Automatic conversion from Fibonacci representation to representation in base φ, and a generalization *

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Corrected final version — 9 February 1998

AMS Mathematics Subject Classification 11A63, 11A67, 11B39, 68Q70.

Abstract

Every positive integer can be written as a sum of Fibonacci numbers; it can also be written as a (finite) sum of (positive and negative) powers of the golden mean φ. We show that there exists a letter-to-letter finite two-tape automaton that maps the Fibonacci representation of any positive integer onto its φ-expansion, provided the latter is folded around the radix point. As a corollary, the set of φ-expansions of the positive integers is a linear context-free language. These results are actually proved in the more general case of quadratic Pisot units.

Résumé

Tout nombre entier positif peut s'écrire comme une somme de nombres de Fibonacci; tout entier peut également s'écrire comme une somme (finie) de puissances (positives et négatives) du "nombre d'or" φ. Nous montrons qu'il existe un automate à deux bandes, fini et lettre-à-lettre, qui envoie la représentation d'un entier en base de Fibonacci sur sa représentation dans la base φ modulo le fait qu'on a replié cette dernière autour du point décimal. On en déduit que l'ensemble des représentations des entiers en base φ est un langage context-free linéaire. Tous ces résultats sont en fait établis dans le cas général où la base considérée est un nombre de Pisot quadratique unitaire.

*This work has been supported by the PRC Mathématiques et Informatique of the Ministère de la Recherche et de la Technologie and by the ESPRIT Basic Research Action 6317 "ASMICS". Research of the first author at MSRI, Berkeley, was supported in part by NSF grant #DMS 9022140.

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Automatic conversion from Fibonacci representation to representation in base φ, and a generalization[1]

1 *Where the reader is introduced to Fibonacci and base φ numeration systems, presented with a small tribute to Marcel-Paul Schützenberger, and asked two questions.*

The writing of numbers, the various ways it can take, have always attracted attention of mathematicians as well as of computer scientists. Some systems — such as the redundant decimal system with digits $\{-6, -5, \ldots, 6\}$ — have been invented in order to implement improved algorithms for some operations (*cf.* [1]). Some have been considered because they bring to light remarkable mathematical objects or properties. This is the case, for instance, of both the Fibonacci numeration system and the golden mean base.

Let $F = \{F_n \mid n \in \mathbb{N}\}$ be the sequence of Fibonacci numbers, defined by the recurrence relation

$$F_{n+2} = F_{n+1} + F_n \qquad (*)$$

and by the "initial conditions" [2]

$$F_0 = 1 \quad , \quad F_1 = 2 \quad .$$

It is well-known[3] that every positive integer can be written as a sum of Fibonacci numbers; the sequence F together with the two-digit alphabet $A = \{0, 1\}$ defines thus the *Fibonacci numeration system*, *i.e.,* every integer is represented by a sequence of 0's and 1's; *e.g.,*

$$24 = F_6 + F_2 \qquad \text{and} \qquad 24 \quad \text{is represented by} \quad 1000100 \quad .$$

In contrast to what happens in the binary numeration system (*i.e.,* the sequence of powers of 2, together with A) the representation of numbers in the Fibonacci system is not unique; *e.g.,*

$$24 = F_5 + F_4 + F_2 \qquad \text{and} \qquad 24 \quad \text{is also represented by} \quad 110100 \quad .$$

[1] A preliminary version of this paper appeared under the title "From the Fibonacci numeration system to the golden mean base and some generalizations" in the *Proceedings of the Conference "Formal Power Series and Algebraic Combinatorics"*, Florence, Italy, June 21–25, 1993, 231–244. In several places this version has been significantly rewritten.

[2] These are *not* the "usual" initial conditions but they happen to be the "good" ones when one wants to turn the Fibonacci sequence into a numeration system.

[3] and usually credited to Zeckendorf [21]; *cf.* also the Exercise 1.2.8.34 in [16].

However, every non-negative integer can be given a *normal representation*, the largest[4] in the lexicographic ordering, which is characterized by the fact it does not contain two consecutive 1's (*cf.* the exercise quoted above). The set of all normal representations of the positive integers is thus

$$R_F = 1A^* \setminus A^*11A^* \quad ,$$

a *rational*[5] set of words of the free monoid A^*, *i.e.,* a set of words *recognized* by a finite automaton.

It seems that it was Schützenberger who first noticed that it is not only true that there exists a finite automaton that *recognizes* the set of all normal representations but there also exists a finite two-tape automaton (an automaton with output) that *computes* the normal representation equivalent to any given representation. Figure 1 shows a *facsimile* of a manuscript[6] of Schützenberger giving such an automaton[7].

Figure 1: A Fibonacci "standardisateur" by Schützenberger.

In the same letter, Schützenberger also conjectured that this property should hold for

[4]when considering representations of the same length after adding leading 0's to the shorter ones.

[5]as we follow the terminology and notation of [18] — which are also those of [6] — we say *rational* rather than *regular* (*cf.* Section 3).

[6]We are thankful to Jean Berstel who kindly gave us a copy of it.

[7]Schützenberger writes numbers least significant digit first, *i.e.,* in the opposite way we are using here, and his automaton performs then the reduction 110 gives 001. "Standardisateur" is a neologism that Schützenberger coined for the occasion and means *normalizer.* Note also that this automaton *is not deterministic* in the input; it is not the "simplest" that performs the Fibonacci normalization (*cf.* [18, p. 44] where a normalizer with 4 states is given) but it is a direct consequence of basic results ([3, Th. IV.2.8]) that such a normalizer *cannot* be deterministic in the input.

any numeration system defined by a linear recurrence relation (with integral coefficients). It is now known that the result is not true in general but, roughly speaking, only for those linear relations that correspond (*via* their characteristic polynomial) to Pisot numbers [9, 13], a statement that is probably even more striking than the original conjecture.

On the other hand, it has been observed that numbers (integers but also real numbers in general) can be represented in (geometric) numeration systems defined by non-integral bases (*cf.* [19]). Such representations form *symbolic dynamical systems* that have been extensively studied.

In particular, let φ be the *golden mean i.e.,* the larger zero of

$$P(X) = X^2 - X - 1 \ ,$$

which is the characteristic polynomial of the recurrence relation ($*$). As above, it is known (*cf.* [16, Exercise 1.2.8.35]) that every number x can be written as a sum of (positive and negative) powers of φ and thus can be represented as a sequence — possibly infinite — of 0's and 1's together with a radix point; *e.g.,*

$$5 = \varphi^3 + \varphi^{-1} + \varphi^{-4} \qquad \text{and} \qquad 5 \quad \text{is represented by} \quad 1000.1001 \ .$$

Such a sequence is called a *φ-representation* of x. For every real number there exists a unique normal φ-representation, called its *φ-expansion*: the one that does not contain two adjacent 1's and does not terminate by the infinite factor 101010 .... From this statement follows that the set of all φ-expansions (of the real numbers) is recognized by a finite automaton (accepting infinite words) and it is not difficult to adapt the Schützenberger normalizer in order to get a two-tape automaton (on infinite words) that computes the φ-expansion equivalent to any given φ-representation. This characterizes the set of the φ-expansions of the *reals*.

The comparison of the two situations leads to the following two questions. Does there exist a characterization of the φ-expansions of the *integers*? And is there any *relationship* between the φ-expansion of an integer and its normal representation in the Fibonacci system?

2 Where the answer is given, the solution that leads to it presented, still on the example of the Fibonacci system, and the domain of validity of the answer precisely delimited.

The answer is *yes*, to both questions, and this is what the paper is all about. The answer is *yes* to the first one, *as a consequence* of the *yes* to the second. The latter was announced in the title: "automatic" is to be understood as "computable by a finite two-tape automaton", just as, for instance, in "automatic group", that are groups in which the multiplication (by a generator) is realized by a (letter-to-letter) finite two-tape automaton (*cf.* [8]).

As we already stated, the set of all normal Fibonacci representations of the positive integers[8] is the rational language

$$R_F = 1A^* \setminus A^*11A^* \quad .$$

To begin with, let us be empirical in approaching the characterization of the set R_φ of the φ-expansions of all positive integers. It first appears that every positive integer has a *finite* φ-expansion (*cf.* Proposition 1). Table 1 below gives the φ-expansion of the first 15 integers together with their Fibonacci normal representation.

The position of the radix point, roughly situated, as Table 1 shows, in the middle of every expansion, suggests that R_φ is not a rational language. It will be eventually shown that R_φ is a linear context-free language[9] (see Corollary 4). This is the consequence of a much more precise result that will require some transformations on R_φ in order to be stated.

Let $f.g$ be the φ-expansion of an integer N, *i.e.,* an element of R_φ; the words f and g belong to $\{0,1\}^*$. It is a classical result ([20]) that R_φ is a linear context-free language if the set

$$S = \{(f, g^t) \mid f.g \in R_\varphi\}$$

is a rational set in $\{0,1\}^* \times \{0,1\}^*$ (g^t denotes the mirror image of g). Moreover, as we have already noted, the lengths of f and of g are approximately equal — the difference of these lengths is indeed bounded by 1 — and this property implies that S is a rational set in $\{0,1\}^* \times \{0,1\}^*$ if, and only if, it is a rational set in $(\{0,1\} \times \{0,1\})^*$ (*cf.* [7, 6, 10]). Such a statement will be made more intelligible by means of the following convention. Every element of $J = \{0,1\} \times \{0,1\}$ will be written as a "vertical double-digit" :

$$J = \left\{ {0 \atop 0}, {0 \atop 1}, {1 \atop 0}, {1 \atop 1} \right\} \quad .$$

Any element of J^* can be read as the superposition of two words of equal length, an "upper word" above a "lower word". If $f.g$ is the φ-expansion of N, its expression $\left({f \atop g^t}\right)$ as an element of J^* will naturally be called the *folded* φ-expansion of N; *e.g.,* the folded φ-expansion of 5 is $\left({1000 \atop 1001}\right)$. Table 1 gives the folded φ-expansion of the 15 first integers as well.

Let T_φ be the set of folded φ-expansions of all positive integers; the announced characterization of R_φ then reads:

[8] It is convenient *not* to deal with 0. Whatever representation is chosen for 0 — 0, to stick to common sense, or the empty word, to be more consistent with the rest of the theory — it will not fit with the general case.

[9] All definitions are postponed to Section 3.

N	Fibonacci representations	φ-expansions	Folded φ-expansions
1	1	1.	1 0
2	10	10.01	1 0 1 0
3	100	100.01	1 0 0 0 1 0
4	101	101.01	1 0 1 0 1 0
5	1000	1000.1001	1 0 0 0 1 0 0 1
6	1001	1010.0001	1 0 1 0 1 0 0 0
7	1010	10000.0001	1 0 0 0 0 0 1 0 0 0
8	10000	10001.0001	1 0 0 0 1 0 1 0 0 0
9	10001	10010.0101	1 0 0 1 0 0 1 0 1 0
10	10010	10100.0101	1 0 1 0 0 0 1 0 1 0
11	10100	10101.0101	1 0 1 0 1 0 1 0 1 0
12	10101	100000.101001	1 0 0 0 0 0 1 0 0 1 0 1
13	100000	100010.001001	1 0 0 0 1 0 1 0 0 1 0 0
14	100001	100100.001001	1 0 0 1 0 0 1 0 0 1 0 0
15	100010	100101.001001	1 0 0 1 0 1 1 0 0 1 0 0

Table 1: Fibonacci representations and φ-expansions of the 15 first integers

PROPOSITION **A** *T_φ is a rational set of J^*.*

Indeed, Proposition A appears as the consequence of a much stronger result that, for every integer N, relates its Fibonacci representation and its folded φ-expansion and which is stated by the following:

THEOREM **B** *There exists a letter-to-letter finite two-tape automaton $\mathcal{A}_\varphi$ that maps the Fibonacci representation of any integer onto its folded φ-expansion.*

The automaton $\mathcal{A}_\varphi$ is not constructed directly. Rather, its construction is broken up into several steps. A major one consists in the fact that normalization — *i.e.,* computation of *the* φ-expansion from *any* φ-representation — can be achieved by a letter-to-letter finite two-tape automaton (*cf.* [9]). A few other ones amount to constructions involving letter-to-letter finite two-tape automata (Propositions 7 and 8).

But the main step in proving Theorem B (later, Theorem 2) is the construction of an automaton $\mathcal{T}_\varphi$ that reads words where the letters have been *grouped into blocks of length* 4, and with the property that there is *at most one digit* 1 *in every block*. As seen on its (deterministic) underlying input automaton shown in Figure 2, this automaton $\mathcal{T}_\varphi$ is remarkably simple. It has 5 states, in a one-to-one correspondence with the above mentioned blocks; it consists in the complete oriented graph with 5 vertices, as indicated in Table 2 which gives the input and output labels of every edge. Every state is final, and denoted as such by an outgoing arrow.

