# Unambiguous automata on bi-infinite words

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**Abstract**

We consider finite automata accepting bi-infinite words. We introduce unambiguous automata where each accepted word is the label of exactly one accepting path. We show that each rational set of bi-infinite words is accepted by such an automaton. This result is a counterpart of McNaughton's theorem for bi-infinite words.

## 1 Introduction

Roughly speaking, an acceptor is unambiguous if each word is accepted in only one way. For instance, a context-free grammar is unambiguous if any word has at most one derivation tree. Similarly a finite state automaton is generally called unambiguous if any finite word labels at most one accepting path. In this area, an important question is whether any acceptor of some type is equivalent to another one of the same type which is unambiguous. For instance it is well known that any Turing machine is equivalent to a deterministic one which is obviously unambiguous [6]. On the contrary, it is not true that any context-free grammar is equivalent to an unambiguous one. Some context-free languages are inherently ambiguous [3, Prop 1.8].

In this paper, we address the problem of ambiguity for automata accepting bi-infinite words. For automata on finite words, any automaton is equivalent to a deterministic one that can be computed using the classical subset construction [9, Thm 2.1]. This gives an equivalent unambiguous automaton. For $\omega$-words, the same result holds but this is already a nontrivial result [8]. Furthermore, Muller automata which have a more powerful acceptance condition must be used and all determinization algorithms of automata over $\omega$-words that have been given so far are complex [11]. Determinization is not the only way to get unambiguous automata on $\omega$-words [1]. In [5], it has been proved that any Büchi is equivalent to an unambiguous one which is actually co-deterministic. The

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determinization of automata has even been extended to automata on transfinite words by Büchi [4].

In this paper, we show that any automaton on bi-infinite words is equivalent to an unambiguous one. We actually prove two results, a specific one for rational sets that are shift invariant and one for all rational sets. The former one holds for automata with a simple and natural acceptance mode which is only suitable for shift invariant sets. The latter one holds for automata with a more powerful acceptance mode suited to all rational sets. This result is also stronger because it states the existence of automata that are unambiguous and complete. There was no counterpart of McNaughton's result for automata on bi-infinite words. Our result fills this gap.

Complementation is often proved using unambiguous automata. For instance, proving that the complement of a rational set of finite words is also rational is usually achieved through the computation of an equivalent deterministic and complete automaton (see [9, Cor 2.3]. For $\omega$-words, complementation becomes also easy if a deterministic Muller automaton is provided. In both cases, the key idea for complementation is to use an automaton where each word labels exactly one path that can be accepting. In deterministic automata, this path is of course the unique path stating in the initial state. Our second result provides unambiguous and complete automata that can be used for complementation.

A rational expression is unambiguous if each word is described in a unique way by the expression. This means that all unions appearing in the expression are disjoint unions and that product, stars and $\omega$-iterations are unambiguous. Recall for instance that a product $XY$ is unambiguous if any word $z$ in $XY$ has a unique factorization $z = xy$ where $x$ and $y$ are in $X$ and $Y$. It is well-known that any rational set of finite words is described by an unambiguous rational expression [9, Prop 4.3]. It follows from Arnold's results [1], that the same result holds for rational sets of infinite words. No similar result is known for sets of bi-infinite words but the unambiguous automata that we have introduced seem to be a step towards such a result.

In symbolic dynamics [7, 2] are studied sets of bi-infinite words that are shift invariant and closed for the usual topology. These sets are called subshifts and they are characterized by the factors of their elements. Recall that a factor of a bi-infinite word is a finite word that appear in it. When its set of factors is rational, a subshift is the set of labels of all bi-infinite paths in an automaton without any acceptance condition and it is called a sofic subshift. Automata without any acceptance condition where each bi-infinite word labels at most one path are called local. They correspond to a strict subclass of sofic subshifts called subshifts of finite type.

The paper is organized as follows. The notion of bi-infinite words is defined in Section 2. Automata accepting these words are introduced in Section 3. The definition of unambiguous automata is given is Section 4. Each of the two main results is stated and proved in a subsection (Subsections 4.1 and 4.2) of this section.

## 2 Bi-infinite Words

In this section, we recall the notion of bi-infinite words. We actually define two notions of bi-infinite words, namely pointed and unpointed words. The latter one refers to shift invariant sets.

