Finite beta-expansions

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Abstract. We characterize numbers having finite β-expansions where β belongs to a certain class of Pisot numbers: when the β-expansion of 1 is equal to \(a_1 a_2 \ldots a_m\) where \(a_1 \geq a_2 \geq \ldots \geq a_m \geq 1\) and when the β-expansion of 1 is equal to \(t_1 t_2 \ldots t_m (t_{m+1})^n\) where \(t_1 \geq t_2 \geq \ldots \geq t_m > t_{m+1} \geq 1\).

1. Introduction

Representations of real numbers with an arbitrary base \(\beta > 1\), called β-expansions, were introduced by Rényi [R]. They arise from the orbits of a piecewise-monotone transformation of the unit interval: \(T_\beta : x \mapsto \beta x \pmod{1}\). Such transformations were extensively studied in ergodic theory (see [P] and the bibliography in [B1]).

Properties of β-expansions are strongly related to symbolic dynamics [B1]. The closure of the set of infinite sequences, appearing as β-expansions, is called the β-shift. It is a symbolic dynamical system, that is, a closed shift-invariant subset of \(\mathcal{A}^\mathbb{N}\), where the alphabet \(\mathcal{A}\) is the set of all possible digits \(\{0, 1, \ldots, [\beta]\}\). A symbolic dynamical system is said to be of finite type if the set of its finite factors is defined by the interdiction of a finite set of words. It is said to be sofic if the set of its finite factors is recognized by a finite automaton. The β-shift has finite type if and only if \(T_\beta^k = 1\) for some \(k\), and it is sofic if and only if the orbit \(\{T_\beta^n 1\}\) is finite (see [B1]).

In this paper we study the set of numbers \(x \geq 0\) having finite β-expansions, which we denote by \(\text{Fin}(\beta)\). It is a subset of \(\text{Per}(\beta)\), the set of numbers having eventually periodic β-expansions. For a standard system of numeration, when \(\beta\) is an integer greater than one, one has \(\text{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{R}_+\) and \(\text{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+\).

Eventually periodic β-expansions were investigated by Bertrand [Be] and Schmidt [S]. An algebraic integer is called a Pisot number if all its Galois conjugates have modulus less than one, and a Salem number if all its conjugates are less or equal than one in modulus and at least one conjugate has modulus one. It is proved in [Be, S] that if \(\beta\) is a Pisot number, then \(\text{Per}(\beta) = \mathbb{Q}(\beta) \cap \mathbb{R}_+\). Conversely, Schmidt proved [S] that if \(\mathbb{Q} \cap [0, 1] = \text{Per}(\beta)\), then \(\beta\) is a Pisot, or a Salem number. It is still unknown whether all rationals have eventually periodic β-expansions if \(\beta\) is a
Salem number. Boyd [Bo] has shown that if $\beta$ is a Salem number of degree four, then the $\beta$-expansion of 1 is eventually periodic.

One of the authors proved in [Fr1] that addition (and multiplication by a fixed integer) in base $\beta$—where $\beta$ is a Pisot number—is a function computable by a finite automaton. In this paper we find two classes of Pisot numbers for which the sum of two finite $\beta$-expansions is again a finite $\beta$-expansion. Examples show that this is false for some Pisot numbers.

We first prove that if $\mathbb{Z}_\leq \subset \text{Fin}(\beta)$ then $\beta$ is a Pisot number or a Salem number, and if $\mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+ \subset \text{Fin}(\beta)$ then $\beta$ is a Pisot number (Lemma 1). It is also possible to show that $\mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+ = \mathbb{Z}_+[\beta^{-1}]$ if and only if $\beta$ is a Perron eigenvalue of a primitive companion matrix (Lemma 2).

The main results of the paper are the following two theorems which give sufficient (but certainly not necessary) conditions for the inclusion $\mathbb{Z}_+[\beta^{-1}] \cap \mathbb{R}_+ \subset \text{Fin}(\beta)$.