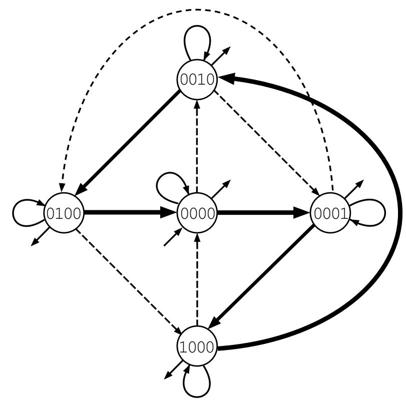

Figure 2: The underlying input automaton of $\mathcal{T}_\varphi$. This is a partial view: the only transitions represented are those labelled by 0 0 0 1 (bold arrows), by 0 0 1 0 (dashed arrows) and by 0 0 0 0 (loops). The transitions labelled by 0 1 0 0 (resp. by 1 0 0 0) are the reverse of those labelled by 0 0 0 1 (resp. by 0 0 1 0).

It should be noted that the output labels of the edges in $\mathcal{T}_\varphi$ are far from being normalized (since digits like 2 or even negative digits like $\bar{1}$ are allowed). It is this freedom in the choice of the output labels that makes possible the construction of a two-tape automaton with such a simple (and deterministic) underlying input automaton, here, and even more strikingly in the general case.

The aim of this paper is to establish a more general version of Theorem B — and thus Proposition A — the generalization consisting of proving the property not only for the golden mean φ but for *any quadratic Pisot unit* θ.

The precise statement requires more definitions and notation that will be given in the next section. The core of the proof will be the complete description of the two-tape automaton $\mathcal{T}_\theta$ in the general case (Sections 6 and 7). This description is made possible by the identification of the underlying input automaton of $\mathcal{T}_\theta$ with a finite Abelian group, the existence of which is "discovered" in Section 6. In Section 5, it is shown how the main theorem (Theorem 2) can be derived from the construction of $\mathcal{T}_\theta$, the idea of which arises — in Section 4 — from the computation (Proposition 5) of the θ-expansion of the elements of the sequence U_θ (that generalizes the Fibonacci sequence F).

	end	0000	0001	0010	0100	1000
origin	label					
0000		$0000 \,/\, \begin{smallmatrix}0000\\0000\end{smallmatrix}$	$0001 \,/\, \begin{smallmatrix}0001\\0000\end{smallmatrix}$	$0010 \,/\, \begin{smallmatrix}0010\\0010\end{smallmatrix}$	$0100 \,/\, \begin{smallmatrix}0100\\0010\end{smallmatrix}$	$1000 \,/\, \begin{smallmatrix}1000\\1001\end{smallmatrix}$
0001		$0100 \,/\, \begin{smallmatrix}1000\\1\bar{1}00\end{smallmatrix}$	$0000 \,/\, \begin{smallmatrix}0001\\1000\end{smallmatrix}$	$1000 \,/\, \begin{smallmatrix}1010\\2000\end{smallmatrix}$	$0010 \,/\, \begin{smallmatrix}0100\\1010\end{smallmatrix}$	$0001 \,/\, \begin{smallmatrix}0010\\1010\end{smallmatrix}$
0010		$1000 \,/\, \begin{smallmatrix}1100\\0000\end{smallmatrix}$	$0010 \,/\, \begin{smallmatrix}0101\\0100\end{smallmatrix}$	$0000 \,/\, \begin{smallmatrix}0010\\0100\end{smallmatrix}$	$0001 \,/\, \begin{smallmatrix}0100\\0010\end{smallmatrix}$	$0100 \,/\, \begin{smallmatrix}1000\\1000\end{smallmatrix}$
0100		$0001 / \begin{smallmatrix}1000\\0\bar{1}00\end{smallmatrix}$	$1000 / \begin{smallmatrix}1101\\1000\end{smallmatrix}$	$0100 \,/\, \begin{smallmatrix}1010\\1000\end{smallmatrix}$	$0000 \,/\, \begin{smallmatrix}0100\\0010\end{smallmatrix}$	$0010 \,/\, \begin{smallmatrix}1000\\1001\end{smallmatrix}$
1000		$0010 \,/\, \begin{smallmatrix}1100\\0000\end{smallmatrix}$	$0100 \,/\, \begin{smallmatrix}1101\\0000\end{smallmatrix}$	$0001 \,/\, \begin{smallmatrix}1010\\0000\end{smallmatrix}$	$1000 \,/\, \begin{smallmatrix}2001\\0010\end{smallmatrix}$	$0000 \,/\, \begin{smallmatrix}1000\\0001\end{smallmatrix}$

Table 2: The labelled edges of the two-tape automaton $\mathcal{T}_\varphi$

As said above, an immediate (and weak) corollary of the generalization of Proposition A states then that the set of θ-expansions of the integers is a *(linear) context-free language*. A short note following this paper ([14]) establishes that, conversely, if the set of θ-expansions of the integers is a context-free language then θ is a quadratic Pisot unit.

3 *Where some definitions are made precise, some notation given, and some classical results recalled, so as to state, at last, the main theorem.*

We first recall classical definitions about finite automata and numeration systems, and we then state results on Pisot numbers upon which this paper is based.

3.1 Finite automata

We basically follow the exposition of [18] or [6] for the definition of finite automata over an alphabet. An *automaton over a finite alphabet* A, $\mathcal{A} = (Q, A, E, I, T)$ is a directed graph labelled by elements of A; Q is the set of *states*, $I \subset Q$ is the set of *initial* states, $T \subset Q$ is the set of *terminal* states and $E \subset Q \times A \times Q$ is the set of labelled *edges*. The automaton $\mathcal{A}$ is *finite* if Q is finite, and this will always be the case in this paper. The *transition function* of $\mathcal{A}$ is the function $\delta : Q \times A \longrightarrow \mathcal{P}(Q)$ defined by $\delta(p, a) = \{q \in Q \mid (p, a, q) \in E\}$. The automaton is *deterministic* if E is the graph of a (partial) function from $Q \times A$ into Q. Note that with these definitions, automata are non-deterministic by default and determinism does not imply completeness.

A *computation* in $\mathcal{A}$ is a finite path in the labelled graph $\mathcal{A}$ and thus the label of a computation is the concatenation (or product) of the labels of the edges. A computation is said to be *successful* if its origin is in I and its end is in T. The subset of A^* consisting of labels of successful computations of $\mathcal{A}$ is called the *set (or language) recognized by* $\mathcal{A}$. A subset of A^* is said to be *rational*[10] if it is recognized by a finite automaton over A.

This definition of automata as labelled graphs extends readily to automata over any monoid M. We shall consider here automata over the monoid $A^* \times B^*$ which are called *two-tape automata*: a two-tape automaton $\mathcal{A} = (Q, A^* \times B^*, E, I, T)$ is a directed graph whose edges are labelled by elements of $A^* \times B^*$. The automaton is *finite* if the set of edges E is finite (and thus Q is finite), and this will always be the case in this paper. In the literature two-tape automata are also often called *non-deterministic generalized sequential machines* or *transducers* (see [3]). The set of labels of successful computations of $\mathcal{A}$ — the *behaviour* of $\mathcal{A}$ — is then a subset of $A^* \times B^*$, *i.e.,* the graph of a relation from A^* into B^*. If it is the behavior of such an automaton, a relation is is said to be *computable by a finite two-tape automaton* — or, often, *rational.* In spite of its conciseness, we do not use the latter word, for it causes an unnecessary interrogation to the mathematically inclined reader, especially when it comes to *functions.* When the relation computed by $\mathcal{A}$ is a function, we also say that $\mathcal{A}$ *realizes* this function, and we sometimes denote this function by $\mathcal{A}$ (in Section 5 and 7).

A *letter-to-letter two-tape automaton* is a two-tape automaton whose edges are labelled in $A \times B$. A letter-to-letter two-tape automaton can thus be viewed as *an automaton over input alphabet* $A \times B$. The composition of two functions realized by letter-to-letter two-tape automata is obviously realized by a letter-to-letter two-tape automaton (*cf.* [6, Sec. IX.7] and [10] for more results on those functions).

Let $\mathcal{A}$ be a letter-to-letter two-tape automaton over $A^* \times B^*$. The automaton over A obtained by taking the projection on A^* of the label of every edge of $\mathcal{A}$ is called the *underlying input automaton* of $\mathcal{A}$. A letter-to-letter two-tape automaton is said to be *(left) sequential* if its underlying input automaton is *deterministic* with every state being final. A sequential two-tape automaton is often defined and denoted in the following way (*cf.* [3, Sec. IV.2]): $\mathcal{A} = (Q, A, B, \delta, \lambda, i)$, where $i \in Q$ is the unique initial state, $\delta : Q \times A \to Q$ is the *transition function* of the underlying input automaton, and $\lambda : Q \times A \to B^*$ is the *output function.* Then, the set of edges of $\mathcal{A}$, seen as a two-tape automaton, is $E = \{(p, (a, \lambda(p, a)), \delta(p, a)) \mid p \in Q,\ a \in A\}$.

Let us end this paragraph with two brief words about infinite words and context-free languages.

If s is a word of A^*, s^ω denotes the infinite word obtained by indefinitely concatenat-

[10] Often *regular* in the literature. As said above, we follow [18] and [6] whose terminology fits well a paper dedicated to M. P. Schützenberger.

ing s. An infinite computation of an automaton $\mathcal{A}$ on A, $\mathcal{A} = (Q, A, E, I, T)$, is an infinite path in the labelled graph $\mathcal{A}$. The computation is *successful* if its origin is in I and if it goes infinitely often through T. This definition of success is usually known as "Bűchi acceptance". The definitions extend, more or less directly, to relations on infinite words, directly in the case of relations realized by letter-to-letter two-tape automata since they are automata over the alphabet of pairs of letters (see [10]).

We shall not make use of context-free languages for more than their mere definition and for that purpose we refer the reader to [3] ot to [15]. Let us just mention that a *linear* context-free language is a language generated by a context-free grammar whose productions have a right-hand side with at most one occurence of a non-terminal symbol.

3.2 Representation of numbers

Two generalizations of representation of numbers in integer base are considered here: general numeration systems for integers and non-integral real bases. All the alphabets we consider are finite. By analogy with the classical decimal or binary systems, we shall say "digit" for a symbol belonging to an alphabet of (possibly negative) integers.

3.2.1 Representation of integers in a numeration system U

Let $U = (u_n)_{n \geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A *representation in the system U* — or a *U-representation* — of a (positive) integer N is a finite sequence of integers $(d_n)_{0 \leq n \leq k(N)}$ such that

$$N = \sum_{n=0}^{k(N)} d_n u_n$$

for a convenient index $k(N) \geq 0$. The sequence $(d_n)_{0 \leq n \leq k(N)}$ will be denoted by the word $d_{k(N)} \cdots d_0$, since numbers are written from left to right, *most significant digit first.*

Among all possible U-representations of a given integer N, one is distinguished and called the *normal U-representation* of N: it is the one given by the classical "greedy algorithm", which as well turns out to be the greatest for the lexicographic ordering, when an adequate number of 0's is added on the left of representations of N so as to make them all of the same length. The normal U-representation of N is denoted by $\langle N \rangle_U$. Under the hypothesis that the ratio u_{n+1}/u_n is bounded as n goes to infinity, the digits of the normal U-representation of any integer N are bounded and are all contained in a minimal alphabet A_U associated with U.

Let B be a finite alphabet of (possibly negative) digits[11]; any finite sequence of digits,

[11] There will be alphabets of various kind in the course of the paper. With the hope it will help the reading, we have sticked to the following conventions. Regardless of superscript or subscript, A will denote canonical alphabets, D alphabets of positive digits, B or C alphabets of possibly negative digits.

or *word* in B^*, is given a *numerical value* by the function $\pi_U \colon B^* \to \mathbb{N}$ which is defined by

$$\pi_U(w) = \sum_{n=0}^{k} d_n u_n \quad \text{where} \quad w = d_k \cdots d_0 \ .$$

Two words v and w of B^* are said to be *equivalent* if they have the same numerical value, *i.e.,* if $\pi_U(u) = \pi_U(v)$. The function that maps any word w of B^* onto the normal U-representation of the integer $\pi_U(w)$ — if $\pi_U(w)$ is positive — is called the *normalization* and is denoted by $\nu_{U,B}$ (since it formally depends on U and B):

$$\nu_{U,B} \colon B^* \longrightarrow A_U^* \ .$$

3.2.2 Representation of real numbers in base θ

Let now θ be a real number larger than 1. A *representation in base* θ — or a θ-*representation* — of a real number x is an infinite sequence $(x_n)_{-\infty \leq n \leq k(x)}$ of integers such that

$$x = \sum_{n=-\infty}^{k(x)} x_n \theta^n$$

for a convenient index $k(x)$ in $\mathbb{Z}$. It is natural to write the sequence $(x_n)_{-\infty \leq n \leq k(x)}$ in the form $x_{k(x)} \cdots x_0 . x_{-1} x_{-2} \cdots$, when $k(x)$ is ≥ 0, and $0.00 \cdots 0 x_{k(x)} x_{k(x)-1} \cdots$, with the adequate number of leading zeroes, when $k(x) < 0$, as one writes of a classical decimal expansion.

