Ergod. Th. & Dynam. Sys. (1992), **12**, 713–723
Printed in Great Britain

Finite beta-expansions

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(*Received* 22 *July* 1991 *and revised* 19 *December* 1991)

Abstract. We characterize numbers having finite β-expansions where β belongs to a certain class of Pisot numbers: when the β-expansion of 1 is equal to $a_1a_2\ldots a_m$, where $a_1 \geq a_2 \geq \cdots \geq a_m \geq 1$ and when the β-expansion of 1 is equal to $t_1t_2\ldots t_m(t_{m+1})^\omega$ where $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} \geq 1$.

1. *Introduction*

Representations of real numbers with an arbitrary base $\beta > 1$, called *β-expansions*, were introduced by Rényi [**R**]. They arise from the orbits of a piecewise-monotone transformation of the unit interval: $T_\beta : x \mapsto \beta x \pmod 1$. Such transformations were extensively studied in ergodic theory (see [**P**] and the bibliography in [**B1**]).

Properties of β-expansions are strongly related to symbolic dynamics [**B1**]. The closure of the set of infinite sequences, appearing as β-expansions, is called the β-shift. It is a symbolic dynamical system, that is, a closed shift-invariant subset of $\mathscr{A}^{\mathbf{N}}$, where the alphabet $\mathscr{A}$ is the set of all possible digits $\{0, 1, \ldots, [\beta]\}$. A symbolic dynamical system is said to be of *finite type* if the set of its finite factors is defined by the interdiction of a finite set of words. It is said to be *sofic* if the set of its finite factors is recognized by a finite automaton. The β-shift has finite type if and only if $T_\beta^k 1 = 0$ for some k, and it is sofic if and only if the orbit $\{T_\beta^k 1\}$ is finite (see [**B1**]).

In this paper we study the set of numbers $x \geq 0$ having finite β-expansions, which we denote by Fin (β). It is a subset of Per (β), the set of numbers having eventually periodic β-expansions. For a standard system of numeration, when β is an integer greater than one, one has Per $(\beta) = \mathbf{Q}(\beta) \cap \mathbf{R}_+$ and Fin $(\beta) = \mathbf{Z}[\beta^{-1}] \cap \mathbf{R}_+$.

Eventually periodic β-expansions were investigated by Bertrand [**Be**] and Schmidt [**S**]. An algebraic integer is called a *Pisot number* if all its Galois conjugates have modulus less than one, and a *Salem number* if all its conjugates are less or equal than one in modulus and at least one conjugate has modulus one. It is proved in [**Be, S**] that if β is a Pisot number, then Per $(\beta) = \mathbf{Q}(\beta) \cap \mathbf{R}_+$. Conversely, Schmidt proved [**S**] that if $\mathbf{Q} \cap [0, 1] \subset$ Per (β), then β is a Pisot, or a Salem number. It is still unknown whether all rationals have eventually periodic β-expansions if β is a

Salem number. Boyd [Bo] has shown that if β is a Salem number of degree four, then the β-expansion of 1 is eventually periodic.

One of the authors proved in [Fr1] that addition (and multiplication by a fixed integer) in base β—where β is a Pisot number—is a function computable by a finite automaton. In this paper we find two classes of Pisot numbers for which the sum of two finite β-expansions is again a finite β-expansion. Examples show that this is false for some Pisot numbers.

We first prove that if $\mathbf{Z}_+ \subset \mathrm{Fin}(\beta)$ then β is a Pisot number or a Salem number, and if $\mathbf{Z}[\beta^{-1}] \cap \mathbf{R}_+ \subset \mathrm{Fin}(\beta)$ then β is a Pisot number (Lemma 1). It is also possible to show that $\mathbf{Z}[\beta^{-1}] \cap \mathbf{R}_+ = \mathbf{Z}_+[\beta^{-1}]$ if and only if β is a Perron eigenvalue of a primitive companion matrix (Lemma 2).

The main results of the paper are the following two theorems which give sufficient (but certainly not necessary) conditions for the inclusion $\mathbf{Z}_+[\beta^{-1}] \cap \mathbf{R}_+ \subset \mathrm{Fin}(\beta)$.

Let β be the positive root of the polynomial $M(X) = X^m - a_1X^{m-1} - a_2X^{m-2} - \cdots - a_m$, $a_i \in \mathbf{Z}$, and $a_1 \geq a_2 \geq \cdots \geq a_m > 0$. Then β is a Pisot number, the β-shift is a system of finite type, and $\mathrm{Fin}(\beta) = \mathbf{Z}[\beta^{-1}] \cap \mathbf{R}_+$ (Theorem 2). The corresponding systems of numeration were considered in [Fra, PT].