Let $\beta$ be the positive root of the polynomial $M(X) = X^m - a_1X^{m-1} - a_2X^{m-2} - \cdots - a_m$, $a_i \in \mathbb{Z}_+$ and $a_i \geq a_{i+1} \geq \cdots \geq a_m > 0$. Then $\beta$ is a Pisot number, the $\beta$-shift is a system of finite type, and $\text{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+$ (Theorem 2). The corresponding systems of numeration were considered in [Fra, PT].

Let $\beta > 1$ be a real number such that the $\beta$-expansion of 1 is equal to $t_1t_2 \cdots t_m(t_{m+1})^\alpha$, with $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} > 0$. Then $\beta$ is a Pisot number, and $\mathbb{Z}_+[\beta^{-1}] \subset \text{Fin}(\beta)$ (Theorem 3).

The numbers $\beta$ in Theorem 2 and in Theorem 3 have a common feature: in fact, $\beta$ belongs to one of these two classes if and only if the orbit of 1 under the $\beta$-transformation is non-increasing.

As a corollary we obtain the following result: for every Pisot number $\beta$ of degree 2, $\mathbb{Z}_+[\beta^{-1}] \subset \text{Fin}(\beta)$.

At the end we give a quantitative version of the results above (Proposition 2) for application to substitution dynamical systems. Let $\zeta$ be the substitution of the alphabet $\{1, 2, \ldots, m\}$:

$$
\zeta(1) = 1 \ldots 1 2, \quad \zeta(2) = 1 \ldots 1 3, \ldots, \zeta(m-1) = 1 \ldots 1 m, \quad \zeta(m) = 1 \ldots 1 1.
$$

This substitution generates a minimal, uniquely ergodic measure-preserving system (‘substitution dynamical system’, see [Q]). In [So2] one of the authors proved that if $\beta$ is a Pisot number of degree $m$ such that the $\beta$-expansion of 1 is equal to $k_1k_2 \cdots k_m$, and if $\mathbb{Z}[\beta^{-1}] \cap \mathbb{R}_+ \subset \text{Fin}(\beta)$, then the substitution dynamical system has purely discrete spectrum. The class of numbers in Theorem 2 fits this scheme.

The properties of periodic and finite expansions are related to the theory of tilings (see [T]). Theorem 2 was used by Praggastis [Pr] to construct Markov partitions for some toral automorphisms.

Some of the results presented here can be found in [Fr2] and in [So1].

2. Representation of numbers

Let $\beta > 1$ be a real number. A representation in base $\beta$ (or a $\beta$-representation) of a real number $x \geq 0$ is an infinite sequence $(x_k)_{k \in \mathbb{Z}}$, $x_k \geq 0$, such that

$$
x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots
$$

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for a certain integer \( k \geq 0 \). It is denoted by
\[
x = x_kx_{k-1}\ldots x_1x_0.\ldots x_{-1}x_{-2}\ldots
\]
A particular \( \beta \)-representation—called the \( \beta \)-expansion—can be computed by the 'greedy algorithm':

Denote by \([y]\) and \( \{y\}\) the integer part and the fractional part of a number \( y \). There exists \( k \in \mathbb{Z} \) such that \( \beta^k \leq x < \beta^{k+1} \). Let \( x_k = \lfloor x/\beta^k \rfloor \), and \( r_k = \{ x/\beta^k \} \). Then for \( k > i \geq -\infty \), put \( x_i = \lfloor r_{i+1} \rfloor \), and \( r_i = \{ r_{i+1} \} \). We get an expansion \( x = x_0\beta^k + x_{-1}\beta^{k-1} + \cdots \). If \( k < 0 \) \((x < 1)\), we put \( x_0 = x_{-1} = \cdots = x_{k+1} = 0 \). If an expansion ends in infinitely many zeros, it is said to be finite, and the ending zeros are omitted.