Let $A$ be a finite alphabet whose elements are called *letter*. A *finite word* over $A$ is a finite sequence $a_1 \dots a_n$ of letters. An *$\omega$-word* $x$ is a infinite sequence $x_0x_1x_2\dots$ of letters. The sets of finite and $\omega$-words are respectively denoted by $A^*$ and $A^\omega$. A *pointed bi-infinite word* $z$ or a *pointed word* for short is a sequence $\dots z_{-2}z_{-1} \cdot z_0z_1z_2 \dots$ of letters indexed by the set $\mathbb{Z}$ of relative integers. A dot is inserted between the letters indexed by $-1$ and $0$ to mark that position. The sets of all pointed bi-infinite words is denoted $A^{\mathbb{Z}}$. It is equipped with the *shift* $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ defined by

$$\sigma((z_k)_{k\in\mathbb{Z}}) = (z_{k-1})_{k\in\mathbb{Z}}.$$

The image by the shift of a pointed word $\dots z_{-3}z_{-2}z_{-1} \cdot z_0z_1z_2 \dots$ is thus the pointed word $\dots z_{-3}z_{-2} \cdot z_{-1}z_0z_1z_2 \dots$ where the marked position has been shift one position to the left. The shift induces the equivalence relation $\sim$ on $A^{\mathbb{Z}}$ defined as follows. Two pointed words $z$ and $z'$ satisfy $z \sim z'$ if $z' = \sigma^n(z)$ for some relative integer $n$. The class of a pointed word $z$ is denoted by $\sigma^{\mathbb{Z}}(z) = \{\sigma^n(z) \mid n \in \mathbb{Z}\}$. If $z$ is equal to $\dots z_{-2}z_{-1} \cdot z_0z_1z_2 \dots$, its class $\sigma^{\mathbb{Z}}(z)$ is written $\dots z_{-2}z_{-1}z_0z_1z_2 \dots$ without any dot. A class $\sigma^{\mathbb{Z}}(z)$ is called an *unpointed bi-infinite word* or an *unpointed word* for short. The set of all unpointed words is denoted by $A^{\zeta}$. For a set $Z$ of pointed words, we denote by $\sigma^{\mathbb{Z}}(Z)$ the set $\{\sigma^{\mathbb{Z}}(z) \mid z \in Z\}$ of unpointed words.

Let $z = (z_i)_{i\in\mathbb{Z}}$ be the pointed word defined by $z_i = a$ if $i$ is even and $z_i = b$ if $i$ is odd. The class $\sigma^{\mathbb{Z}}(z)$ of $z$ only contains the two pointed words $z = \dots abab \cdot abab \dots = (ab)^{\tilde{\omega}} \cdot (ab)^\omega$ and $\sigma(z) = \dots baba \cdot baba \dots = (ba)^{\tilde{\omega}} \cdot (ba)^\omega$. This class is the unpointed word $\dots ababab \dots$ which is denoted $(ab)^{\zeta}$.

A set $Z$ of pointed words is *shift invariant* if it satisfies $\sigma(Z) = Z$. If $Z$ is shift invariant, each class $\sigma^{\mathbb{Z}}(z)$ is either contained in $Z$ if $z \in Z$ or disjoint from $Z$ if $z \notin Z$. The set $Z$ is then equal to the union $\bigcup_{z\in Z} \sigma^{\mathbb{Z}}(z)$ of classes. This means that a shift invariant set $Z$ can be identified with the set $\sigma^{\mathbb{Z}}(Z)$ of unpointed words.

Let $x = x_0x_1x_2\dots$ and $y = y_0y_1y_2\dots$ be two $\omega$-words. We denote by $\tilde{x} \cdot y$ the pointed word $z = \dots x_2x_1x_0 \cdot y_0y_1y_2 \dots$ obtained by concatenating the mirror image of $x$ with $y$. The corresponding unpointed word $\dots x_2x_1x_0y_0y_1y_2 \dots$ is denoted by $\tilde{x}y$. For two sets $X$ and $Y$ of $\omega$-words, we respectively denote by $\tilde{X} \cdot Y$ and $\tilde{X}Y$ the sets $\{\tilde{x} \cdot y \mid x \in X \text{ and } y \in Y\}$ and $\{\tilde{x}y \mid x \in X \text{ and } y \in Y\}$. Note that $\sigma^{\mathbb{Z}}(\tilde{X} \cdot Y) = \tilde{X}Y$.