Let $\beta > 1$ be a real number such that the β-expansion of 1 is equal to $t_1t_2 \ldots t_m(t_{m+1})^\omega$, with $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} > 0$. Then β is a Pisot number, and $\mathbf{Z}_+[\beta^{-1}] \subset \mathrm{Fin}(\beta)$ (Theorem 3).

The numbers β in Theorem 2 and in Theorem 3 have a common feature: in fact, β belongs to one of these two classes if and only if the orbit of 1 under the β-transformation is non-increasing.

As a corollary we obtain the following result: for every Pisot number β of degree 2, $\mathbf{Z}_+[\beta^{-1}] \subset \mathrm{Fin}(\beta)$.

At the end we give a quantitative version of the results above (Proposition 2) for application to substitution dynamical systems. Let ζ be the substitution of the alphabet $\{1, 2, \ldots, m\}$:

$$\zeta(1) = \underbrace{1\ldots1}_{k_1}2,\ \zeta(2) = \underbrace{1\ldots1}_{k_2}3, \ldots, \zeta(m-1) = \underbrace{1\ldots1}_{k_{m-1}}m,\ \zeta(m) = \underbrace{1\ldots1}_{k_m}.$$

This substitution generates a minimal, uniquely ergodic measure-preserving system ('substitution dynamical system', see [Q]). In [So2] one of the authors proved that if β is a Pisot number of degree m such that the β-expansion of 1 is equal to $k_1k_2 \ldots k_m$, and if $\mathbf{Z}[\beta^{-1}] \cap \mathbf{R}_+ \subset \mathrm{Fin}(\beta)$, then the substitution dynamical system has purely discrete spectrum. The class of numbers in Theorem 2 fits this scheme.

The properties of periodic and finite expansions are related to the theory of tilings (see [T]). Theorem 2 was used by Praggastis [Pr] to construct Markov partitions for some toral automorphisms.

Some of the results presented here can be found in [Fr2] and in [So1].

2. *Representation of numbers*

Let $\beta > 1$ be a real number. A *representation in base* β (or a β-representation) of a real number $x \geq 0$ is an infinite sequence $(x_i)_{k \geq i \geq -\infty}$, $x_i \geq 0$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain integer $k \geq 0$. It is denoted by

$$x = x_k x_{k-1} \ldots x_1 x_0 \,.\, x_{-1} x_{-2} \ldots .$$

A particular β-representation—called the *β-expansion*—can be computed by the 'greedy algorithm':

Denote by $[y]$ and $\{y\}$ the integer part and the fractional part of a number y. There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = [x/\beta^k]$, and $r_k = \{x/\beta^k\}$. Then for $k > i \geq -\infty$, put $x_i = [\beta r_{i+1}]$, and $r_i = \{\beta r_{i+1}\}$. We get an expansion $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $k < 0$ $(x < 1)$, we put $x_0 = x_{-1} = \cdots = x_{k+1} = 0$. If an expansion ends in infinitely many zeros, it is said to be finite, and the ending zeros are omitted.

The digits x_i obtained by this algorithm are integers from the set $\mathscr{A} = \{0, \ldots, \beta - 1\}$ if β is an integer, or the set $\mathscr{A} = \{0, \ldots, [\beta]\}$ if β is not an integer. We will sometimes omit the splitting point between the integer part and the fractional part of the β-expansion; then the infinite sequence is just an element of $\mathscr{A}^{\mathbb{N}}$.

For numbers $x < 1$, the expansion defined above coincides with the β-expansion of Rényi [R], which can be defined by means of the *β-transformation* of the unit interval

$$T_\beta x = \beta x (\bmod 1), \quad x \in [0, 1].$$

In fact, for $x \in [0, 1[$, we have $x_k = [\beta T_\beta^{k-1} x]$. However, for $x = 1$ the two algorithms differ: our expansion is just $1 = 1.$, while the Rényi expansion is

$$d(1, \beta) = .t_1 t_2 \ldots, \quad t_k = [\beta T_\beta^{k-1} 1],$$

(the point is usually omitted).

Let D_β be the set of β-expansions of numbers of $[0, 1[$, and let $d\colon [0, 1] \to D_\beta \cup \{d(1, \beta)\}$ be the function mapping $x \neq 1$ onto its β-expansion, and 1 onto $d(1, \beta)$. Clearly, if $x = x_k \ldots x_0 \,.\, x_{-1} \ldots$ is a β-expansion, then $x/\beta^{k+1} = .\, x_k \ldots x_0 x_{-1} \ldots$ belongs to D_β.