As above, the greatest in the lexicographic ordering of all θ-representations of a given positive real number x is distinguished as the *normal θ-representation* of x, usually called the *θ-expansion* of x. The θ-expansion of a real x can be computed by the following greedy algorithm (see [19]):

> Denote by $\lfloor x \rfloor$ and by $\{x\}$ the integer part and the fractional part of a number x. There exists $k \in \mathbb{Z}$ such that $\theta^k \leq x < \theta^{k+1}$. Let $x_k = \lfloor x/\theta^k \rfloor$, and $r_k = \{x/\theta^k\}$. Then for $k > i \geq -\infty$, put $x_i = \lfloor \theta r_{i+1} \rfloor$, and $r_i = \{\theta r_{i+1}\}$.

We get an expansion $x = x_k \theta^k + x_{k-1} \theta^{k-1} + \cdots$. If $k < 0$ (*i.e.,* $x < 1$), we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. The θ-expansion of x is denoted by $\langle x \rangle_\theta$. It follows from the algorithm that every digit x_i of the θ-expansion of a number x is smaller than θ, *i.e.,* is an element of the set

$$A_\theta = \{0, \ldots, \lfloor \theta \rfloor\} \ ,$$

called the *canonical alphabet for* θ .[12]

[12] This holds indeed when θ is not an integer; when θ is an integer, $A_\theta = \{0, \cdots, \theta - 1\}$ — but this latter case will never occur here.

An expansion ending with infinitely many zeroes is said to be *finite*, and the trailing zeroes are omitted.

By convention (see [19], [17]) — and slight abuse —, we shall call *θ-expansion* of 1 , and denote it by $d(1,\theta)$, the largest θ-representation of 1 in the lexicographic ordering *which is smaller than "1." i.e.,* the largest sequence of integers $(t_n)_{n\geq 1}$ such that

$$1 = \sum_{n\geq 1} t_n \theta^{-n} \quad .$$

Let us introduce another definition: for every k in $\mathbb{Z}$, the *k-th initial section* of $\mathbb{Z}$ is the set of all integers smaller than or equal to k. The set of all initial sections of $\mathbb{Z}$ is denoted by $\mathbb{Z}_w$. Let B be any finite alphabet of (possibly negative) digits. The set of sequences $(x_n)_{-\infty\leq n\leq k}$ with x_i in B is thus denoted by $B^{\mathbb{Z}_w}$. It is a natural convention to consider that any finite sequence $(y_m)_{l\leq m\leq k}$ of elements in B is also an infinite sequence $(y_m)_{-\infty\leq m\leq k}$ of $B^{\mathbb{Z}_w}$ with $y_m = 0$ for all $m < l$.

Any element of $B^{\mathbb{Z}_w}$ is given a *numerical value* by the function $\pi_\theta : B^{\mathbb{Z}_w} \longrightarrow \mathbb{R}$ which is defined by

$$\pi_\theta(s) = \sum_{n=k}^{-\infty} s_n \theta^n \quad \text{where} \quad s = (s_n)_{-\infty\leq n\leq k} \quad .$$

Two infinite words s and y of $B^{\mathbb{Z}_w}$ are said to be *equivalent* if they have the same numerical value. The function that maps any element s of $B^{\mathbb{Z}_w}$ onto the θ-expansion of the real $\pi_\theta(s)$ — if $\pi_\theta(s) \geq 0$ — is called the *normalization* and is denoted by $\nu_{\theta,B}$:

$$\nu_{\theta,B} : B^{\mathbb{Z}_w} \longrightarrow A_\theta^{\mathbb{Z}_w} \quad .$$

3.3 Pisot numbers

A polynomial $P(X) = a_n X^n + \cdots + a_0$ in $\mathbb{Z}[X]$ is said to be *monic* if $a_n = 1$. An *algebraic integer* is a zero of a monic polynomial in $\mathbb{Z}[X]$ which can be supposed irreducible; its *algebraic conjugates* are the other zeroes of this polynomial. A zero θ of $P(X) = 0$ is said to be *dominant* when every other zero is strictly smaller than θ in modulus. A *Pisot number* is an algebraic integer such that all its algebraic conjugates have modulus smaller than 1 (it is thus larger than 1).

An algebraic integer is said to be a *unit* if the constant term a_0 of its minimal polynomial $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ is equal to ± 1. The minimal polynomial of a *quadratic Pisot unit* θ is thus of the form:

$$P(X) = X^2 - rX - \varepsilon$$

with either $r \geq 1$ and $\varepsilon = +1$, or $r \geq 3$ and $\varepsilon = -1$, cases which will be referred to as Case 1 and Case 2 respectively throughout the paper.

3.3.1 Representation of integers in base θ

When θ is not an integer, the θ-expansion of a positive integer is, in general, an infinite sequence over the alphabet A_θ. It turns out, however, that for certain Pisot numbers θ, the θ-expansion of every integer is finite. As stated by the following, this is the case for the quadratic Pisot numbers on which we shall concentrate in the sequel of this paper.

PROPOSITION 1 [12]
If θ is a quadratic Pisot number, then every integer has a finite θ-expansion.

3.3.2 Linear numeration systems associated to Pisot numbers

A very fundamental property of Pisot numbers (as far as θ-expansions are concerned) is given by the following:

THEOREM 1 [4]
If θ is a Pisot number, then $d(1,\theta)$, the θ-expansion of 1, is eventually periodic.

Indeed, this property makes it possible to canonically associate a linear recurrent sequence U_θ with every Pisot number θ. This system U_θ is characterized by the fact that normal U_θ-representations and θ-expansions are defined by the same set of forbidden words (they define indeed the same dynamical system). Two cases have to be considered, according to whether $d(1,\theta)$ is finite or infinite. We give here the construction of the sequence U_θ for the case of quadratic Pisot units we shall be studying. The general case is analoguous.

DEFINITION 1 [5]
Case 1. *($\varepsilon = +1$, $r \geq 1$; i.e., θ is the dominant root of $X^2 - rX - 1 = 0$.) Then*

$$A_\theta = \{0, \cdots, r\} \quad \textit{and} \quad d(1,\theta) = r\,1 \ .$$

The linear recurrent sequence $U_\theta = (u_k)_{k\geq 0}$ associated with θ is defined by

$$u_{k+2} = r u_{k+1} + u_k, \quad k \geq 0 \quad \text{and} \quad u_0 = 1, \quad u_1 = r+1 \ .$$

Case 2. *($\varepsilon = -1$, $r \geq 3$; i.e., θ is the dominant root of $X^2 - rX + 1 = 0$.) Then*

$$A_\theta = \{0, \cdots, r-1\} \quad \text{and} \quad d(1,\theta) = r-1\,(r-2)^\omega \ .$$

The linear recurrent sequence $U_\theta = (u_k)_{k\geq 0}$ associated with θ is defined by

$$u_{k+2} = r u_{k+1} - u_k, \quad k \geq 0 \quad \textit{and} \quad u_0 = 1, \quad u_1 = r \ .$$

In both cases, the sequence U_θ, together with the alphabet A_θ, define the linear numeration system associated with θ.

A brief word on what is known, in general, on the θ-expansions and on the representations in the associated system U_θ. In Case 1, an infinite sequence (resp. a finite word) over A_θ is a θ-expansion (resp. is a U_θ-representation) if and only if this sequence and all the shifted ones are lexicographically smaller than $(r0)^\omega$. The associated dynamical system is a *subshift of finite type*. Similarly in Case 2, an infinite sequence (resp. a finite word) over A_θ is a θ-expansion (resp. is a U_θ-representation) if and only if this sequence and all the shifted ones are lexicographically smaller than $d(1, \theta) = (r-1)(r-2)^\omega$. The associated dynamical system is a *sofic subshift* (see [17] and [5]).

3.3.3 Normalization in base θ

The fundamental property that relates representation of numbers in a Pisot base and automata theory is given by the following:

PROPOSITION 2 [9] *If θ is a Pisot number, then for every finite alphabet B, normalization on $B^{\mathbb{N}}$ in base θ is a function computable by a letter-to-letter finite two-tape automaton.*

Let us make three comments. This statement is the one that requires the definition of functions on infinite words. In the course of the paper, the normalization will be applied on finite words only. This is the reason why we did not find necessary to give more details on this definition in Section 3.1.

In [9], Proposition 2 is proved in the case where every element of B is non-negative. The proof extends readily to alphabets containing both positive and negative digits. As a matter of fact, the converse of this result holds as well (see [2]), but this will not be used here.

Normalization on $B^{\mathbb{Z}_w}$ is slightly different from normalization on $B^{\mathbb{N}}$, because of the presence of negative digits. We shall deal with this problem at Section 5.

3.4 Main result

After all these reminders we still have to introduce one more new operation on θ-representations (already sketched in the introduction), in order to state the main result.

3.4.1 Folded θ-representation

Let B be an arbitrary alphabet of digits containing 0, and let $B_\rho = \{{a \atop b} \mid a, b \in B\}$ be the alphabet of pairs of elements of B, conveniently written one above the other, and called "double-digits". The mirror image of a word v is denoted by v^t. Any element w of B_ρ^*

can be written as $w = {u \atop v}$, where $u, v \in B^*$ and $|u| = |v|$. The *upper* part of w will be denoted by $\overleftarrow{w} = u$, and the *lower* part of w by $\overrightarrow{w} = v^t$. For instance, if $A = \{0, 1\}$ then $A_\rho = \{{0 \atop 0}, {0 \atop 1}, {1 \atop 0}, {1 \atop 1}\}$. Let $w = {100101 \atop 100100}$; then $\overleftarrow{w} = 100101$ and $\overrightarrow{w} = 001001$.

Let $s = f.g$, with $f, g \in B^*$; by completing the shorter of f and g with enough 0's (either at the left for f, or at the right for g), one can assume that $|f| = |g|$. Such an s will be called a *balanced (θ-)representation.* The *folding operation* ρ maps any balanced representation $s = f.g$ onto the element $\rho(s) = {f \atop g^t}$ of B_ρ^*. Conversely, the inverse of ρ, ρ^{-1}, called the *unfolding* operation, maps every element $w = {u \atop v}$ of B_ρ^* onto the balanced representation $\rho^{-1}(w) = \rho^{-1}({u \atop v}) = u.v^t$. Thus $\overleftarrow{\rho(f.g)} = f$, $\overrightarrow{\rho(f.g)} = g$, and $\rho^{-1}(w) = \overleftarrow{w}.\overrightarrow{w}$.

The numerical value function π_θ extends to folded representations : if w is a word on B_ρ^*, then, by definition $\pi_\theta(w) = \pi_\theta(\overleftarrow{w}.\overrightarrow{w})$.

With these definitions and notations, a classical result in formal language theory (*cf.* [3, Prop. V.6.5], [20]) that we have already quoted in the introduction can be stated as follows.

Proposition 3 *Let B be an arbitrary alphabet and let B_ρ be the alphabet of "double-digits". Let K be a rational set of B_ρ^*. Then $\rho^{-1}(K)$ is a linear context-free language of $(B \cup \{.\})^*$.*

3.4.2 The result

Theorem 2 *Let θ be a quadratic Pisot unit and let D be an arbitrary finite alphabet of non-negative digits. The function $\mu_{\theta,D}$ that maps any word w on D^* onto the folded θ-expansion of $\pi_{U_\theta}(w)$, the integer represented by w in the linear numeration system (U_θ, D), is computable by a letter-to-letter two-tape automaton.*

Since the image of D^* by a function computable by a letter-to-letter two-tape automaton is a rational language, it then follows immediately from Theorem 2 and Proposition 3 that we have:

Corollary 4 *Let θ be a quadratic Pisot unit. The set of folded θ-expansions of all integers is a rational language. The set of θ-expansions of all integers is a linear context-free language.* ∎

4 Where the θ-expansion of the elements of the linear recurrent sequence U_θ is computed, which leads to the reduction of the problem to a smaller set of words and, at the same time, puts the reader on the track of a finite two-tape automaton.