Let $A$ be the alphabet $\{a, b\}$ and let $X$ and $Y$ be the sets $A^\omega$ and $aA^\omega$ of $\omega$-words. The set $\tilde{X} \cdot Y = A^{\tilde{\omega}} \cdot aA^\omega$ is the set of pointed words having an $a$ at position 0 whereas $\tilde{X}Y = A^{\tilde{\omega}}aA^\omega$ is the set of unpointed words having at least one occurrence of an $a$.

### 2.1 Rational sets of bi-infinite words

A set $Z$ of pointed (respectively unpointed) words is said to be *rational* if $Z$ is equal to a finite union $Z = \bigcup_{i=1}^{n} \tilde{X}_i \cdot Y_i$ (respectively $Z = \bigcup_{i=1}^{n} \tilde{X}_i Y_i$) where each set $X_i$ or $Y_i$ is a rational set of $\omega$-words. For a shift invariant set $Z$, $Z$ is rational as a set of pointed words if and only if $\sigma^{\mathbb{Z}}(Z)$ is rational as a set of unpointed words. In the sequel, we need the following result.

**Theorem 1** *The complement of a rational set of pointed (respectively unpointed) words is also rational.*

We refer the reader to [10, chap. 9] for a proof of this result.

# 3 Automata

In this section, we recall the notion of automata accepting bi-infinite words. These automata are a natural extension of Büchi automata to bi-infinite words. We refer the reader to [10, chap. 9] for a complete introduction to these notions.

As we shall see, automata equipped with the most natural acceptance mode accept unpointed words. For pointed words, an additional set of states called middle states is needed to define the accepting paths.

An *automaton* over the alphabet $A$ is given by a set $Q$ of states, a set $E \subseteq Q \times A \times Q$ of *transitions* and by sets $I \subseteq Q$ and $F \subseteq Q$ of *initial* and *final* states. For an automaton accepting pointed words, a set $M \subseteq Q$ of *middle* states is also given. Such an automaton $\mathcal{A}$ is denoted $(Q, A, E, I, F)$ or $(Q, A, E, I, M, F)$ in the latter case. Each transition $(p, a, q)$ is denoted by $p \xrightarrow{a} q$ and the letter $a$ is called its *label*.

Generally speaking, a path in $\mathcal{A}$ is a sequence of consecutive transitions. As for words, we consider infinite and bi-infinite paths. An infinite (respectively bi-infinite) *path* is an infinite (respectively bi-infinite) sequence of consecutive transitions. The label of a path is the concatenation of the labels of its transitions. The label of an infinite (respectively bi-infinite) path is thus an infinite (respectively bi-infinite) word over $A$. More formerly, the *label* of an infinite path

$$q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \cdots$$

is the $\omega$-word $a_0 a_1 a_2 \ldots$ and the *label* of a bi-infinite path $\gamma$

$$\cdots q_{-2} \xrightarrow{a_{-2}} q_{-1} \xrightarrow{a_{-1}} q_0 \xrightarrow{a_0} q_1 \xrightarrow{a_1} q_2 \xrightarrow{a_2} q_3 \cdots$$

is the pointed bi-infinite word $z = (a_k)_{k \in \mathbb{Z}}$. Note that the pointed bi-infinite word $\sigma^n(z)$ is the label of the path $\sigma^n(\gamma)$ obtained by shifting the path. Therefore, we say that the unpointed word $\sigma^{\mathbb{Z}}(z)$ is also the *label* of the path $\gamma$.

An infinite path is said to be *initial* if its first state is initial. This definition is of course not relevant to bi-infinite paths which do not have any first state. A bi-infinite path is said to be *initial* if at least one initial state occurs infinitely often on the left.

More formerly, let us define the left limit $\lim_{-\infty}\gamma$ of a bi-infinite path $\gamma$ by

$$\lim_{-\infty}\gamma = \{q \mid \forall n \in \mathbb{N}\ \exists k\ \ k < -n \text{ and } q_k = q\}.$$

A bi-infinite path $\gamma$ is *initial* if $\lim_{-\infty}\gamma \cap I$ is non-empty. An infinite or bi-infinite path is final if at least one final state occurs infinitely often on the right. Let us define similarly the right limit of an infinite or bi-infinite path $\gamma$ by

$$\lim_{+\infty}\gamma = \{q \mid \forall n \in \mathbb{N}\ \exists k\ \ k > n \text{ and } q_k = q\}.$$

An infinite or bi-infinite path is *final* if $\lim_{+\infty}\gamma \cap F$ is non-empty.