The digits \( x_i \) obtained by this algorithm are integers from the set \( \mathcal{A} = \{0, \ldots, \beta - 1\} \) if \( \beta \) is an integer, or the set \( \mathcal{A} = \{0, \ldots, \lfloor \beta \rfloor \} \) if \( \beta \) is not an integer. We will sometimes omit the splitting point between the integer part and the fractional part of the \( \beta \)-expansion; then the infinite sequence is just an element of \( \mathcal{A}^\mathbb{N} \).

For numbers \( x < 1 \), the expansion defined above coincides with the \( \beta \)-expansion of Rényi [R], which can be defined by means of the \( \beta \)-transformation of the unit interval
\[
T_\beta x = \beta x \mod 1, \quad x \in [0, 1].
\]
In fact, for \( x \in [0, 1[ \), we have \( x_k = \lfloor \beta T_\beta^{k-1} x \rfloor \). However, for \( x = 1 \) the two algorithms differ: our expansion is just \( 1 = 1 \), while the Rényi expansion is
\[
d(1, \beta) = \cdot t_1t_2\ldots, \quad t_k = \lfloor \beta T_\beta^{k-1} \rfloor,
\]
(the point is usually omitted).

Let \( D_\beta \) be the set of \( \beta \)-expansions of numbers of \([0, 1]\], and let \( d : [0, 1] \to D_\beta \cup \{ d(1, \beta) \} \) be the function mapping \( x \neq 1 \) onto its \( \beta \)-expansion, and 1 onto \( d(1, \beta) \). Clearly, if \( x = x_k \ldots x_0 \ldots x_{-1} \ldots \) is a \( \beta \)-expansion, then \( x/\beta^{k+1} = . x_k \ldots x_0 x_{-1} \ldots \) belongs to \( D_\beta \).

Recall some results concerning the set \( D_\beta \). The set \( \mathcal{A}^\mathbb{N} \) is endowed with the lexicographical order (notation \( <_{\text{lex}} \)), the product topology, and the (one-sided) shift \( \sigma \). The set \( D_\beta \) is shift-invariant. The \( \beta \)-shift \( S_\beta \) is the closure of \( D_\beta \), it is a subshift of \( \mathcal{A}^\mathbb{N} \) (see [Be] and [B1]). We have \( d \circ T_\beta = \sigma \circ d \) on \([0, 1]\]. Recall that the \( \beta \)-shift \( S_\beta \) is a system of finite type if and only if \( d(1, \beta) \) is finite [P] (such numbers were called simple \( \beta \)-numbers by Parry). The \( \beta \)-shift \( S_\beta \) is a sofic system if and only if \( d(1, \beta) \) is eventually periodic (Bertrand, see [B1]).

We recall the characterization of the set \( D_\beta \) [P]. By \( x^\omega \) will be denoted the sequence \( xxx \ldots \).

**Theorem 1.** Let \( \beta \) be a real number greater than one, and let \( d(1, \beta) = t_1t_2\ldots \). Let \( s \) be an infinite sequence of positive integers.

(i) If \( d(1, \beta) \) is infinite, the condition
\[
\forall p \geq 0, \quad \sigma^p(s) <_{\text{lex}} d(1, \beta)
\]
is necessary and sufficient for \( s \) to belong to \( D_\beta \).

(ii) If \( d(1, \beta) \) is finite, \( d(1, \beta) = t_1\ldots t_n t_n \), then \( s \in D_\beta \) if and only if
\[
\forall p \geq 0, \quad \sigma^p(s) <_{\text{lex}} d^n(1, \beta) = (t_1\ldots t_n (t_n-1))^\omega.
\]
3. Finite expansions

Let \( \text{Per}(\beta) \) be the set of all numbers \( x \geq 0 \) having eventually periodic \( \beta \)-expansions. It is clear that \( \text{Per}(\beta) \cap [0, 1[ \) is the set of points whose orbits under \( T_\beta \) are finite.

