(D\) has two equivalent states, one of them can be eliminated, and the transitions into this state can be changed to go to the other one. This will not affect the accepted language.

(⇐) Assume that in \(D\) all states are pairwise distinguishable. Let \(D\) have \(k\) states. Consider any other \(D\) with \(l < k\) states. We need to prove that \(L(D) \neq L(\overline{D})\).

For each state \(q\) of \(D\), choose arbitrarily one string \(w_q\) such that \(\delta(q_0, w_q) = q\). That such \(w_q\) exists for each \(q\) follows from the assumption that all states are accessible. Since \(l < k\), there are two states \(p \neq q\) of \(D\) such that in \(\overline{D}\) we have \(\delta(q_0, w_p) = \delta(q_0, w_q)\). Since \(p, q\) are distinguishable in \(D\), there is a string \(w\) such that \(\delta(q_w) \in F\) but \(\delta(p_w) \notin F\) (or vice versa, but if so, we can always swap \(p\) and \(q\)). This means that \(D\) accepts \(w_p w\) but not \(w_q w\). But in \(\overline{D}\), we have \(\delta(q_0, w_p w) = \delta(q_0, w_q w)\), so \(\overline{D}\) either accepts both \(w_p w\) and \(w_q w\) or rejects both. Therefore \(L(\overline{D}) \neq L(D)\), as claimed.

Lemma 3.2. State indistinguishability is an equivalence relation.

Lemma 3.3. Let \(\delta(p, a) = p'\) and \(\delta(q, a) = q'\). Then, if \(p', q'\) are distinguishable then so are \(p, q\).

To minimize automaton \(A\), after removing unreachable states, we will find all equivalence class of the indistinguishability relation and join all states in each class into one state of the new automaton \(\overline{A}\).

To determine which states are equivalent, we will use the following algorithm. Instead of trying to figure out which states are indistinguishable, we will try to figure out which states are distinguishable. The complete algorithm is given below.

Algorithm 1 The classical algorithm

Require: Deterministic finite automaton \(A\)
Ensure: Minimal automaton \(\overline{A}\) equivalent to \(A\)
1: Remove inaccessible states
2: Mark all pairs \(p, q\), where \(p \in F\) and \(q \notin F\)
3: repeat
4: for all non-marked pairs \(p, q\) do
5: for all symbol \(a\) do
6: if the pair \(\delta(p, a), \delta(q, a)\) is marked then
7: mark \(p, q\)
8: end if
9: end for
10: end for
11: until no new pairs are marked
12: Construct the reduced automaton \(\overline{A}\)

Theorem 3.1. Minimization algorithm 3.1 is correct, that is \(L(\overline{A}) = L(A)\) and \(\overline{A}\) is minimal.

Proof. We prove all the required conditions, one by one.

The equivalence are correct The proof is quite easy, by induction on the length of the shortest word that distinguishes \(p, q\), using Lemma 3.3
is well defined The fact that indistinguishability is an equivalence relation from Lemma 3.2 implies that the states of $\tilde{A}$ are well-defined.

$L(\tilde{A}) = L(A)$ For each $w$ we have $\tilde{\delta}(q_0, w) \in \tilde{\delta}(\tilde{q}_0, w)$. Thus, by the definition of the final states of $\tilde{A}$, $\tilde{A}$ accepts $w$ if and only if $A$ accepts $w$.

$\tilde{A}$ is minimal This is where Lemma 3.1 is helpful. This is quite obvious from the construction that all states are distinguishable in $\tilde{A}$.

#### Theorem 3.2

The running time of Algorithm 3.1 is $O(n^2)$.

### 3.2 The Hopcroft’s algorithm

#### Lemma 3.4

Let $D = (Q, \Sigma, \delta, q_0, F)$ and a series $\rho_i$ ($i \geq 0$) of equivalence relations on $Q$ be defined as follows:

$$\rho_0 = \{(p, q)|p, q \in F\} \cup \{(p, q)|p, q \in Q - F\},$$

$$\rho_{i+1} = \{(p, q) \in \rho_i|(\forall a \in \Sigma)(\delta(p, a), \delta(q, a)) \in \rho_i\}. $$

Then the following holds:

- $\rho_0 \supseteq \rho_1 \supseteq \cdots$
- If $\rho_i = \rho_{i+1}$ then $\rho_i = \rho_j$ for all $j > i$.
- There exists $0 \leq k \leq |Q|$ such that $\rho_k = \rho_{k+1}$.