From now on, θ is a quadratic Pisot unit, the dominant zero of $P(X) = X^2 - rX - \varepsilon$; and $U_\theta = (u_n)_{n\in\mathbb{N}}$ is the linear recurrent sequence associated to θ as above. The result relies indeed on the very regular expression of the elements of U_θ in terms of the powers of θ, as stated in the following :

Proposition 5 **Case 1**. *For every k in $\mathbb{N}$,*

$$\begin{aligned} u_{2k} &= \theta^{2k} + (r-1)\theta^{2k-2} + \theta^{2k-4} + \cdots + (r-1)\theta^{-2k+2} + \theta^{-2k} \\ &= \left(\sum_{0\le j\le k} \theta^{2k-4j}\right) + \left((r-1)\sum_{1\le j\le k} \theta^{2k+2-4j}\right) , \qquad \textit{and} \\ u_{2k+1} &= \theta^{2k+1} + (r-1)\theta^{2k-1} + \theta^{2k-3} + \cdots + (r-1)\theta^{-2k-1} + \theta^{-2k-2} \\ &= \left(\sum_{0\le j\le k} \theta^{2k+1-4j}\right) + \theta^{-2k-2} + \left((r-1)\sum_{0\le j\le k} \theta^{2k-1-4j}\right) . \end{aligned}$$

Case 2. *For every k in $\mathbb{N}$,*

$$u_k = \theta^k + \theta^{k-2} + \cdots + \theta^{-k} = \sum_{0\le j\le k} \theta^{k-2j} .$$

Proof. **Case 1**. For every j in $\mathbb{Z}$, the equality

$$\theta^{j+2} = r\theta^{j+1} + \theta^j \tag{1}$$

holds, and, as stated in Definition 1, the sequence $U_\theta = (u_k)_{k\ge 0}$ is defined by

$$u_{k+2} = ru_{k+1} + u_k, \quad k \ge 0 \qquad \text{and} \qquad u_0 = 1, \quad u_1 = r+1 .$$

Equation 1 gives (for $j = -2$ and $j = -1$) $1 = r\theta^{-1} + \theta^{-2}$ and $r = \theta - \theta^{-1}$ from which one gets

$$u_1 = r + 1 = \theta - \theta^{-1} + r\theta^{-1} + \theta^{-2} = \theta + (r-1)\theta^{-1} + \theta^{-2} \tag{2}$$

Together with $u_0 = \theta^0$, this shows the property for $k = 0$.

By induction, let us suppose that the statement holds for u_{2k} and u_{2k+1} . Then

$$\begin{aligned}
u_{2k+2} &= r u_{2k+1} + u_{2k} \\
&= r \left(\theta^{2k+1} + (r-1)\theta^{2k-1} + \theta^{2k-3} + \cdots + \theta^{-2k+1} + (r-1)\theta^{-2k-1} + \theta^{-2k-2} \right) \\
&\quad + \theta^{2k} + (r-1)\theta^{2k-2} + \theta^{2k-4} + \cdots + (r-1)\theta^{-2k+2} + \theta^{-2k} \\
&= r\theta^{2k+1} + (r-1)r\theta^{2k-1} + r\theta^{2k-3} + \cdots + r\theta^{-2k+1} + (r-1)r\theta^{-2k-1} + r\theta^{-2k-2} \\
&\quad + \theta^{2k} + (r-1)\theta^{2k-2} + \theta^{2k-4} + \cdots + (r-1)\theta^{-2k+2} + \theta^{-2k}
\end{aligned}$$

Grouping together terms of the form $r\theta^{j-1} + \theta^{j-2}$, for j ranging from $-2k+2$ to $2k+2$ yields

$$u_{2k+2} = \theta^{2k+2} + (r-1)\theta^{2k} + \theta^{2k-2} + \cdots + \theta^{-2k+2} + (r-1)r\theta^{-2k-1} + r\theta^{-2k-2}$$

and thus

$$u_{2k+2} = \theta^{2k+2} + (r-1)\theta^{2k} + \theta^{2k-2} + \cdots + \theta^{-2k+2} + (r-1)\theta^{-2k-1} + \theta^{-2k-2}$$

since

$$(r-1)r\theta^{-2k-1} + r\theta^{-2k-2} = (r-1)\left(r\theta^{-2k-1} + \theta^{-2k-2}\right) + \theta^{-2k-2} = (r-1)\theta^{-2k} + \theta^{-2k-2}$$

The statement holds for u_{2k+2}. The computation of u_{2k+3} is then possible (and similar):

$$\begin{aligned}
u_{2k+3} &= r u_{2k+2} + u_{2k+1} \\
&= r \left(\theta^{2k+2} + (r-1)\theta^{2k} + \theta^{2k-2} + \cdots + (r-1)\theta^{-2k} + \theta^{-2k-2} \right) \\
&\quad + \theta^{2k+1} + (r-1)\theta^{2k-1} + \theta^{2k-3} + \cdots + (r-1)\theta^{-2k-1} + \theta^{-2k-2} \\
&= r\theta^{2k+2} + (r-1)r\theta^{2k} + r\theta^{2k-2} + \cdots (r-1)r\theta^{-2k} + r\theta^{-2k-2} \\
&\quad + \theta^{2k+1} + (r-1)\theta^{2k-1} + \theta^{2k-3} + \cdots + (r-1)\theta^{-2k-1} + \theta^{-2k-2}
\end{aligned}$$

Grouping together terms of the form $r\theta^{j-1} + \theta^{j-2}$, for j ranging from $-2k-1$ to $2k+3$, yields

$$u_{2k+3} = \theta^{2k+3} + (r-1)\theta^{2k+1} + \theta^{2k-1} + \cdots (r-1)\theta^{-2k+1} + (r+1)\theta^{-2k-2}$$

and thus

$$u_{2k+3} = \theta^{2k+3} + (r-1)\theta^{2k+1} + \theta^{2k-1} + \cdots (r-1)\theta^{-2k+1} + \theta^{-2k-1} + (r-1)\theta^{-2k-3} + \theta^{-2k-4}$$

since

$$(r+1)\theta^{-2k-2} = \theta^{-2k-1} + (r-1)\theta^{-2k-3} + \theta^{-2k-4} \qquad \text{by multiplication of (2) by } \theta^{-2k-2} .$$

The statement holds for u_{2k+3}.

Case 2. For every j in $\mathbb{Z}$, the equation

$$\theta^{j+2} = r\theta^{j+1} - \theta^{j} \tag{3}$$

holds, and, as stated in Definition 1, the sequence $U_\theta = (u_k)_{k\geq 0}$ is defined by

$$u_{k+2} = ru_{k+1} - u_k, \quad k \geq 0 \qquad \text{and} \qquad u_0 = 1, \quad u_1 = r \ .$$

Equation 3 (for $j = -1$) gives

$$r = \theta + \theta^{-1} \ , \tag{4}$$

which shows, together with $u_0 = \theta^0$, the property for $k = 0$ and $k = 1$.

The induction step is similar to (and easier than) the one for Case 1. Suppose that the statement holds for u_k and u_{k+1}. Then

$$\begin{aligned} u_{k+2} &= ru_{k+1} - u_k \\ &= r\left(\theta^{k+1} + \theta^{k-1} + \cdots + \theta^{-k+1} + \theta^{-k-1}\right) \\ &\qquad - \theta^{k} - \theta^{k-2} - \cdots - \theta^{-k} \\ &= r\theta^{k+1} + r\theta^{k-1} + \cdots + r\theta^{-k+1} + r\theta^{-k-1} \\ &\qquad - \theta^{k} - \theta^{k-2} - \cdots - \theta^{-k} \end{aligned}$$

Grouping together terms of the form $r\theta^{j+1} - \theta^{j}$, for j ranging from $-k$ to k yields

$$\begin{aligned} u_{k+2} &= \theta^{k+2} + \theta^{k} + \cdots + \theta^{-k+2} + r\theta^{-k-1} \\ &= \theta^{k+2} + \theta^{k} + \cdots + \theta^{-k+2} + \theta^{-k} + \theta^{-k-2} \end{aligned}$$

since

$$r\theta^{-k-1} = \theta^{-k} + \theta^{-k-2} \qquad \text{by multiplication of (4) by } \theta^{-k-1} \ .$$

The statement holds for u_{k+2}. ■

In the case where θ is equal to the golden mean φ , Proposition 5 takes an even simpler form for the Fibonacci numbers (for which, to our surprise, we have not found any reference):

Corollary 6 *For every k in $\mathbb{N}$,*

$$F_{2k} = \varphi^{2k} + \varphi^{2k-4} + \cdots + \varphi^{-2k-4} + \varphi^{-2k} = \sum_{0\leq j\leq k} \varphi^{2k-4j} \ ,$$

and

$$F_{2k+1} = \varphi^{2k+1} + \varphi^{2k-3} + \cdots + \varphi^{-2k+1} = \left(\sum_{0\leq j\leq k} \varphi^{2k+1-4j}\right) + \varphi^{-2k-2} \ .$$

Proposition 5 can be rewritten in terms of the θ-expansions of the elements of U_θ:

PROPOSITION 5 **Case 1**. *For every k in $\mathbb{N}$,*

$$\begin{aligned}
\langle u_{4k}\rangle_\theta &= 1\,(0\ r{-}1\ 0\ 1)^k.(0\ r{-}1\ 0\ 1)^k \\
\langle u_{4k+1}\rangle_\theta &= 1\,0\,(r{-}1\ 0\ 1\ 0)^k.(r{-}1\ 0\ 1\ 0)^k r{-}1\,1 \\
\langle u_{4k+2}\rangle_\theta &= 1\,0\,r{-}1\,(0\ 1\ 0\ r{-}1)^k.(0\ 1\ 0\ r{-}1)^k 0\,1 \\
\langle u_{4k+3}\rangle_\theta &= (1\ 0\ r{-}1\ 0)(1\ 0\ r{-}1\ 0)^k.(1\ 0\ r{-}1\ 0)^k\,1\ 0\ r{-}1\ 1
\end{aligned}$$

Case 2. *For every k in $\mathbb{N}$,*

$$\langle u_{2k}\rangle_\theta = 1\,(0\,1)^k.(0\,1)^k \qquad \langle u_{2k+1}\rangle_\theta = (1\,0)^{k+1}.(1\,0)^{k+1}$$

Proposition 5 can be rewritten again in terms of the *folded* θ-expansions of the elements of U_θ:

PROPOSITION 5 **Case 1**. *For every k in $\mathbb{N}$,*

$$\rho(\langle u_{4k}\rangle_\theta) = \left(\begin{smallmatrix}0&0&0&1\\0&0&0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&r-1&0&1\\1&0&r-1&0\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+2}\rangle_\theta) = \left(\begin{smallmatrix}0&1&0&r-1\\0&0&1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&r-1\\r-1&0&1&0\end{smallmatrix}\right)^k$$

$$\rho(\langle u_{4k+1}\rangle_\theta) = \left(\begin{smallmatrix}0&0&1&0\\0&0&1&r-1\end{smallmatrix}\right)\left(\begin{smallmatrix}r-1&0&1&0\\0&1&0&r-1\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+3}\rangle_\theta) = \left(\begin{smallmatrix}1&0&r-1&0\\1&r-1&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&r-1&0\\0&r-1&0&1\end{smallmatrix}\right)^k$$

Case 2. *For every k in $\mathbb{N}$,*

$$\rho(\langle u_{4k}\rangle_\theta) = \left(\begin{smallmatrix}0&0&0&1\\0&0&0&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&1\\1&0&1&0\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+2}\rangle_\theta) = \left(\begin{smallmatrix}0&1&0&1\\0&0&1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}0&1&0&1\\1&0&1&0\end{smallmatrix}\right)^k$$

$$\rho(\langle u_{4k+1}\rangle_\theta) = \left(\begin{smallmatrix}0&0&0&1\\0&0&1&0\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right)^k \qquad \rho(\langle u_{4k+3}\rangle_\theta) = \left(\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right)\left(\begin{smallmatrix}1&0&1&0\\0&1&0&1\end{smallmatrix}\right)^k$$

This series of equations strongly suggests writing words of A_θ^* — and, for coherence, writing words on any alphabet of digits D as well — as the *concatenation (or product) of blocks of length* 4, the words having been first padded on the left by the adequate number of 0's to make the length a multiple of 4 . It is then convenient to have alphabets of blocks. For the sequel of the paper, let

$$X = \{z, a, b, c, d\}$$

be the alphabet of *basic blocks*, with

$$z = 0\,0\,0\,0\,, \quad a = 0\,0\,0\,1\,, \quad b = 0\,0\,1\,0\,, \quad c = 0\,1\,0\,0 \quad \text{and} \quad d = 1\,0\,0\,0\ .$$

For instance, the normal U_θ-representation of the numbers u_n,

$$\langle u_n\rangle_{U_\theta} = 10^n\ ,$$

can be written as words on the block alphabet X :

$$\langle u_{4k}\rangle_{U_\theta} = az^k\,, \quad \langle u_{4k+1}\rangle_{U_\theta} = bz^k\,, \quad \langle u_{4k+2}\rangle_{U_\theta} = cz^k \quad \text{and} \quad \langle u_{4k+3}\rangle_{U_\theta} = dz^k \ .$$