In a Büchi automaton $\mathcal{A} = (Q, A, E, I, F)$, an infinite path is *accepting* if it is both initial and final. This means that its first state is initial and that it satisfies the well-known Büchi condition [12, p. 136]. An $\omega$-word is *accepted* by a Büchi automaton if it is the label of an accepting path.

In the figures, initial and final states are respectively marked by a small incoming and outgoing arrow.

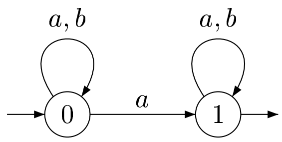


Figure 1: An automaton for unpointed words

**Example 2** Consider the automaton pictured in Figure 1 as a Büchi automaton. It accepts the set $b^*aA^\omega$ of $\omega$-words having at least one occurrence of an $a$.

In an automaton $\mathcal{A} = (Q, A, E, I, F)$ a bi-infinite path is *accepting* if it is both initial and final. This means that its left part satisfies the Büchi condition with respect to initial states and that its right part satisfies the Büchi condition with respect to final states. An unpointed word $z$ is *accepted* by an automaton if it is the label of an accepting path.

**Example 3** Consider again the automaton pictured in Figure 1. It accepts the set $A^{\tilde{\omega}}aA^\omega$ of unpointed words having at least one occurrence of an $a$.

This acceptance mode is the most natural one for bi-infinite words because it simply generalizes the Büchi acceptance condition used for infinite words. For unpointed words, this mode is suitable since there is a Kleene-like theorem. A set of unpointed words is rational if and only if it is accepted by a finite automaton. For pointed words, this acceptance mode is unsuitable for the following reason. If $\gamma$ is an accepting path, the path $\sigma(\gamma)$ is also accepting. Therefore, if the pointed word $z$ is accepted, the word $\sigma(z)$ must also be accepted. Its turns out that only shift invariant sets can be accepted by automata with this acceptance mode.

In an automaton $\mathcal{A} = (Q, A, E, I, M, F)$ with a set $M$ of middle states, a bi-infinite path $\gamma$ is *accepting*, if it is both initial and final and if the state $q_0$ at position 0 in $\gamma$ is a middle state.

In the figures, the middle states are marked by a double circle.

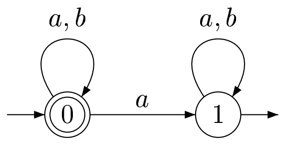


Figure 2: An automaton for pointed words

**Example 4** Consider the automaton pictured in Figure 2. It accepts the set $A^{\tilde{\omega}} \cdot aA^{\omega}$ of pointed words having an $a$ at position 0.

# 4 Unambiguous automata

In this section, we introduce the notion of unambiguous automata on bi-infinite words. We first give the definitions of unambiguity and completeness for these automata. These definitions make sense for both automata on pointed bi-infinite words and automata on unpointed bi-infinite words. However, the results that can be obtained are different. For pointed words, the main result is that any rational set can be accepted by an automaton which is both unambiguous and complete. For unpointed words, it is only possible to accept any rational set by an unambiguous automaton. The automaton cannot be complete in general. This comes intrinsically from the acceptance mode.

We begin with the definition and we provide some examples and counter-examples.

**Definition 5** *An automaton $\mathcal{A}$ is* $\zeta$-unambiguous *(respectively* $\zeta$-complete*) if any bi-infinite word is the label of at most (respectively at least) one path which is initial and final.*

Note first that the initial and final paths of an automaton only depend on its initial and final states. Therefore, this definition can be applied to both automata with or without middle states. Second, an automaton is unambiguous (respectively complete) for pointed words if and only if it is unambiguous (respectively complete) for unpointed words. This is due to the fact that if a bi-infinite word $z$ labels a path, the word $\sigma(z)$ labels the path $\sigma(\gamma)$.

To illustrate this definition, we provide examples of $\zeta$-ambiguous and $\zeta$-unambiguous automata.