Let \( Q(\beta) \) denote the smallest field containing the field of rational numbers \( Q \) and \( \beta \). It is proved in \([\text{Be}, S]\) that if \( \beta \) is a Pisot number, then \( \text{Per}(\beta) = Q(\beta) \cap \mathbb{R}^+ \).

Conversely, Schmidt proved \([S]\) that if \( Q \cap [0, 1[ \subset \text{Per}(\beta) \), then \( \beta \) is a Pisot, or a Salem number.

Now, let \( \text{Fin}(\beta) \) stand for the set of \( x \geq 0 \) having finite \( \beta \)-expansions. Note that \( \text{Fin}(\beta) \cap [0, 1[ \) consists of those points whose orbits under \( T_\beta \) end up at zero.

For \( \lambda > 0 \), let \( \mathbb{Z}[\lambda] \) denote the ring of polynomials in \( \lambda \) with integral coefficients, \( \mathbb{Z}_+(\lambda) \) the cone of polynomials with non-negative coefficients, and set \( (\mathbb{Z}[\lambda])_+ = \mathbb{Z}[\lambda] \cap \mathbb{R}_+ \).

If \( \beta > 1 \) is an integer, then \( \text{Fin}(\beta) = (\mathbb{Z}[\beta^{-1}])_+ \). We will address the problems: for which \( \beta \) all integers have finite \( \beta \)-expansions? when \( \text{Fin}(\beta) = (\mathbb{Z}[\beta^{-1}])_+ \)? More specifically, consider the following conditions on \( \beta \):

\[
(F_1) \quad \mathbb{Z}_+ \subset \text{Fin}(\beta);
(F_2) \quad \mathbb{Z}_+[\beta^{-1}] \subset \text{Fin}(\beta);
(F_3) \quad (\mathbb{Z}[\beta^{-1}])_+ \subset \text{Fin}(\beta).
\]

Note that condition \((F_1)\) implies that \( \beta \) is an algebraic integer (consider the \( \beta \)-expansion of \( x = [\beta] + 1 \)). If \( \beta \) is an algebraic integer, then \( \text{Fin}(\beta) = \mathbb{Z}[\beta^{-1}] = \mathbb{Z}[\beta^{-1}]_+ \), so \((F_1)\) actually means that \( \text{Fin}(\beta) = (\mathbb{Z}[\beta^{-1}])_+ \).

**Lemma 1.** (a) If \( \mathbb{Z}_+ \subset \text{Fin}(\beta) \), then \( \beta \) is a Pisot number or a Salem number; (b) condition \((F_1)\) implies that \( \beta \) is a Pisot number.

**Proof.** (a) is proved similar to \([S, 2.4]\) and even easier since we deal with finite expansions. We present the proof for convenience of the reader.

Suppose that \( \beta \) has a Galois conjugate \( \gamma, |\gamma| > 1 \). Let \( \eta = \max(|\beta|^{-1}, |\gamma|^{-1}) \), and \( C = 2|\beta| \eta(1 - \eta)^{-1} \). It is easy to see that one can find \( m \in \mathbb{Z}_+ \) so that \( |\beta^m - \gamma^m| > C \). Then take \( x = [\beta^m] + 1 \in \mathbb{Z}_+ \). If \( x \) has a finite \( \beta \)-expansion, it must be of the following form:

\[
x = \beta^m + \varepsilon_1 \beta^{m-1} + \varepsilon_2 \beta^{-2} + \cdots + \varepsilon_k \beta^{-k}.
\]

Since \( x \in \mathbb{Z}_+ \), all Galois conjugates of \( \beta \) satisfy the same equation, and \( x = \gamma^m + \varepsilon_1 \gamma^{m-1} + \varepsilon_2 \gamma^{-2} + \cdots + \varepsilon_k \gamma^{-k} \). Subtracting these two representations of \( x \) and using that \( |\varepsilon_i| \leq |\beta| \), we come to a contradiction.