Recall some results concerning the set D_β. The set $\mathscr{A}^{\mathbb{N}}$ is endowed with the *lexicographical order* (notation $<_{lex}$), the product topology, and the (one-sided) shift σ. The set D_β is shift-invariant. The *β-shift* S_β is the closure of D_β, it is a subshift of $\mathscr{A}^{\mathbb{N}}$ (see [Be] and [B1]). We have $d \circ T_\beta = \sigma \circ d$ on $[0, 1[$. Recall that the β-shift S_β is a system of finite type if and only if $d(1, \beta)$ is finite [P] (such numbers were called *simple β-numbers* by Parry). The β-shift S_β is a sofic system if and only if $d(1, \beta)$ is eventually periodic (Bertrand, see [B1]).

We recall the characterization of the set D_β [P]. By x^ω will be denoted the sequence $xxx\ldots$.

THEOREM 1. *Let β be a real number greater than one, and let $d(1, \beta) = t_1 t_2 \ldots$ Let s be an infinite sequence of positive integers.*

(i) *If $d(1, \beta)$ is infinite, the condition*

$$\forall p \geq 0, \qquad \sigma^p(s) <_{lex} d(1, \beta)$$

is necessary and sufficient for s to belong to D_β.

(ii) *If $d(1, \beta)$ is finite, $d(1, \beta) = t_1 \ldots t_{n-1} t_n$, then $s \in D_\beta$ if and only if*

$$\forall p \geq 0, \qquad \sigma^p(s) <_{lex} d^*(1, \beta) = (t_1 \ldots t_{n-1}(t_n - 1))^\omega.$$

3. *Finite expansions*

Let Per (β) be the set of all numbers $x \geq 0$ having eventually periodic β-expansions. It is clear that Per $(\beta) \cap [0, 1[$ is the set of points whose orbits under T_β are finite.

Let $\mathbf{Q}(\beta)$ denote the smallest field containing the field of rational numbers $\mathbf{Q}$ and $\beta > 1$. It is proved in [Be, S] that if β is a Pisot number, then Per $(\beta) = \mathbf{Q}(\beta) \cap \mathbf{R}_+$. Conversely, Schmidt proved [S] that if $\mathbf{Q} \cap [0, 1[\subset$ Per (β), then β is a Pisot, or a Salem number.

Now, let Fin (β) stand for the set of $x \geq 0$ having finite β-expansions. Note that Fin $(\beta) \cap [0, 1[$ consists of those points whose orbits under T_β end up at zero.

For $\lambda > 0$, Let $\mathbf{Z}[\lambda]$ denote the ring of polynomials in λ with integral coefficients, $\mathbf{Z}_+[\lambda]$ the cone of polynomials with non-negative coefficients, and set $(\mathbf{Z}[\lambda])_+ = \mathbf{Z}[\lambda] \cap \mathbf{R}_+$.

If $\beta > 1$ is an integer, then Fin $(\beta) = (\mathbf{Z}[\beta^{-1}])_+$. We will address the problems: for which β all integers have finite β-expansions? when Fin $(\beta) = (\mathbf{Z}[\beta^{-1}])_+$? More specifically, consider the following conditions on β:

$$(F_1) \quad \mathbf{Z}_+ \subset \mathrm{Fin}\,(\beta);$$

$$(F_2) \quad \mathbf{Z}_+[\beta^{-1}] \subset \mathrm{Fin}\,(\beta);$$

$$(F_3) \quad (\mathbf{Z}[\beta^{-1}])_+ \subset \mathrm{Fin}\,(\beta).$$

Note that condition (F_1) implies that β is an algebraic integer (consider the β-expansion of $x = [\beta] + 1$). If β is an algebraic integer, then Fin $(\beta) \subset \mathbf{Z}[\beta, \beta^{-1}] = \mathbf{Z}[\beta^{-1}]$, so (F_3) actually means that Fin $(\beta) = (\mathbf{Z}[\beta^{-1}])_+$.

Lemma 1. (a) *If* $\mathbf{Z}_+ \subset$ Fin (β), *then* β *is a Pisot number or a Salem number*; (*b*) *condition* (F_3) *implies that* β *is a Pisot number.*

Proof. (a) is proved similar to [S, 2.4] and even easier since we deal with finite expansions. We present the proof for convenience of the reader.

Suppose that β has a Galois conjugate γ, $|\gamma| > 1$. Let $\eta = \max(|\beta|^{-1}, |\gamma|^{-1})$, and $C = 2[\beta]\eta(1-\eta)^{-1}$. It is easy to see that one can find $m \in \mathbf{Z}_+$ so that $|\beta^m - \gamma^m| > C$. Then take $x = [\beta^m] + 1 \in \mathbf{Z}_+$. If x has a finite β-expansion, it must be of the following form:

$$x = \beta^m + \varepsilon_1\beta^{-1} + \varepsilon_2\beta^{-2} + \cdots + \varepsilon_k\beta^{-k}.$$

Since $x \in \mathbf{Z}$, all Galois conjugates of β satisfy the same equation, and $x = \gamma^m + \varepsilon_1\gamma^{-1} + \varepsilon_2\gamma^{-2} + \cdots + \varepsilon_k\gamma^{-k}$. Subtracting these two representations of x and using that $|\varepsilon_i| \leq [\beta]$, we come to a contradiction.