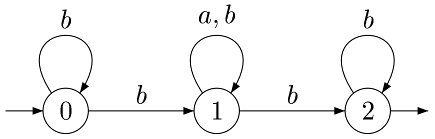


Figure 3: $\zeta$-ambiguous automaton of Example 6

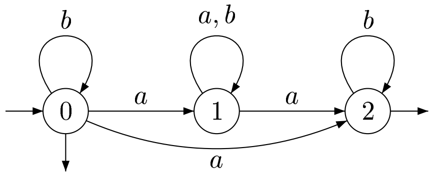


Figure 4: $\zeta$-unambiguous automaton of Example 6

**Example 6** The automaton pictured in Figure 3 accepts the set $b^{\tilde{\omega}}(a+b)^*b^{\omega}$ of unpointed words having finitely many $a$. This automaton is $\zeta$-ambiguous since the word $b^{\zeta}$ is the label of the two accepting paths $0^{\tilde{\omega}}12^{\omega}$ and $0^{\tilde{\omega}}112^{\omega}$. The automaton pictured in Figure 4 accepts the set same set but it is $\zeta$-unambiguous. This automaton is however not $\zeta$-complete since the word $a^{\zeta}$ is not the label of a path.

From the previous example, it might seem obvious to find an unambiguous automaton recognizing a given set of unpointed words. The following example shows that it is not always so easy.

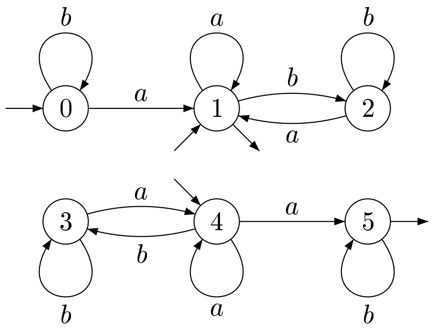


Figure 5: $\zeta$-unambiguous automaton of Example 7

**Example 7** The set $(ab^*)^{\tilde{\omega}}(a+b)^{\omega} + (a+b)^{\tilde{\omega}}(b^*a)^{\omega}$ of unpointed words having infinitely many $a$ is accepted by the automaton pictured in Figure 5. This automaton is $\zeta$-unambiguous. This automaton is also not $\zeta$-complete since the word $b^{\zeta}$ is not the label of a path.

## 4.1 Pointed words

In this section, we focus our attention to pointed bi-infinite words. We first show that unambiguous and complete automata behave well with respect to complementation. Then, we state and we prove the main result for pointed words.

Complementation becomes easy with unambiguous and complete automata on pointed words. It suffices to exchange middle states and non-middle states to get an automaton accepting the complement set.

**Proposition 8** *Let $\mathcal{A} = (Q, A, E, I, M, F)$ be a $\zeta$-unambiguous and $\zeta$-complete automaton accepting a subset $Z$ of bi-infinite pointed words. The automaton $(Q, A, E, I, Q \setminus M, F)$ accepts the complement of $Z$.*

In the previous examples, we have seen that many typical rational sets of unpointed bi-infinite words can accepted by a $\zeta$-unambiguous automata. The following theorem states that this is actually true for any rational set of pointed bi-infinite words. This is one of the main results of the paper.

**Theorem 9** *Any rational set of pointed bi-infinite words is accepted by a $\zeta$-unambiguous and $\zeta$-complete automaton.*

The result of McNaughton states that any Büchi automaton can be replaced by an equivalent automaton which is deterministic and thus unambiguous. The previous theorem is thus the counterpart of McNaughton's result for bi-infinite words.

The proof of Theorem 9 is based on a similar result concerning automata on $\omega$-words. This result involves the notion of an unambiguous automaton on $\omega$-words that we now define.

**Definition 10** *An automaton $\mathcal{A} = (Q, A, E, I, F)$ is said to be* $\omega$-unambiguous *(respectively* $\omega$-complete*) if any $\omega$-word is the label of at most (respectively at least) one final path.*

The main difference with the definitions on unambiguity and completeness for automata on bi-infinite words is that final paths are considered instead of paths which are both initial and final.

We can now state the analogous result for $\omega$-words.

**Theorem 11 ([5])** *Any rational set of $\omega$-words is accepted by an $\omega$-unambiguous and $\omega$-complete automaton.*

Let $\mathcal{A} = (Q, A, E, I, F)$ and $\mathcal{A}' = (Q', A, E', I', F')$ be two automata on $\omega$-words. The (synchronized) product $\tilde{A} \times \mathcal{A}'$ is the automaton $\mathcal{B}$ defined as follows. The set of states of $\mathcal{B}$ is $Q \times Q'$, its sets of initial and final states are respectively $F \times Q'$ and $Q \times F'$ and its set $T$ of transitions is defined by

$$T = \{(p, q) \xrightarrow{a} (p', q') \mid p' \xrightarrow{a} p \in E \text{ and } q \xrightarrow{a} q' \in E'\}.$$

Note that the transitions of $\mathcal{A}$ are used backwards in the automaton $\tilde{\mathcal{A}} \times \mathcal{A}'$.