Relations and functions defined on words of D^*, such as the numerical value π_θ or as the mapping onto the folded θ-expansion $\mu_{\theta,D}$, as well as the definition of letter-to-letter two-tape automaton, naturally extend to words of X^*. With these conventions, Proposition 5 may be rewritten (for the last time):

Proposition 5 **Case 1**. *For every k in $\mathbb{N}$,*

$$\mu_{\theta,D}(az^k) = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&r-1&0&1 \\ 1&0&r-1&0 \end{pmatrix}^k \qquad \mu_{\theta,D}(cz^k) = \begin{pmatrix} 0&1&0&r-1 \\ 0&0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&0&r-1 \\ r-1&0&1&0 \end{pmatrix}^k$$

$$\mu_{\theta,D}(bz^k) = \begin{pmatrix} 0&0&1&0 \\ 0&0&1&r-1 \end{pmatrix}\begin{pmatrix} r-1&0&1&0 \\ 0&1&0&r-1 \end{pmatrix}^k \qquad \mu_{\theta,D}(dz^k) = \begin{pmatrix} 1&0&r-1&0 \\ 1&r-1&0&1 \end{pmatrix}\begin{pmatrix} 1&0&r-1&0 \\ 0&r-1&0&1 \end{pmatrix}^k$$

Case 2. *For every k in $\mathbb{N}$,*

$$\mu_{\theta,D}(az^k) = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}\begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \end{pmatrix}^k \qquad \mu_{\theta,D}(cz^k) = \begin{pmatrix} 0&1&0&1 \\ 0&0&1&0 \end{pmatrix}\begin{pmatrix} 0&1&0&1 \\ 1&0&1&0 \end{pmatrix}^k$$

$$\mu_{\theta,D}(bz^k) = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \end{pmatrix}\begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \end{pmatrix}^k \qquad \mu_{\theta,D}(dz^k) = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \end{pmatrix}\begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \end{pmatrix}^k$$

Hence the restriction of $\mu_{\theta,D}$ to the subset of words Xz^* is clearly realized by a letter-to-letter two-tape automaton, the one given in Figure 3.

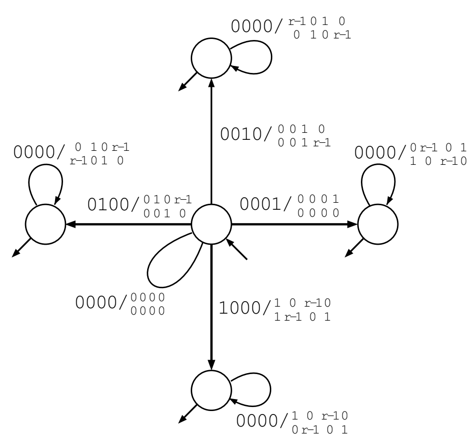

Figure 3: An automaton realizing the restriction of $\mu_{\theta,D}$ to z^*Xz^* .

In the case where θ is the golden mean φ, the automaton of Figure 3 corresponds to the first row and the diagonal of Table 2 that describes the automaton $\mathcal{T}_\varphi$ in the introduction. The core of the paper — developed in sections 5 and 6 — consists in showing that this restriction of $\mu_{\theta,D}$ extends to all words of X^*, that is, more precisely and with the current notation:

Theorem 3 *There exist an alphabet of digits B_θ and a letter-to-letter two-tape automaton $\mathcal{T}_\theta$, with output alphabet ${B_\theta}^\rho$, that maps any word f of X^* onto a folded θ-representation of $\pi_{U_\theta}(f)$, the integer represented by f in the numeration system U_θ.*

5 Where it is shown how Theorem 3 implies the main result. For that purpose, we make the most of the properties of letter-to-letter two-tape automata, by means of a new operation on words: the digit-addition.

Let $f = f_n \cdots f_0$ and $g = g_n \cdots g_0$ be two words of equal length on any alphabet of digits B. The *digit-addition* of f and g is the word $f \oplus g = (f_n + g_n) \cdots (f_0 + g_0)$ over the new alphabet of digits $B' = B \oplus B$ obtained by adding pairs of elements of B. This definition naturally extends to words over alphabets of blocks of digits of fixed length, as well as to words over alphabets B_ρ of pairs of digits.

Example 1 : With the notation above we have:

$$ac \oplus da = 1\,0\,0\,1\,0\,1\,0\,1$$

and

$$\begin{smallmatrix} 0\,0\,0\,1 & 1\,0\,0\,0 \\ 0\,0\,0\,0 & 1\,\bar{1}\,0\,0 \end{smallmatrix} \oplus \begin{smallmatrix} 1\,0\,0\,0 & 1\,0\,1\,0 \\ 1\,0\,0\,1 & 0\,0\,0\,0 \end{smallmatrix} = \begin{smallmatrix} 1\,0\,0\,1 & 2\,0\,1\,0 \\ 1\,0\,0\,1 & 1\,\bar{1}\,0\,0 \end{smallmatrix} \quad . \qquad \square$$

Let $D \subseteq \{0, 1, \ldots, m\}$ be an arbitrary finite alphabet of non-negative digits with greatest element m. The following then clearly holds.

Fact 1 *Any word of D^*, the length of which is a multiple of 4, can be obtained by the digit-addition of at most $4m$ words of X^*.*

Example 2 : With the notation above we have:

$$3\,0\,2\,1\,2\,1\,1\,3 = da \oplus da \oplus dd \oplus ba \oplus ac \oplus bd \oplus zb \quad . \qquad \square$$

Another obvious fact is that if f and f', respectively g and g', are equivalent θ-representations, then $f \oplus g$ and $f' \oplus g'$ are equivalent θ-representations. This property extends to mappings that preserve the numerical value, with a little preparation.

A function (or a relation) $\alpha : B^* \longrightarrow A^*$ from an alphabet of digits onto another one is said to be *conservative* if any word of B^* is mapped onto an equivalent word (onto a set of equivalent words) of A^*. A two-tape automaton is said to be *conservative* as well if the relation it realizes is conservative. The following property is then a simple exercise in automata theory.

PROPOSITION 7 *Let $\mathcal{A}$ and $\mathcal{B}$ be two conservative letter-to-letter two-tape automata. There exists a (conservative letter-to-letter) two-tape automaton, denoted by $\mathcal{A} \oplus \mathcal{B}$, such that, for every f, g, f' and g' with $f' \in \mathcal{A}(f)$ and $g' \in \mathcal{B}(g)$, we have $f' \oplus g' \in \mathcal{A} \oplus \mathcal{B}(f \oplus g)$, and conversely, if $h' \in \mathcal{A} \oplus \mathcal{B}(h)$ then there exist f, f', g, g' such that $f' \in \mathcal{A}(f)$, $g' \in \mathcal{B}(g)$, $h = f \oplus g$, and $h' = f' \oplus g'$.*

Proof. [Idea]. Let us first remark that it is always possible to assume that a relation ψ realized by a (conservative) letter-to-letter two-tape automaton has the property that if f' is in $\psi(f)$ then $0^k f' \in \psi(0^k f)$ for any integer k. In such an automaton, called a *padding automaton*, every initial state bears a loop with label $(0, 0)$.

Let $\mathcal{A} = (Q, B \times A, E, I, T)$ and $\mathcal{B} = (R, B \times A, F, J, U)$ be two conservative padding letter-to-letter two-tape automata. The automaton $\mathcal{C} = \mathcal{A} \oplus \mathcal{B}$ is defined as follows:

$$\mathcal{C} = (Q \times R, (B \oplus B) \times (A \oplus A), H, I \times J, T \times U)$$

the edges which are made by the "addition" of the edges of $\mathcal{A}$ with those of $\mathcal{B}$:

$$H = \{((p, r), (x, y), (q, s)) \mid (p, (i, j), q) \in E,\ (r, (k, l), s) \in F \text{ and } x = i + k,\ y = j + l\}.$$

It is clear that any two successful computations of $\mathcal{A}$ and $\mathcal{B}$, that can be supposed to be of the same length since $\mathcal{A}$ and $\mathcal{B}$ are padding automata, can be "added" edge by edge to give a successful computation of $\mathcal{A} \oplus \mathcal{B}$. Conversely, any (successful) computation of $\mathcal{A} \oplus \mathcal{B}$ can be "decomposed" — in possibly several different ways — into a pair of (successful) computations of $\mathcal{A}$ and $\mathcal{B}$. ∎

The next result deals with the "transfer" of transformations of θ-representations to transformations of *folded* θ-representations.

PROPOSITION 8 *Let $\psi : B^{\mathbb{Z}_w} \longrightarrow B^{\mathbb{Z}_w}$ be a relation realized by a letter-to-letter finite two-tape automaton. Then the relation $\psi^\rho : B_\rho^* \longrightarrow B_\rho^*$ defined by $\psi^\rho = \rho \circ \psi \circ \rho^{-1}$ is also realized by a letter-to-letter finite two-tape automaton.*

The statement makes use of the convention we mentioned in Section 3.2.2 : if $w \in B_\rho^*$ then $\rho^{-1}(w)$ is a finite sequence considered as an element of $B^{\mathbb{Z}_w}$. It is also understood that the relation ψ has the property that an infinite sequence the elements of which are all equal to 0 from a certain rank on is mapped onto sequences with the same property. Then $\psi(\rho^{-1}((w))$ is indeed a finite representation, that can be balanced and then folded.

Proof. [Sketch]. The relation ψ is realized by an automaton $\mathcal{A} = (Q, B \times B, E, I, T)$. Two automata $\mathcal{A}_1 = (Q, B_\rho \times B_\rho, E_1, I_1, T_1)$ and $\mathcal{A}_2 = (Q, B_\rho \times B_\rho, E_2, I_2, T_2)$ are then built in the following way: for every edge $(p, (i, x), q)$ in E and every j and k in B, let $(p, (\binom{i}{j}, \binom{x}{k}), q)$ be an edge in E_1 and let $(q, (\binom{j}{i}, \binom{k}{x}), p)$ be an edge in E_2. Up to some adequate tuning of I_1, I_2, T_1, T_2 (that depends indeed on the way the radix point is treated by ψ), it is then easy to check that ψ^ρ is equal to the composition of the relation realized by $\mathcal{A}_1$ with the relation realized by $\mathcal{A}_2$. ■

Proof of Theorem 2. Let m be the greatest element of the digit alphabet D. Let $\mathcal{T}_\theta$ be the two-tape automaton the existence of which is given by Theorem 3 and let $\mathcal{N}_\theta$ be the "sum" — in the sense of Proposition 7 — of $4m$ copies of $\mathcal{T}_\theta$. Let f be a word in D^*; it is, in several ways, the digit-sum of at most $4m$ words of X^*. The image of f by $\mathcal{N}_\theta$ is a set of folded θ-representations of $\pi_{U_\theta}(f)$ written on the pairs of digits of $C_\theta = 4m\, B_\theta$ (which stands for $B_\theta \oplus B_\theta \oplus \cdots \oplus B_\theta$, $4m$ times).

Let ν_θ be the normalization in base θ on $C_\theta^{\mathbb{N}}$. By Proposition 2, ν_θ is realized by a letter-to-letter finite two-tape automaton. By a simple shift (to the right) of the radix point, this ν_θ transfers into a *quasi*-normalization ν'_θ from $C_\theta^{\mathbb{Z}_w}$ onto $A_\theta^{\mathbb{Z}_w}$ that is realized by *the same* letter-to-letter finite two-tape automaton as ν_θ. It is not quite a normalization anymore because the output may begin with a sequence of leading zeros — this may happen because C_θ contains *negative* digits. By Proposition 8, ${\nu'_\theta}^\rho = \rho \circ \nu'_\theta \circ \rho^{-1}$ is realized by a letter-to-letter finite two-tape automaton.

Now, $\mu_{\theta,D}$ is the composition of $\mathcal{N}_\theta$, ${\nu'_\theta}^\rho$, and possibly the function ζ that erases the leading zeros and which is obviously realized by a finite two-tape automaton. Hence $\mu_{\theta,D}$ is realized by a finite two-tape automaton and we are almost done, but for the fact that, since ζ is not "length-preserving", we have not yet proved that $\mu_{\theta,D}$ is realized by a *letter-to-letter* finite two-tape automaton. It would be tedious to prove it directly, *i.e.,* by stating properties of the actual output of $\mathcal{N}_\theta$, so we rather prove that last step by an "external" argument.

LEMMA 9 *For any f in D^*, the difference between the lengthes of f and $\mu_{\theta,D}(f)$ is bounded (independently of f).*

Proof. Let f be a word of length $k+1$ that does not begin with a 0, and let $N = \pi_{U_\theta}(f)$. Then $u_k \leq N \leq m(u_k + \cdots + u_0)$. Let ξ be the algebraic conjugate of θ. It is known that for every $n \geq 0$, $u_n = \alpha\theta^n + \beta\xi^n$, where α and β are real constants.