(b) It remains to exclude the case of Salem numbers. As mentioned above, $\mathbf{Z}_+ \subset$ Fin (β) implies that β is an algebraic integer, and so $\beta \in \mathbf{Z}[\beta^{-1}]$. Thus, $\beta - [\beta] \in$ Fin (β), and so β is a root of a polynomial

$$X^m - k_1X^{m-1} - k_2X^{m-2} - \cdots - k_m, \quad k_i \in \mathbf{Z}_+.$$

It follows that β has no positive Galois conjugates. On the other hand, it is known that every Salem number α is reciprocal (see [Sa, p. 26]), and hence has a conjugate $\alpha^{-1} > 0$. Thus β is a Pisot number. □

We do not know whether conditions (F_1) and (F_2) are equivalent. The relation between (F_2) and (F_3) can be understood with the help of the following lemma.

LEMMA 2. *Let $\lambda > 0$. The following are equivalent:*
(a) $(\mathbf{Z}[\lambda^{-1}])_+ = \mathbf{Z}_+[\lambda^{-1}]$;
(b) *λ is a Perron eigenvalue of a primitive companion matrix.*

It is possible that this result is known. By a recent result of Handelman [**H**], condition (b) is equivalent to λ being a Perron number with no positive conjugates.

Proof of Lemma 2. (a)$\Rightarrow$(b). If $\lambda < 1$, then (a) is impossible since then $\mathbf{Z}_+[\lambda^{-1}] \cap]0, 1[= \emptyset$, while $\lambda^{-1} - [\lambda^{-1}] \in (\mathbf{Z}[\lambda^{-1}])_+$. Let $\lambda \geq 1$. Then (a) implies that

$$1 - \lambda^{-1} = c_1\lambda^{-1} + c_2\lambda^{-2} + \cdots + c_k\lambda^{-k}, \qquad c_i \in \mathbf{Z}_+, \quad c_k > 0, \quad k \geq 1.$$

Then λ is a Perron eigenvalue of the companion matrix A of the polynomial $p(X) = X^k - (c_1+1)X^{k-1} - c_2X^{k-2} - \cdots - c_k$. One can check that A^k is strictly positive, so A is primitive.

(b)$\Rightarrow$(a). Suppose that λ is the Perron eigenvalue of a primitive companion matrix

$$M = \begin{bmatrix} 0 & 0 & \cdots & 0 & a_k \\ 1 & 0 & \cdots & 0 & a_{k-1} \\ 0 & 1 & \cdots & 0 & a_{k-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & a_1 \end{bmatrix}.$$

Let $x \in (\mathbf{Z}[\lambda^{-1}])_+$. Using the 'recurrence relation'

$$\lambda^r = a_1\lambda^{r-1} + a_2\lambda^{r-2} + \cdots + a_k\lambda^{r-k}, \quad r \in \mathbf{Z},$$

one can express x as an integral linear combination of k consecutive powers of λ. More precisely, there exist $s \in \mathbf{Z}_+$ and $B \in \mathbf{Z}^k$ such that

$$x = \lambda^{-s}\Lambda \cdot B, \quad \Lambda = [\lambda^{-k+1}, \ldots, \lambda^{-1}, 1],$$

('$\cdot$' denotes the usual scalar product in $\mathbf{R}^k$). Since Λ is the left Perron eigenvector for M, one has

$$x = \lambda^{-s-l}\Lambda \cdot M^l B. \tag{1}$$

By the Perron–Frobenius theory (see [**Se**]),

$$\lim_{l\to\infty} \lambda^{-l}M^l B = \frac{\Lambda \cdot B}{\Lambda \cdot R} R = \frac{x\lambda^s}{\Lambda \cdot R} R > 0,$$

where R is a strictly positive right eigenvector for M. Hence $M^l B = [d_i]_{i=1}^k > 0$ for l sufficiently large, and by virtue of (1),

$$x = \sum_{i=1}^{k} d_i\lambda^{-s-l-k+i} \in \mathbf{Z}_+[\lambda^{-1}]. \qquad \square$$

COROLLARY. *The following are equivalent:*
(i) *β satisfies condition (F_3);*
(ii) *β satisfies condition (F_2) and β is a simple β-number ($d(1, \beta)$ is finite).*

Proof. If β satisfies condition (F_3) then $\beta-[\beta]\in \text{Fin}(\beta)$, $\sigma(d(1,\beta))=d(\beta-[\beta])$ hence $d(1,\beta)$ is finite. Conversely, if $d(1,\beta)=a_1a_2\dots a_m$, then β is the positive root of the polynomial $x^k-a_1x^{k-1}-\cdots-a_k$, which has a primitive companion matrix because $a_1=[\beta]\geq 1$. Thus by Lemma 2, $(\mathbb{Z}[\beta^{-1}])_+=\mathbb{Z}_+[\beta^{-1}]$, and conditions (F_2) and (F_3) are identical. □

The following examples show that there are Pisot numbers such that $d(1,\beta)$ is finite but $\mathbb{Z}_+\not\subset \text{Fin}(\beta)$.