Now consider the situation where $\rho_i \neq \rho_{i+1}$.

$$\rho_i \neq \rho_{i+1} \iff (\exists p, q \in Q, \ a \in \Sigma) \ (p, q) \in \rho_i \ \text{and} \ (\delta(p, a), \delta(q, a)) \notin \rho_i$$

$$\iff (\exists U \subseteq Q/\rho_i, \ a \in \Sigma) \ p, q \in U \ \text{and} \ (\delta(p, a), \delta(q, a)) \notin \rho_i$$

$$\iff (\exists U, V \subseteq Q/\rho_i, \ a \in \Sigma) \ p, q \in U \ \text{and} \ \delta(p, a) \in V \ \text{and} \ \delta(q, a) \notin V$$

$$\iff (\exists U, V \subseteq Q/\rho_i, \ a \in \Sigma) \ \delta(U, a) \cap V \neq \emptyset \ \text{and} \ \delta(U, a) \subseteq V$$

(1)

Using Lemma 3.4 an equivalent relation on $D$ can be computed by the algorithm below:

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**Algorithm 2 Computing equivalent relation using refinements**

1. $Q/\theta \leftarrow \{F, Q - F\}$
2. while $(\exists U, V \subseteq Q/\theta, \ a \in \Sigma)$ s.t. Equation (1) holds do
3. $Q/\theta \leftarrow (Q/\theta - \{U\}) \cup \{U \cap \delta^{-1}(V, a), U - U \cap \delta^{-1}(V, a)\}$
4. end while

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As such, it is yet rather abstract, and in particular we have to decide how to efficiently find some triple $U, V, a$ for which Equation (1) holds.

#### Lemma 3.5

Let $D = (Q, \Sigma, \delta, q_0, F)$ and $U \in Q/\theta$. Suppose we refine $U$ into $U'$ and $U''$. Then, for any $a \in \Sigma$, refining all the classes of $\theta$ with respect to $(U', a)$ and $(U'', a)$ yields the same result as refining $\theta$ with respect to $(U, a)$, $(U', a)$ and $(U'', a)$.

#### Lemma 3.6

Let $D = (Q, \Sigma, \delta, q_0, F)$, $Q/\theta = \{F, Q - F\}$, and $a \in \Sigma$. Then refining $\theta$ with respect to either $(F, a)$ or $(Q - F, a)$ yields the same result as refining $\theta$ with respect to both of them.
Algorithm 3 Hopcroft’s algorithm

1: \( W \leftarrow \{F, Q - F\} \)
2: \( P \leftarrow \{F, Q - F\} \)
3: while \( W \) is not empty do
4:   select and remove \( S \) from \( W \)
5:   for all \( a \in \Sigma \) do
6:     \( l_a \leftarrow \delta^{-1}(S, a) \)
7:     for all \( R \) in \( P \) such that \( R \cap l_a \neq \emptyset \) and \( R \not\subseteq l_a \) do
8:       partition \( R \) into \( R_1 \) and \( R_2 \); \( R_1 \leftarrow R \cap l_a \) and \( R_2 \leftarrow R - R_1 \)
9:       replace \( R \) in \( P \) with \( R_1 \) and \( R_2 \)
10:      if \( R \in W \) then
11:        replace \( R \) in \( W \) with \( R_1 \) and \( R_2 \)
12:      else
13:        if \( |R_1| \leq |R_2| \) then
14:          add \( R_1 \) to \( W \)
15:        else
16:          add \( R_2 \) to \( W \)
17:        end if
18:      end if
19:   end for
20: end for
21: end while

Lemmas 3.5 and 3.6 provide a possibility to reduce calculation. Algorithm 3.2 is the Hopcroft’s algorithm:

Theorem 3.3. Let \( U = \{U_1, U_2, \ldots, U_m\} \) (\( 1 \leq m \leq |Q| \)) be the set of all built classes created in the minimization algorithm. Then

\[ |U| = \sum_{i=1}^{m} |U_i| \leq |Q| \log |Q| \]

Theorem 3.4. The running time of Algorithm 3.2 is \( O(n \log n) \).

References