For Case 1, $\xi = -\theta^{-1}$. Since $\alpha + \beta = u_0 = 1$, and $\alpha\theta - \beta\theta^{-1} = u_1 = r + 1$, an easy computation shows that $\alpha = \frac{\theta^2+\theta}{\theta^2+1} > 1$ and $\beta < 0$. Then

$$\begin{aligned} m(u_k + \cdots + u_0) &< m\alpha(\theta^k - 1)/(\theta - 1) + m|\beta|(1 + \theta^{-1} + \theta^{-2} + \theta^{-3} + \cdots) \\ &< m\alpha\theta^k/(\theta - 1) + m(\alpha - 1)\theta/(\theta - 1) \\ &< m\alpha(\theta^k + \theta)/(\theta - 1) \leq m\alpha\theta^{k+1}/(\theta - 1) \ . \end{aligned}$$

For Case 2, $\xi = \theta^{-1}$. From $\alpha + \beta = 1$, and $\alpha\theta + \beta\theta^{-1} = u_1 = r$, it follows that $\alpha = \frac{\theta^2}{\theta^2-1} > 1$ and $\beta < 0$. Then

$$m(u_k + \cdots + u_0) < m\alpha(\theta^k - 1)/(\theta - 1) + m\beta\theta/(\theta - 1) < m\alpha\theta^k/(\theta - 1) \ .$$

Thus, in both cases, $N < m\alpha\theta^{k+1}/(\theta-1)$. It follows that $N < \theta^{k+p}$ holds, with $p = \lfloor \log_\theta(m\alpha\theta/(\theta-1)) \rfloor + 2$. And then, recalling that $\theta^k < u_N \le N$, it holds:

$$|f| - 1 \le |\mu_{\theta,D}(f)| \le |f| + p \ . \qquad \blacksquare$$

It is then a known result (*cf.* [7], [10, Cor. 2.5]), that a relation "with bounded length difference" that is realized by a finite two-tape automaton is realized by a letter-to-letter finite two-tape automaton. And the proof of Theorem 2 — assuming Theorem 3 — is thus complete. ■

The results established in this section call for some comments.

Remark 1 Proposition 8 no longer holds if ψ is realized by a two-tape automaton which is not assumed to be letter-to-letter. This is the step in the proof that makes it necessary to specify throughout the paper that the relations we are dealing with are actually realized by *letter-to-letter* two-tape automata.

Remark 2 The construction involved in the proof of Theorem 2 is far from being optimal (in the sense of the number of states) for the building of $\mathcal{N}_\theta$ from $\mathcal{T}_\theta$. The precise study of the complexity of the construction remains to be done.

Remark 3 Proposition 7, stated here for ancillary purpose, also yields simplified proofs for already known results in the domain of numeration systems and automata theory. Although it does not pertain to the rest of the paper, let us state, for later reference, a striking application (*cf.* [13]).

Proposition 10 *Let U be a linear numeration system and let $A_U = \{0, 1, \cdots, m\}$ be the canonical alphabet. Let us assume that the characteristic polynomial of U has a dominant zero larger than 1. The normalization $\nu_{U,D}$ over any alphabet of non-negative digits D is realized by a letter-to-letter two-tape automaton if and only if the normalization $\nu_{U,A'}$ over $A' = \{0, 1, \cdots, m+1\}$ is realized by a letter-to-letter two-tape automaton.*

Proof. First, if the normalization $\nu_{U,D}$ is realized by a letter-to-letter two-tape automaton then, for every subalphabet $C \subset D$, $\nu_{U,C}$ is realized by a letter-to-letter two-tape automaton as well. This gives the necessary part of the statement as well as the assurance that it is sufficient to consider alphabet of digits that are intervals of the integers.

Conversely, let $\mathcal{N}$ be the letter-to-letter two-tape automaton that realizes $\nu_{U,A'}$ and let $\mathcal{I}_k$ be the (1-state letter-to-letter) two-tape automaton that realizes the identity mapping on the words on $\{0, \cdots, k\}$. Then $\mathcal{N} \oplus \mathcal{I}_k$ maps any word on $\{0, \cdots, m+k+1\}$ onto an equivalent one on $\{0, \cdots, m+k\}$. The normalization on the alphabet $\{0, \cdots, m+k+1\}$ is obtained by the composition of $\mathcal{N} \oplus \mathcal{I}_k$, $\mathcal{N} \oplus \mathcal{I}_{k-1}$, ..., $\mathcal{N} \oplus \mathcal{I}_1$, and $\mathcal{N}$ and the result follows. ■

A result analogous to Proposition 10 holds for normalization in base θ (when θ is the dominant zero of an irreducible polynomial).

6 *Where a finite Abelian group is discovered and then computed to serve as the underlying input automaton of $\mathcal{T}_\theta$.*

Let us come back to Proposition 5 and to the "obvious" two-tape automaton $\mathcal{T}'_\theta$ it suggests for the computation of a folded equivalent θ-expansion of words of the form xz^k, $x \in X$. In $\mathcal{T}'_\theta$, the reading of the letter a induces a transition from the initial state to a certain state, say $\hat{a}$. In state $\hat{a}$, the reading of letter z $(= 0\,0\,0\,0)$ causes $\mathcal{T}'_\theta$

i) to stay in $\hat{a}$;

ii) to output the "letter" $\left(\begin{smallmatrix} 0 & r-1 & 0 & 1 \\ 1 & 0 & r-1 & 0 \end{smallmatrix}\right)$ [if we are in Case 1; the letter $\begin{smallmatrix} 0\,1\,0\,1 \\ 1\,0\,1\,0 \end{smallmatrix}$ if in Case 2].

If we thus keep reading z, $\mathcal{T}'_\theta$ *stays* in $\hat{a}$ and keeps outputting $\left(\begin{smallmatrix} 0 & r-1 & 0 & 1 \\ 1 & 0 & r-1 & 0 \end{smallmatrix}\right)$ (resp. $\begin{smallmatrix} 0\,1\,0\,1 \\ 1\,0\,1\,0 \end{smallmatrix}$).

Proof of Theorem 3 amounts to building a two-tape automaton $\mathcal{T}_\theta$ that extends (the definition domain of) $\mathcal{T}'_\theta$ to *all words* of X^* . We shall assume that two properties — that are met by $\mathcal{T}'_\theta$ — hold in $\mathcal{T}_\theta$:

(H1) $\mathcal{T}_\theta$ is (left) sequential;

(H2) in every state $\hat{s}$ of $\mathcal{T}_\theta$, the reading of z causes $\mathcal{T}_\theta$ *to stay* in state $\hat{s}$.

Thus (H1) leads to use notation of [3] that we have recalled in Section 3 : $\mathcal{T}_\theta = (Q, X, B, \delta, \lambda, i)$, δ is the *transition function* and λ is the *output function* of $\mathcal{T}_\theta$. (H2) then reads:

(H2) in every state $\hat{s}$ of $\mathcal{T}_\theta$, $\delta(\hat{s}, z) = \hat{s}$.

It turns out that these two hypotheses can be met but also lead naturally to a two-tape automaton $\mathcal{T}_\theta$ that solves the problem — and that is remarkably simple. Let us explore $\mathcal{T}_\theta$ "outside" $\mathcal{T}'_\theta$ and consider the reading of a word w of the form

$$w = abz^k \quad .$$

The reading of a puts $\mathcal{T}_\theta$ in state $\hat{a}$, then the reading of b puts it in a certain state, say $\hat{s}$. Let us try to compute $\lambda(\hat{s}, z)$ and let us remark for that purpose that w can be written as

$$w = azz^k \oplus zbz^k \tag{5}$$

from which follows that $\lambda(\hat{s}, z)$ has to be *the sum* of $\lambda(\hat{a}, z)$ and $\lambda(\hat{b}, z)$.

Let us be more specific (we suppose that we are in "Case 1" for the next paragraph). Proposition 5 yields:

$$\begin{aligned}
\langle azz^k \rangle_\theta &= (0\,0\,0\,1)(0\,r{-}1\,0\,1)(0\,r{-}1\,0\,1)^k.(0\,r{-}1\,0\,1)^k(0\,r{-}1\,0\,1)(0\,0\,0\,0) \\
\langle zbz^k \rangle_\theta &= (0\,0\,0\,0)(0\,0\,1\,0)(r{-}1\,0\,1\,0)^k.(r{-}1\,0\,1\,0)^k(r{-}1\,1\,0\,0)(0\,0\,0\,0)
\end{aligned}$$

and thus, by addition,

$$\langle abz^k\rangle_\theta \;= (\,0\;0\;0\;1\,)(\,0\;r{-}1\;1\;1\,)(\,r{-}1\;r{-}1\;1\;1\,)^k.(\,r{-}1\;r{-}1\;1\;1\,)(\,r{-}1\;r\;0\;1\,)^k(\,0\;0\;0\;0\,) \tag{6}$$

which *implies*, (going back to the folded θ-representations)

$$\lambda(\hat{s}, z) = \left(\begin{smallmatrix} r-1 & r-1 & 1 & 1 \\ 1 & 1 & r-1 & r-1 \end{smallmatrix}\right) \; .$$

It seems then adequate to identify

$$\hat{a} \;\text{ to }\; 0\;r{-}1\;0\;1\;, \qquad \hat{b} \;\text{ to }\; r{-}1\;0\;1\;0\;, \quad\text{and}\quad \hat{s} \;\text{ to }\; r{-}1\;r{-}1\;1\;1\;.$$

The idea behind the building of the underlying input automaton of $\mathcal{T}_\theta$ is to maintain this identification between the states and the elements of $\mathbb{Z}^4$, the reading of a letter of X being equivalent to an addition in $\mathbb{Z}^4$. The successive additions would yield an infinite number of states if it was not taken into account that expressions such as in (6) are θ-representations and that two θ-representations are equivalent if they give the same *numerical value*. This equivalence, transfered on the factors of length 4 gives the following equalities[13]:

$$1\;\bar{r}\;\bar{\varepsilon}\;0 \;=\; \bar{r}\;\bar{\varepsilon}\;0\;1 \;=\; \bar{\varepsilon}\;0\;1\;\bar{r} \;=\; 0\;1\;\bar{r}\;\bar{\varepsilon} \;=\; 0\;0\;0\;0$$

Let us denote by γ_θ the congruence of $\mathbb{Z}^4$ generated by these equalities.

Hypotheses (H1) and (H2) have thus led us to choose as underlying input automaton of $\mathcal{T}_\theta$ the submonoid[14] G_θ of $\mathbb{Z}^4/\gamma_\theta$ generated by $\hat{a}$, $\hat{b}$, $\hat{c}$ and $\hat{d}$, the transition function being defined by the canonical morphism $\delta : X^* \longrightarrow G_\theta$ ($\delta(a) = \hat{a}$, *etc.*). We compute G_θ in the remainder of this section and we complete the description of $\mathcal{T}_\theta$ in the next section.

In order to give precise and complete statements, we have to specify the case we are in.

Case 1. ($\varepsilon = +1$, $r \geq 1$). θ is the zero larger than 1 of $P(X) = X^2 - rX - 1$. The discriminant of $P(X)$ is $\Delta = r^2 + 4$.

Proposition 11

(i) *if r is odd, then $G_\theta \simeq \mathbb{Z}/\Delta\mathbb{Z}$;*

(ii) *if r is even, and*

a) *if $r = 4m$, then $G_\theta \simeq \mathbb{Z}/(\frac{1}{2}\Delta)\mathbb{Z}$;*

b) *if $r = 4m + 2$, then $G_\theta \simeq \mathbb{Z}/(\frac{1}{4}\Delta)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.*

Case 2. ($\varepsilon = -1$, $r \geq 3$). θ is the zero larger than 1 of $P(X) = X^2 - rX + 1$. The discriminant of $P(X)$ is $\Delta = r^2 - 4$.

[13] With the convention that if n is an integer, $\bar{n}$ denotes $-n$, as already used in the introduction.

[14] We do not know yet that it is a subgroup.