Example 1. Let β be the dominant root of the polynomial X^3-3X^2+2X-2. Then β is a Pisot number, $d(1,\beta)=2102$, and the β-expansion of 6 is $6=20.210(00112)^\omega$.

Example 2. Let β be the positive root of the polynomial X^4-2X^3-X-1. Then β is a unitary Pisot number, $d(1,\beta)=2011$, and the β-expansion of 3 is $3=10.111(00012)^\omega$.

4. *Systems of finite type*

THEOREM 2. *Let β be the positive root of the polynomial $M(X)=X^m-a_1X^{m-1}-a_2X^{m-2}-\cdots-a_m$, $a_i\in\mathbb{Z}$, and $a_1\geq a_2\geq\cdots\geq a_m>0$. Then β is a Pisot number, $d(1,\beta)=a_1a_2\cdots a_m$, and* $\text{Fin}(\beta)=(\mathbb{Z}[\beta^{-1}])_+$.

The special case of the Theorem when $M(X)=X^2-X-1$, and $\beta=(1+\sqrt{5})/2$ is the golden ratio, is known and seems to belong to folklore. Note also that the case of $M(X)=X^2-nX-1$ is implicitly contained in [S, 3.4].

Proof. Since the sequence $a_1a_2\dots a_m$ is lexicographically greater than its shifts, one has from [P] that $d(1,\beta)=a_1a_2\dots a_m$. From [Br, Th. 2] it follows that β is a Pisot number with the minimal polynomial $M(X)$. Using Lemma 2 and dividing by β^k if necessary, we see that one needs to prove only that $\mathbb{Z}_+[\beta^{-1}]\cap[0,1[\subset \text{Fin}(\beta)$. Let $x\in\mathbb{Z}_+[\beta^{-1}]$, $0\leq x<1$. Then x has a β-representation which ends in infinitely many zeros. Of course, it does not have to be the β-expansion of x. Let us say that a sequence is *β-admissible* if it corresponds to a β-expansion of some number. We are going to apply an algorithm which modifies a finite β-representation of x, aiming at a β-admissible sequence.

First we formulate the algorithm, and then prove that the process stops after finitely many passes. The following notations will be used: if $w=w_1w_2\dots$ is a sequence, then $s\oplus_k w$ denotes the sequence $s_1s_2\dots s_{k-1}(s_k+w_1)(s_{k+1}+w_2)\dots$ The factor $s_i\dots s_{i+k}$ of s will be denoted by $s[i;i+k]$.

It follows from Theorem 1 that a sequence $s=s_1s_2\dots s_r0^w$ is β-admissible if and only if for all k holds

$$s[k+1,k+m]<_{lex}a_1\dots a_m.$$

ALGORITHM. Applied to a sequence $s=s_1s_2\dots s_r0^w$, $s_i\in\mathbb{Z}_+$, such that $x=\sum_{i=1}^\infty s_i\beta^{-i}<1$.

Step 1. If there is a factor $s[k+1,k+m]\geq a_1a_2\dots a_m$ (termwise), let

$$s'=s\bigoplus_k 1(-a_1)\cdots(-a_m).$$

Repeat this as long as possible, then go to Step 2.

Step 2. Find the least k such that

$$s[k+1, k+m] >_{lex} a_1 a_2 \dots a_m. \tag{2}$$

If there is no such k, the sequence s is β-admissible, and the process stops (since Step 1 has been performed before, equality in (2) is impossible). Observe that $k \geq 1$, since $1 > x > s_1 \beta^{-1} \Rightarrow s_1 \leq [\beta] = a_1$.