The following lemma allows us to combine unambiguous and complete automata on $\omega$-words to make unambiguous and complete automata on bi-infinite words.

**Lemma 12** *If $\mathcal{A}$ and $\mathcal{A}'$ are two $\omega$-unambiguous and $\omega$-complete automata, the synchronized product $\tilde{\mathcal{A}} \times \mathcal{A}'$ is $\zeta$-unambiguous and $\zeta$-complete.*

The main idea of the proof of Theorem 9 is to use the synchronized product of automata on $\omega$-words and Lemma 12.

### 4.2 Unpointed words

In this section, we focus our attention to unpointed words.

**Theorem 13** *Any rational set of unpointed bi-infinite words is accepted by a $\zeta$-unambiguous automaton.*

Note that the automaton given by the previous theorem cannot be complete since it would accept the full set $A^{\zeta}$.

The result can however be applied to both a set $X$ and its complement $\complement X = A^{\zeta} \setminus X$ to get two unambiguous automata $\mathcal{A}_X$ and $\mathcal{A}_{\complement X}$ recognizing $X$ and $\complement X$. The automaton $\mathcal{A} = \mathcal{A}_X \cup \mathcal{A}_{\complement X}$ is still unambiguous and it is also complete. One obtains an unambiguous and complete automaton which is naturally divided in two parts. The unique path labeled by any unpointed bi-infinite word $x$ is either in the first part of the automaton if $x$ belongs to $X$ and in the second part otherwise. If for instance the automata pictured in Figure 4 and 5 are joined, the resulting automaton is unambiguous and complete.

The construction used in the proof of Theorem 13 is illustrated by the following example.

**Example 14** Consider again the automaton of Example 3 pictured in Figure 1. It accepts the set $X = A^{\tilde{\omega}} a A^{\omega}$ of bi-infinite words having at least one occurrence of $a$. The construction described in the proof of Theorem 13 can be used to get an unambiguous automaton recognizing $X$. The complement of $X$ is the set $\complement X = b^{\zeta}$. The unambiguous and complete automata $\mathcal{A}$ (in the Figure is actually pictured the automaton $\tilde{\mathcal{A}}$ obtained by reversing the transitions of $\mathcal{A}$) and $\mathcal{A}'$ pictured in Figure 6 accept the sets $b^{\omega}$ and $b^{\omega}$. The automaton $\tilde{\mathcal{A}} \times \mathcal{A}'$ is unambiguous and complete. One gets an unambiguous automata recognizing $X$ by removing the state 02 from it.

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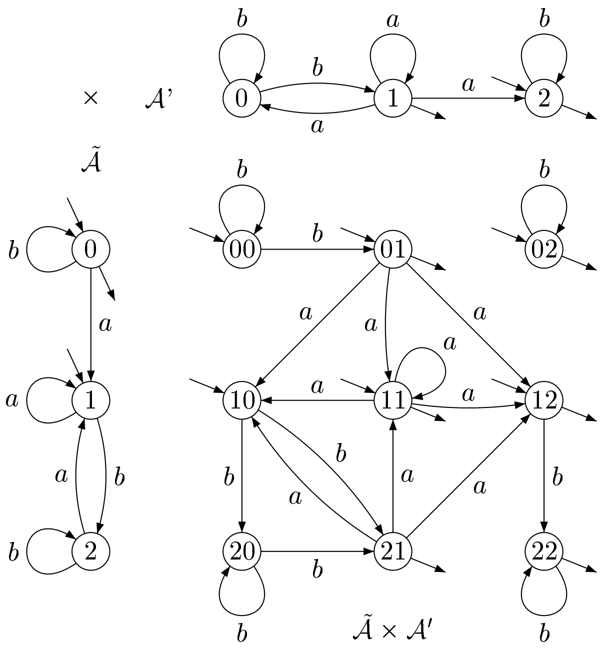


Figure 6: Product automaton of Example 14

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