PROPOSITION 12

(i) *if r is odd, then* $G_\theta \simeq \mathbb{Z}/\Delta\mathbb{Z}$;

(ii) *if r is even, then* $G_\theta \simeq \mathbb{Z}/(\frac{1}{2}\Delta)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof of Proposition 11. By definition, γ_θ is generated by the following relations:

$$1\ \bar{r}\ \bar{1}\ 0 \ = 0\ 0\ 0\ 0 \tag{7}$$
$$\bar{r}\ \bar{1}\ 0\ 1 \ = 0\ 0\ 0\ 0 \tag{8}$$
$$\bar{1}\ 0\ 1\ \bar{r} \ = 0\ 0\ 0\ 0 \tag{9}$$
$$0\ 1\ \bar{r}\ \bar{1} \ = 0\ 0\ 0\ 0 \tag{10}$$

which imply:

$$0\ \bar{r}\ 0\ \bar{r} \ = 0\ 0\ 0\ 0 \tag{11}$$
$$\bar{r}\ 0\ \bar{r}\ 0 \ = 0\ 0\ 0\ 0 \tag{12}$$

The four generators of G_θ are

$$\hat{a} = \ 0\ r{-1}\ 0\ 1\ , \qquad \hat{b} = \ r{-1}\ 0\ 1\ 0\ , \qquad \hat{c} = \ 0\ 1\ 0\ r{-1}\ , \quad \text{and} \quad \hat{d} = \ 1\ 0\ r{-1}\ 0\ .$$

Thus

$$\hat{a} + \hat{c} = \ 0\ r\ 0\ r\ = 0\ 0\ 0\ 0 \qquad \text{and} \qquad \hat{b} + \hat{d} = \ r\ 0\ r\ 0\ = 0\ 0\ 0\ 0\ , \tag{13}$$

and G_θ is a subgroup, quotient of $\mathbb{Z}^2$, with generators $\hat{a}$ and $\hat{b}$. We have now to distinguish between the cases where r is odd or even.

i) r is odd.

Claim 1

$$r\hat{a} - 2\hat{b} = \ 0\ 0\ 0\ 0 \tag{14}$$

Proof. Let $r = 2n+1$. It comes

$$\begin{array}{rllllll}
r\hat{a} - 2\hat{b} = & \overline{2(r{-1})} & r(r{-1}) & \overline{2} & r & & \\
= & \overline{4n} & 2rn & \overline{2} & r & & \\
= & \overline{2n} & 0 & \overline{2n+2} & r & & \text{by (7) , } 2n \text{ times,} \\
= & \overline{2n+1} & 0 & \overline{2n+1} & 0 & & \text{by (9) ,} \\
= & 0 & 0 & 0 & 0 & & \text{by (12) .}
\end{array}$$
■

The circular permutation on elements of $\mathbb{Z}^4$, applied to (14) , gives

$$r\hat{b} - 2\hat{c} = \ 0\ 0\ 0\ 0$$

which, by (13) , reads

$$r\hat{b} + 2\hat{a} = \ 0\ 0\ 0\ 0 \tag{15}$$

It is an easy exercise to show the following.

LEMMA 13 *Let x and y be two generators of $\mathbb{Z}^2$. The quotient of $\mathbb{Z}^2$ by the relation $px + qy = 0$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, where d is the gcd of p and q. If u is a generator of $\mathbb{Z}$ and t is a generator of $\mathbb{Z}/d\mathbb{Z}$, a possible isomorphism is defined by $x \mapsto (-(q/d)u, 0)$ and $y \mapsto ((p/d)u, t)$.* ∎

Since r and 2 are relatively prime,

$$\mathbb{Z}^2/[r\hat{a} - 2\hat{b} = 0] \simeq \mathbb{Z}$$

with the isomorphism defined by $\hat{a} \mapsto 2u$ and $\hat{b} \mapsto ru$. From (15) it follows that $(r^2 + 4)u = 0$ and thus

$$G_\theta \simeq \mathbb{Z}/\Delta\mathbb{Z} \ .$$

ii) $r = 2n$ is even.

Claim 2

$$(n+1)\hat{a} + (n-1)\hat{b} = \ 0\ 0\ 0\ 0 \tag{16}$$

Proof.

$$\begin{array}{rlllll}
(n+1)\hat{a} + (n-1)\hat{b} = & (n-1)(r-1) & (n+1)(r-1) & n-1 & n+1 & \\
= & r(n-1)+1 & n-1 & \bar{1} & n+1 & \text{by (7) , } n \text{ times} \\
= & r(n-1) & n-1 & 0 & \overline{n-1} & \text{by (9)} \\
= & 0 & 0 & 0 & 0 & \text{by (8) , } n-1 \text{ times}
\end{array}$$

∎

The circular permutation on elements of $\mathbb{Z}^4$, applied to (16) , gives

$$(n+1)\hat{b} + (n-1)\hat{c} = \ 0\ 0\ 0\ 0$$

which, by (13) , reads

$$(n+1)\hat{b} - (n-1)\hat{a} = \ 0\ 0\ 0\ 0 \tag{17}$$

Two cases are to be considered, according to whether r is equal to 0 or to 2 modulo 4 .

a) $r = 2n = 4m$. Equations (16) and (17) become

$$(2m+1)\hat{a} + (2m-1)\hat{b} = 0\ 0\ 0\ 0 \qquad (18)$$

$$(2m+1)\hat{b} - (2m-1)\hat{a} = 0\ 0\ 0\ 0 \qquad (19)$$

As $2m+1$ and $2m-1$ are relatively prime,

$$\mathbb{Z}^2/[(2m+1)\hat{a} + (2m-1)\hat{b} = 0] \simeq \mathbb{Z}$$

with the isomorphism defined by $\hat{a} \mapsto -(2m-1)u$ and $\hat{b} \mapsto (2m+1)u$. From (19) it follows that

$$\left((2m+1)^2 + (2m-1)^2\right) u = (\frac{1}{2}\Delta)u = 0$$

and thus

$$G_\theta \ \simeq \mathbb{Z}/(\frac{1}{2}\Delta)\mathbb{Z} \ \ .$$

b) $r = 2n = 4m+2$. Equations (16) and (17) become

$$(2m+2)\hat{a} + 2m\hat{b} = 0\ 0\ 0\ 0 \qquad (20)$$

$$(2m+2)\hat{b} - 2m\hat{a} = 0\ 0\ 0\ 0 \qquad (21)$$

As $2m+2$ and $2m$ have gcd 2,

$$\mathbb{Z}^2/[(2m+2)\hat{a} + 2m\hat{b} = 0] \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

with the isomorphism defined by $\hat{a} \mapsto (-mu, 0)$ and $\hat{b} \mapsto ((m+1)u, 1)$. From (21) it follows that

$$((2m+2)(m+1)u, 0) + \left(2m^2u, 0\right) = (0, 0)$$

i.e.,

$$(4m^2 + 4m + 2)u = (\frac{1}{4}\Delta)u = 0 \ \ ,$$

and thus

$$G_\theta \ \simeq \mathbb{Z}/(\frac{1}{4}\Delta)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ \ . \qquad \blacksquare$$

Proof of Proposition 12. In this case, γ_θ is generated by the following relations:

$$1\ \bar{r}\ 1\ 0\ = 0\ 0\ 0\ 0 \quad (22)$$
$$\bar{r}\ 1\ 0\ 1\ = 0\ 0\ 0\ 0 \quad (23)$$
$$1\ 0\ 1\ \bar{r}\ = 0\ 0\ 0\ 0 \quad (24)$$
$$0\ 1\ \bar{r}\ 1\ = 0\ 0\ 0\ 0 \quad (25)$$

The generators of G_θ are

$$\hat{a} = 0\ 1\ 0\ 1 \qquad \text{and} \qquad \hat{b} = 1\ 0\ 1\ 0$$

and the equalities $\hat{c} = \hat{a}$ and $\hat{d} = \hat{b}$ hold: G_θ is a quotient of $\mathbb{Z}^2$.

Claim 3

$$r\hat{a} - 2\hat{b} = 0\ 0\ 0\ 0 \quad (26)$$

Proof.

$$\begin{aligned} r\hat{a} - 2\hat{b} &= \bar{2}\ r\ \bar{2}\ r \\ &= \bar{1}\ 0\ \bar{1}\ r && \text{by (22)} \\ &= 0\ 0\ 0\ 0 && \text{by (24)} \end{aligned}$$

■

By circular permutation:

$$r\hat{b} - 2\hat{a} = 0\ 0\ 0\ 0 \ . \quad (27)$$

We have to distinguish again between the cases where r is odd or even.

i) r is odd. Since r and 2 are relatively prime,

$$\mathbb{Z}^2/[r\hat{a} - 2\hat{b} = 0] \simeq \mathbb{Z}$$

with the isomorphism defined by $\hat{a} \mapsto 2u$ and $\hat{b} \mapsto ru$. From (27) it follows that $(r^2+4)u = 0$ and thus

$$G_\theta \simeq \mathbb{Z}/\Delta\mathbb{Z} \ .$$

ii) $r = 2n$ is even. Then

$$\mathbb{Z}^2/[r\hat{a} - 2\hat{b} = 0] \simeq \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$$

with the isomorphism defined by $\hat{a} \mapsto (u, 0)$ and $\hat{b} \mapsto (nu, 1)$. From (27) it follows that

$$(rnu, 0) - (2u, 0) = (0, 0) \quad \text{i.e.,} \quad (2n^2 - 2)u = (\frac{1}{2}\Delta)u = 0$$

and thus

$$G_\theta \simeq \mathbb{Z}/(\frac{1}{2}\Delta)\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ .$$

■

REMARK 4 The above construction can be given an interpretation that brings it closer to the area of θ-expansions.

i) Let $\hat{s}$ be a state of $\mathcal{T}_\theta$; we have identified $\hat{s}$ to an element of $\mathbb{Z}^4$, denoted $\hat{s}$ as well, such that

$$\lambda(\hat{s}, z) = \rho(\hat{s}.\hat{s})$$

If, as in state $\hat{s}$, one keeps reading z, $\mathcal{T}_\theta$ keeps outputting $\lambda(\hat{s}, z)$. One thus could say that $\hat{s}$ "potentially contains" the word $\hat{s}^k.\hat{s}^k$ for any k and it would have been as legitimate to identify the state $\hat{s}$ with the *bi-infinite word*

$$^{\omega}\hat{s}.\hat{s}^{\omega} \qquad (*)$$

which is periodic (of period 4) up to the radix point. The circular permutation on words of length 4 corresponds to the *shift* on bi-infinite words.

In this setting, $\mathbb{Z}^4$ is isomorphic to the set Y of periodic bi-infinite words on $\mathbb{Z}$ of period 4.

ii) It is not only the group G_θ that is finite but the whole group Y/γ_θ a description of which can be given by the definition of a *normal form* of its elements.

Let K_θ be a set of "reduced words", the exact description of which depends upon which "case" we consider:

Case 1. Let θ be the root greater than 1 of $X^2 - rX - 1 = 0$, with $r \geq 1$. Then K_θ is the set of words of A_θ^4 with the property that they, and all their conjugates, are strictly smaller, in the lexicographical ordering, than $r0r0$.

Case 2. Let θ be the root greater than 1 of $X^2 - rX + 1 = 0$, with $r \geq 3$. Then K_θ is the set of words of A_θ^4 with the property that they, and all their conjugates, are different from $(r-2)(r-2)(r-2)(r-2)$ and strictly smaller, in the lexicographical ordering, than $(r-1)(r-2)(r-2)(r-2)$.

PROPOSITION 14

Every class of Y modulo γ_θ contains exactly one element represented by a word of K_θ.

iii) Although it is not possible to give a *numerical value* to bi-infinite words such as $(*)$, γ_θ corresponds to a "numerical value equivalence" and Proposition 14 happens to be the exact counterpart of a result of Parry characterizing the θ-expansions of real numbers ([17]). Proposition 14 is completely independent from the rest of the paper : Proposition 11 and Proposition 12 prove that G_θ is finite and that is enough for the construction of $\mathcal{T}_\theta$. Its proof is purely combinatorial and a bit lengthy. For these reasons, we have decided to publish it elsewhere ([11]).

7 Where the description of $\mathcal{T}_\theta$ is completed.

As announced, $\mathcal{T}_\theta$ is a *sequential* (letter-to-letter) two-tape automaton and will be denoted as such:

$$\mathcal{T}_\theta = (G_\theta, X, B_\theta, \delta_\theta, \lambda_\theta, 0)$$

To lighten the notation, and if there is no ambiguity, we write δ and λ instead of δ_θ and λ_θ respectively.

The group G_θ is:

i) the subgroup generated by the images $\hat{a}$, $\hat{b}$, $\hat{c}$, and $\hat{d}$ of X ,

ii) in the quotient of $\mathbb{Z}^4$ by γ_θ .

By i), the canonical morphism from X^* into G_θ is surjective and, for coherence, every element G_θ is denoted as $\hat{f}$, where f is an element of X^*, and it holds:

$$\forall f, g \in X^* \qquad \widehat{fg} = \hat{f} + \hat{g}$$

The identity element of G_θ is denoted by 0 and $\widehat{1_{X^*}} = \hat{z} = 0$.

The transition function δ is the *(right) action* of X^* over G_θ (defined by the canonical morphism):

$$\forall \hat{g} \in G_\theta\,, \quad \forall f \in X^* \qquad \delta(\hat{g}, f) = \hat{g} + \hat{f}$$

By ii), every element $\hat{g}$ of G_θ can be identified with an element of $\mathbb{Z}^4$, a fixed representative of its class modulo γ_θ , chosen[15] once for all and also denoted by $\hat{g}$.