It follows from (2) that there exists $l \geq k+1$ such that

$$s[k+1, l-1] = a_1 a_2 \dots a_{l-k-1}, \quad s_l > a_{l-k}.$$

Since we have already performed all possible operations in Step 1, $l \leq k+m-1$. Let

$$s' = s \bigoplus_k [1(-a_1) \cdots (-a_m)] \bigoplus_l [(-1)a_1 \dots a_m].$$

More explicitly, we have

$$s'_i = \begin{cases} s_i, & \text{for } i \leq k-1, \text{ and } i \geq l+m+1; \\ s_i + 1, & \text{for } i = k; \\ 0, & \text{for } k+1 \leq i \leq l-1; \\ s_i - a_{l-k} - 1, & \text{for } i = l; \\ s_i - a_{i-k} + a_{i-l}, & \text{for } l+1 \leq i \leq k+m \\ s_i + a_{i-l}, & \text{for } k+m+1 \leq i \leq l+m. \end{cases} \tag{3}$$

Go to Step 1.

It is clear that in both steps of the Algorithm s' is non-negative. The fact that s' is a representation of x in base β follows from the equation $\beta^r = a_1 \beta^{r-1} + a_2 \beta^{r-2} + \cdots + a_m \beta^{r-m}$. Thus is remains to check that the process stops after finitely many passes of the algorithm.

Observe that after each operation in Step 1, the sum of all digits decreases: $\sum_i s'_i = \sum_i s_i - (a_1 + \cdots + a_m - 1)$. It follows that the operation in Step 1 can be applied only finitely many times.

Let us analyze what is going on in Step 2. First we note that the operation in Step 2 does not change the sum of the digits: $\sum_i s'_i = \sum_i s_i$. Let l be the index in Step 2. The quantity

$$\mathrm{Adm}_x s = \sum_{i=1}^{l-1} s_i \beta^{-i},$$

will be called the *β-admissible part of x with respect to s.* Note that l is the maximal index such that the sequence $s_1 s_2 \dots s_{l-1} 0^w$ is β-admissible. We have to consider two cases.

Case 1. The sequence $s'_1 s'_2 \dots s'_{l-1} 0^\omega$ is β-admissible.

$$\mathrm{Adm}_x s' \geq \sum_{i=1}^{l-1} s'_i \beta^{-i} = \mathrm{Adm}_x s + \beta^{-k} - a_1 \beta^{-k-1} - \cdots - a_{l-k} \beta^{-l+1} \geq \mathrm{Adm}_x s + \beta^{-l}.$$

Case 2. The sequence $s'_1 s'_2 \dots s'_{l-1} 0^\omega$ is not β-admissible.

Then from (3) and the choice of k and l in Step 2 it follows that $s'[k-m+1, k] = a_1 \dots a_{m-1}a_m$, and the operation in Step 1 will be applied in the next pass of the algorithm.

Conclusion of the proof. Let $s^{(0)} = s$, and let $s^{(j)}$ be the representations of x obtained by repeated application of the Algorithm. If the process stops at $s^{(N)}$, then it is β-admissible, and we get a finite β-expansion of x because $s_i^{(N)} = 0$ for i sufficiently large. Suppose that the process goes on indefinitely. Since the sum of the digits remains unchanged in Step 2 and decreases in each operation of Step 1, the operation in Step 1 can be applied only finitely many times. Choose K so that for $j > K$, only the operation in Step 2 was applied. Then for $j > K$ we always find ourselves in Case 1. Let l_j be the index l for the sequence $s^{(j)}$,

$$l_j = \max\{r : s_1^{(j)} s_2^{(j)} \dots s_{r-1}^{(j)} 0^\omega \text{ is } \beta\text{-admissible}\}.$$

We have for $j > K$,

$$x \geq \mathrm{Adm}_x\, s^{(j+1)} \geq \mathrm{Adm}_x\, s^{(j)} + \beta^{-l_j}.$$

Therefore, $l_j \to \infty$ as $j \to \infty$. Looking at (3) we see that $s_i^{(j+1)} = s_i^{(j)}$ for $i < l_j - 2m$, so there exists the limiting sequence $\tilde{s}$ such that $\tilde{s}_i = s_i^{(j)}$ for all j sufficiently large. Since $\sum_i s_i^{(j)} \leq \sum_i s_i = S$, we conclude that

$$x = \lim_{j\to\infty} \sum_{i=1}^{\infty} s_i^{(j)} \beta^{-i} = \sum_{i=1}^{\infty} \tilde{s}_i \beta^{-i},$$

and also $\sum_i \tilde{s}_i \leq S$. But $\tilde{s}_i \in \mathbb{Z}_+$, so there exists K_1 such that $\tilde{s}_i = 0$ for $i > K_1$. Let j_0 be such that $s_i^{(j)} = \tilde{s}_i$ for $i \leq K_1$, $j \geq j_0$. Then we have

$$x = \sum_{i=1}^{K_1} \tilde{s}_i \beta^{-i} = \sum_{i=1}^{\infty} s_i^{(j)} \beta^{-i}, \quad j \geq j_0,$$

whence $s_i^{(j)} = \tilde{s}_i$, $j \geq j_0$, for all i. But this means that $s_i^{(j+1)} = s_i^{(j)}$, so no operation was applied. This is a contradiction. We have shown that the process stops, and the proof of Theorem 2 is now complete. □