Example 3 : $\varepsilon = +1$, $r = 3$, $\tau = \frac{3+\sqrt{13}}{4}$ is the dominant root of $X^2 - 3X - 1 = 0$. $G_\tau \simeq \mathbb{Z}/13\mathbb{Z}$ and $\hat{a} = 0\,2\,0\,1$. A set of representatives[16] of G_τ in $\mathbb{Z}^4$ and the action of X on G_τ is exhibited in Figure 4 . □

With these notation, the following lemma is a consequence of Propositions 11 and 12 and their proof.

Lemma 15 *For any f, g, and h in X^* such that*

$$\hat{h} = \hat{g} + \hat{f}$$

in G_θ , there exists an element u in $\mathbb{Z}^4$ such that

$$\hat{h} = \hat{g} \oplus \hat{f} \oplus u$$

in $\mathbb{Z}^4$, which is a linear combination *of the left-hand side of the defining relations of γ_θ (equations (7) to (10) — Case 1 — or (22) to (25) — Case 2).* ■

[15] Proposition 14 tells what such a choice can be, but it is obviously immaterial to the proof.

[16] Chosen according to Proposition 14.

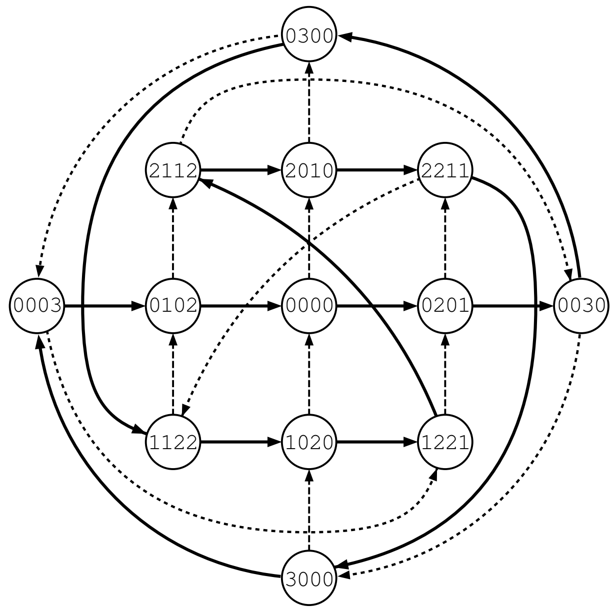

Figure 4: The action of X on G_τ: the only transitions represented are those labelled by $\hat{a} = 0\,2\,0\,1$ (bold arrows), and by $\hat{b} = 2\,0\,1\,0$ (dashed arrows).

Exemple 3 (continued): Let $\hat{g} = 1\,1\,2\,2$ and $\hat{a} = 0\,2\,0\,1$. Then

$$\begin{aligned} 1\,1\,2\,2\ +0\,2\,0\,1 &= 1\,3\,2\,3 \\ &= 2\,0\,1\,3 && \text{by (7)} \\ &= 1\,0\,2\,0 = \widehat{ga} && \text{by (9)} \end{aligned}$$

Thus let

$$v = 1\,\bar{3}\,\bar{1}\,0 \oplus \bar{1}\,0\,1\,\bar{3} = 0\,\bar{3}\,0\,\bar{3}$$

and the equation

$$\widehat{ga} = \hat{g} \oplus \hat{a} \oplus v$$

holds. □

As we have seen in Section 4, Proposition 5 defines $\lambda(0, x)$ for every x in X and in Section 6 we have defined $\lambda(\hat{g}, z)$ to be

$$\lambda(\hat{g}, z) = \rho(\hat{g}.\hat{g})$$

for every $\hat{g}$ in G_θ .

Exemple 3 (continued):

$$\mu_{\tau,X}(az^k) = \left(\begin{smallmatrix} 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & r-1 & 0 & 1 \\ 1 & 0 & r-1 & 0 \end{smallmatrix}\right)^k, \quad \lambda_\tau(0, a) = \begin{smallmatrix} 0\,0\,0\,1 \\ 0\,0\,0\,0 \end{smallmatrix}, \quad \lambda_\tau(\hat{a}, z) = \begin{smallmatrix} 0 & r-1 & 0 & 1 \\ 1 & 0 & r-1 & 0 \end{smallmatrix}$$

□

The output function is then given by the following:

Lemma 16 *For every $\hat{g}$ in G_θ and every x in X there exists a double-digit word of length 4, $\lambda(\hat{g}, x)$ (on a certain alphabet B_θ), with the property that, for every integer k, the equation*

$$\pi_\theta(\lambda(\hat{g}, x)\lambda(\hat{h}, z)^k) = \pi_\theta(\lambda(\hat{g}, z)^{k+1}) + \pi_\theta(\lambda(0, x)\lambda(\hat{x}, z)^k) \tag{28}$$

holds, with $\hat{h} = \delta(\hat{g}, x)$.

Proof. Let us first rewrite (28) for unfolded θ-representations:

$$\pi_\theta(\overleftarrow{\lambda(\hat{g}, x)}\hat{h}^k.\hat{h}^k\overrightarrow{\lambda(\hat{g}, x)}) = \pi_\theta(\hat{g}^{k+1}.\hat{g}^{k+1}) + \pi_\theta(\overleftarrow{\lambda(0, x)}\hat{x}^k.\hat{x}^k\overrightarrow{\lambda(0, x)}) \tag{29}$$

The essence of the proof is to show that the word

$$\lambda(\hat{g}, x) = \rho(\overleftarrow{\lambda(\hat{g}, x)}.\overrightarrow{\lambda(\hat{g}, x)})$$

in (28) is independent of k.

Let us now consider the analogous of the defining relations for γ_θ but "expanded to the order k" on both sides of the radix point and "completed" on both sides to have full words $1\,\bar{r}\,\bar{\varepsilon}\,0$ as factors:

$$(0\,0\,0\,0)(1\,\bar{r}\,\bar{\varepsilon}\,0)^k.(1\,\bar{r}\,\bar{\varepsilon}\,0)^k(0\,0\,0\,0) \qquad (7)_k$$

$$(0\,0\,0\,1)(\bar{r}\,\bar{\varepsilon}\,0\,1)^k.(\bar{r}\,\bar{\varepsilon}\,0\,1)^k(\bar{r}\,\bar{\varepsilon}\,0\,0) \qquad (8)_k$$

$$(0\,0\,1\,\bar{r})(\bar{\varepsilon}\,0\,1\,\bar{r})^k.(\bar{\varepsilon}\,0\,1\,\bar{r})^k(\bar{\varepsilon}\,0\,0\,0) \qquad (9)_k$$

$$(0\,0\,0\,0)(0\,1\,\bar{r}\,\bar{\varepsilon})^k.(0\,1\,\bar{r}\,\bar{\varepsilon})^k(0\,0\,0\,0) \qquad (10)_k$$

For any k, the numerical value π_θ of any of these words is 0 .

Let $u = u(\hat{g}, x)$ be the element of $\mathbb{Z}^4$ such that

$$\hat{h} = \hat{g} \oplus \hat{x} \oplus u \ .$$

As stated in Lemma 15, u is a linear combination of the defining relations of γ_θ . The *same* linear combination of the words $(7)_k$ to $(10)_k$ gives a word

$$u' u^k.u^k u''$$

with numerical value 0 .

Let us set

$$\overleftarrow{\lambda(\hat{g}, x)} = \overleftarrow{\lambda(0, x)} \oplus \hat{g} \oplus u' \quad \text{and} \quad \overrightarrow{\lambda(\hat{g}, x)} = \overrightarrow{\lambda(0, x)} \oplus \hat{g} \oplus u''$$

and the verification of (29) is straightforward. ∎

EXEMPLE 3 (continued): In this example, $(7)_k$ and $(9)_k$ read

$$(0\,0\,0\,0)(1\,\bar{2}\,\bar{1}\,0)^k.(1\,\bar{2}\,\bar{1}\,0)^k(0\,0\,0\,0) \qquad (7)_k$$

$$(0\,0\,1\,\bar{2})(\bar{1}\,0\,1\,\bar{2})^k.(\bar{1}\,0\,1\,\bar{2})^k(\bar{1}\,0\,0\,0) \qquad (9)_k$$

and thus

$$v'v^k.v^kv'' = (0\,0\,1\,\bar{2})(0\,\bar{2}\,0\,\bar{2})^k.(0\,\bar{2}\,0\,\bar{2})^k(\bar{1}\,0\,0\,0)$$

which yields

$$\overleftarrow{\lambda_\tau(1\,1\,2\,2,0\,2\,0\,1)} = 0\,0\,0\,1 \oplus 1\,1\,2\,2 \oplus 0\,0\,1\,\bar{2} = 1\,1\,3\,0$$
$$\overrightarrow{\lambda_\tau(1\,1\,2\,2,0\,2\,0\,1)} = 0\,0\,0\,0 \oplus 1\,1\,2\,2 \oplus \bar{1}\,0\,0\,0 = 0\,1\,2\,2$$

that is

$$\lambda_\tau(1\,1\,2\,2,0\,2\,0\,1) = {}^{1\,1\,3\,0}_{2\,2\,1\,0} \qquad \square$$

The alphabet B_θ is the set of all double-digits that appear in such computation of $\lambda(\hat{g},x)$ when $\hat{g}$ ranges over G_θ and x over X. We are now in a position to give an explicit statement for Theorem 3:

THEOREM 3 *Let $\mathcal{T}_\theta = (G_\theta, X, B_\theta, \delta, \lambda, 0)$ be the sequential letter-to-letter two-tape automaton defined by the functions δ and λ as above. The two-tape automaton $\mathcal{T}_\theta$ maps every word of X^* onto a folded equivalent θ-representation , that is*

$$\forall f \in X^* \qquad \pi_\theta(\mathcal{T}_\theta(f)) = \pi_U(f) \ .$$

Proof. By induction on $|f|$, we prove a more general relation:

$$\forall f \in X^*, \quad \forall k \in \mathbb{N} \qquad \pi_\theta(\mathcal{T}_\theta(fz^k)) = \pi_U(fz^k) \ . \qquad (30)$$

By construction of $\mathcal{T}_\theta$, it holds

$$\forall f \in X^*, \quad \forall k \in \mathbb{N} \qquad \mathcal{T}_\theta(fz^k)) = \mathcal{T}_\theta(f)\rho(\hat{f}^k.\hat{f}^k) \ , \qquad (31)$$

and, also by construction, Proposition 5 yields (30) for $|f| = 1$.

We need two more pieces of notation: let $Z = {}^{0\,0\,0\,0}_{0\,0\,0\,0} = \lambda(0,z)$ be the block of four null double-digits and let us denote by $\mathcal{T}_\theta(\hat{h}, f)$ the output of $\mathcal{T}_\theta$ when reading the word f from the state $\hat{h}$ taken as initial state. It then comes:

$$\begin{aligned}
\forall x \in X, \quad \pi_U(fxz^k) &= \pi_U(fz^{k+1}) + \pi_U(xz^k) && \text{by induction hypothesis,}\\
&= \pi_\theta(\mathcal{T}_\theta(fz^{k+1})) + \pi_U(xz^k) && \text{by construction and Proposition 5,}\\
&= \pi_\theta(\mathcal{T}_\theta(f)\lambda(\hat{f}, z^{k+1})) + \pi_\theta(\lambda(0,x)\lambda(\hat{x}, z^k))\\
&= \pi_\theta(\mathcal{T}_\theta(f)Z^{k+1}) + \pi_\theta(\lambda(\hat{f}, z^{k+1})) + \pi_\theta(\lambda(0,x)\lambda(\hat{x}, z^k)) && \text{by (28) ,}\\
&= \pi_\theta(\mathcal{T}_\theta(f)Z^{k+1}) + \pi_\theta(\lambda(\hat{f}, x)\lambda(\widehat{fx}, z^k))\\
&= \pi_\theta(\mathcal{T}_\theta(f)Z^{k+1}) + \pi_\theta(\mathcal{T}_\theta(\hat{f}, xz^k))\\
&= \pi_\theta(\mathcal{T}_\theta(f)\mathcal{T}_\theta(\hat{f}, xz^k)) = \pi_\theta(\mathcal{T}_\theta(fxz^k)) \ .
\end{aligned}$$

■

We have established in Section 5 that Theorem 3 proves Theorem 2.

Acknowledgements. The first version of this paper was completed while the first author was visiting the Mathematical Sciences Research Institute at Berkeley, by invitation of the Program on Symbolic Dynamics. The friendly atmosphere, the excellent working conditions, and the exciting intellectual environment of MSRI are gratefully acknowledged.

The criticism and advice of David Klarner who first rewied the paper led the authors to correct several inaccuracies in some statements and proofs. The authors are pleased to thank also two anonymous careful referees who were kind enough to correct the many errors in spelling and grammar that spoiled the version that was sent to them, and Chritian Choffrut who pointed to a last (?) mistake.

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