5. *Sofic systems*

THEOREM 3. *Let $\beta > 1$ be a real number such that $d(1, \beta) = t_1 t_2 \dots t_m (t_{m+1})^\omega$, with $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} > 0$. Then β is a Pisot number, and $\mathbb{Z}_+[\beta^{-1}] \subset \mathrm{Fin}(\beta)$.*

Proof. From the equality $d(1, \beta) = t_1 t_2 \dots t_m (t_{m+1})^\omega$ one deduces that β is a real root of the polynomial

$$M(X) = X^{m+1} - (t_1 + 1)X^m + (t_1 - t_2)X^{m-1} + \cdots + (t_{m-1} - t_m)X + (t_m - t_{m+1}).$$

From a result of Perron (see [**Br**, II]) one has that β is a Pisot number and that M is the minimal polynomial for β.

Let $x \in \mathbb{Z}_+[\beta^{-1}]$. As in the proof of Theorem 2, one can assume that $x < 1$. Then x has a β-representation $x = \sum_{i=1}^{r} s_i \beta^{-i}$, $s_i \in \mathbb{Z}_+$. We shall use an algorithm which reduces a finite β-representation to a finite β-expansion. Set $t = d(1, \beta) = (t_i)_{i=1}^{\infty}$, keeping in mind that $t_i = t_{m+1}$ for $i \geq m+1$.

ALGORITHM. Applied to a sequence $s = s_1 s_2 \dots s_r 0^\omega$, $s_i \in \mathbb{Z}_+$, such that $x = \sum_i s_i \beta^{-i}$. Find the least k such that

$$s[k+1, \infty[\; >_{lex} d(1, \beta) = t.$$

If not found, the sequence s is β-admissible, and the process stops. Otherwise, let $l \geq k+1$ be such that

$$s[k+1, l-1] = t_1 \dots t_{l-k-1}, \quad s_l > t_{l-k}.$$

Put

$$s' = s \bigoplus_k [1(-t_1)(-t_2)\dots] \bigoplus_l [(-1)t_1 t_2 \dots],$$

or more explicitly,

$$s'_i = \begin{cases} s_i, & \text{for } i \leq k-1, \text{ and } i \geq l+m+1; \\ s_i + 1, & \text{for } i = k; \\ 0, & \text{for } k+1 \leq i \leq l-1; \\ s_i - t_{l-k} - 1, & \text{for } i = l; \\ s_i - t_{i-k} + t_{i-l}, & \text{for } l+1 \leq i \leq l+m. \end{cases}$$

Set $s := s'$, and go to the beginning of the Algorithm.

From the choice of k and l, and the monotonic property of the t_i it follows that s' is a non-negative sequence. This sequence is a β-representation of x because $t = d(1, \beta)$. Finally, since $t_i = t_{m+1}$ for $i \geq m+1$, we have

$$\sum_{i=1}^{\infty} s'_i = \sum_{i=1}^{\infty} s_i - (l-k)t_{m+1} \leq \sum_{i=1}^{\infty} s_i - 1.$$

Therefore s'_i terminates with infinitely many zeros, and the sum of all digits has decreased. So the process starting with $s^{(0)} = s$, $s^{(j+1)} = (s^{(j)})'$, stops at $s^{(N)}$, $N \leq \sum_i s_i$, which provides a finite β-expansion of x. □

As a corollary of Theorem 2 and Theorem 3 one has the following

PROPOSITION 1. *For every Pisot number β of degree* 2, $\mathbb{Z}_+[\beta^{-1}] \subset \operatorname{Fin}(\beta)$.

Proof of Proposition 1 comes from the following characterization of Pisot numbers of degree 2.

LEMMA 3. *The only Pisot numbers of degree* 2 *are the dominant roots of the following polynomials in* $\mathbb{Z}[X]$

$$X^2 - aX - b \text{ with } a \geq b \geq 1;$$

$$X^2 - aX - b \text{ with } a \geq 3 \quad \text{and } -a+2 \leq b \leq -1.$$

Proof. Let $P(X) = X^2 - aX - b$, with $b \neq 0$, and $a \neq 0$ (else its root cannot be a Perron number). Since one of the roots is positive, the discriminant $\Delta > 0$, and let $x_1 = (a+\sqrt{\Delta})/2$ and $x_2 = (a - \sqrt{\Delta})/2$.

First, we note that x_1 is a Perron number if and only if $a > 0$, thus if $a \geq 1$. Second, x_1 is a Pisot number if and only if $-2 + a < \Delta < a+2$.

We have $\sqrt{\Delta} < a+2$ if and only if $b < a$.

Next, $-2+a<\sqrt{\Delta}$ if $a\leq 2$. If $a\geq 3$, then $-2+a<\sqrt{\Delta}$ if and only if $2-a\leq b$. □

Proof of Proposition 1. The case $a\geq b\geq 1$ has been discussed in Theorem 2. The second case gives $d(1,\beta)=(a-1)(a+b-1)^{\omega}$, which has been treated in Theorem 3. □

Finally, we give a 'quantitative version' of the results above, which is needed for the application to substitution dynamical systems. Condition (F_2) may be restated as follows: finite β-expansions can be added producing a finite β-expansion. Condition (F_3) means that subtraction is also allowed within finite expansions. One may ask if there is a bound on the increase of the length of the expansion in these operations. The answer is 'yes' if the base is a Pisot number. Let $\mathrm{Fin}_N(\beta)$ be the set of numbers x such that in the β-expansion $x_{-k}=0$ for $k>N$.

PROPOSITION 2. *Let β be a Pisot number. There exists $L=L(\beta)$ having the following property. Let $x, y\in \mathrm{Fin}_N(\beta)$, $x>y$. If $x+y\in \mathrm{Fin}(\beta)$ then $x+y\in \mathrm{Fin}_{N+L}(\beta)$, and if $x-y\in \mathrm{Fin}(\beta)$ then $x-y\in \mathrm{Fin}_{N+L}(\beta)$.*

This property can be deduced from a result of the first author [**Fr1**] which says that, if β is a Pisot number, then addition (and subtraction) in base β is computable by a finite 2-automaton with bounded delay, that is a finite 2-automaton such that the distance between the two heads keeps bounded. We give here a direct proof, which is a simple modification of the Schmidt's argument in [**S**, p. 271].

Proof of Proposition 2. Let $z=x\pm y$. Dividing x and y by β^k, if necessary, we can assume that $z\in\,]0,1[$. The hypotheses of the Proposition imply that

$$z=\sum_{i=1}^{N} A_i\beta^{-i},\quad \text{where } A_i\in\mathbf{Z},\ |A_i|\leq 2[\beta].$$

Let $z=\varepsilon_1\varepsilon_2\ldots\varepsilon_M$ be the β-expansion. Consider

$$\rho^{(N)}=\left(z-\sum_{i=1}^{N}\varepsilon_i\beta^{-i}\right)\beta^N=T_\beta^N z.$$

Let m be the degree of β, and let $\beta_2, \beta_3, \ldots, \beta_m$ denote the conjugate roots of $\beta=\beta_1$. Clearly, $\rho^{(N)}\in\mathbf{Z}[\beta]\subset\mathbf{Z}[\beta^{-1}]$, so one can write

$$\rho^{(N)}=\sum_{i=1}^{m} r_i\beta^{-i},\quad r_i\in\mathbf{Z}.$$

We claim that $|r_i|\leq C(\beta)$, where the constant $C(\beta)$ does not depend on N. Let

$$\rho_j^{(N)}=\sum_{i=1}^{m} r_i\beta_j^{-i},\quad j=1,2,\ldots,m. \tag{4}$$

Note that β satisfies the equation:

$$\sum_{i=1}^{m} r_iX^{-i}=\sum_{i=1}^{N}(A_i-\varepsilon_i)X^{N-i}.$$

This implies that

$$\rho_j^{(N)}=\sum_{i=1}^{m} r_i\beta_j^{-i}=\sum_{i=1}^{N}(A_i-\varepsilon_i)\beta_j^{N-i},\quad 1\leq j\leq m.$$

Therefore, for $j \geq 2$,

$$|\rho_j^{(N)}| \leq [\max(A_i) + \max(\varepsilon_i)](1-\eta)^{-1} \leq 3[\beta](1-\eta)^{-1},$$

where $\eta = \max_{j\geq 2}|\beta_j| < 1$. On the other hand, $|\rho_1^{(N)}| = T_\beta^N z < 1$. Since the matrix $[\beta_j^{-k}]_{j,k\leq m}$ is nonsingular, it follows from (4) that $|r_i| \leq C(\beta)$, $i = 1, \ldots, m$. The claim is proved. Consider the finite set $G = \{t = \sum_{i=1}^m d_i\beta^{-i},\ d_i \in \mathbb{Z},\ |d_i| \leq C(\beta)\}$, and let $L = L(\beta)$ be the maximal length of β-expansions for numbers $t \in G \cap \mathrm{Fin}(\beta)$. This L is the desired one, since $\rho^{(N)} \in G \cap \mathrm{Fin}(\beta)$, and has the β-expansion $\rho^{(N)} = \varepsilon_{N+1}\varepsilon_{N+2} \ldots \varepsilon_M$. □

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