(b) It remains to exclude the case of Salem numbers. As mentioned above, \( \mathbb{Z}_+ \subset \text{Fin}(\beta) \) implies that \( \beta \) is an algebraic integer, and so \( \beta \in \mathbb{Z}[\beta^{-1}] \). Thus, \( \beta - [\beta] \in \text{Fin}(\beta) \), and so \( \beta \) is a root of a polynomial

\[
X^m - k_1 X^{m-1} - k_2 X^{m-2} - \cdots - k_m, \quad k_i \in \mathbb{Z}_+.
\]

It follows that \( \beta \) has no positive Galois conjugates. On the other hand, it is known that every Salem number \( \alpha \) is reciprocal (see \([S_a, p. 26]\)), and hence has a conjugate \( \alpha^{-1} > 0 \). Thus \( \beta \) is a Pisot number. \( \square \)
We do not know whether conditions \((F_1)\) and \((F_2)\) are equivalent. The relation between \((F_2)\) and \((F_3)\) can be understood with the help of the following lemma.

**Lemma 2.** Let \(\lambda > 0\). The following are equivalent:

(a) \((\mathbb{Z}[\lambda^{-1}])_+ = \mathbb{Z}_+[\lambda^{-1}]\);

(b) \(\lambda\) is a Perron eigenvalue of a primitive companion matrix.

It is possible that this result is known. By a recent result of Handelman [H], condition (b) is equivalent to \(\lambda\) being a Perron number with no positive conjugates.

**Proof of Lemma 2.** (a) \(\Rightarrow\) (b). If \(\lambda < 1\), then (a) is impossible since then \(\mathbb{Z}_+[\lambda^{-1}] \cap [0, 1] = \emptyset\), while \(\lambda^{-1} - [\lambda^{-1}] \in (\mathbb{Z}[\lambda^{-1}])_+\). Let \(\lambda \geq 1\). Then (a) implies that

\[
1 - \lambda^{-1} = c_0 \lambda^{-1} + c_2 \lambda^{-2} + \cdots + c_k \lambda^{-k}, \quad c_i \in \mathbb{Z}_+, \quad c_k > 0, \quad k \geq 1.
\]

Then \(\lambda\) is a Perron eigenvalue of the companion matrix \(A\) of the polynomial

\[
p(X) = X^k - (c_1 + 1)X^{k-1} - c_2 X^{k-2} - \cdots - c_k.
\]

One can check that \(A^k\) is strictly positive, so \(A\) is primitive.

(b) \(\Rightarrow\) (a). Suppose that \(\lambda\) is the Perron eigenvalue of a primitive companion matrix

\[
M = \begin{bmatrix}
0 & 0 & \cdots & 0 & a_k \\
1 & 0 & \cdots & 0 & a_{k-1} \\
& \ddots & \ddots & \ddots & \ddots \\
& & 0 & 1 & 0 \\
& & & 0 & 1
\end{bmatrix}
\]

Let \(x \in (\mathbb{Z}[\lambda^{-1}])_+\). Using the 'recurrence relation'

\[
\lambda^r = a_1 \lambda^{r-1} + a_2 \lambda^{r-2} + \cdots + a_r \lambda^{-k}, \quad r \in \mathbb{Z},
\]

one can express \(x\) as an integral linear combination of \(k\) consecutive powers of \(\lambda\). More precisely, there exist \(s \in \mathbb{Z}_+\) and \(B \in \mathbb{Z}^k\) such that

\[
x = \lambda^{-s} \Lambda \cdot B, \quad \Lambda = [\lambda^{-k+1}, \ldots, \lambda^{-1}, 1],
\]

(\(\cdot\) denotes the usual scalar product in \(\mathbb{R}^k\)). Since \(\Lambda\) is the left Perron eigenvector for \(M\), one has

\[
x = \lambda^{-s} \Lambda \cdot M^s B.
\]

By the Perron-Frobenius theory (see [Se]),

\[
\lim_{l \to \infty} \lambda^{-s} M^l B = \frac{\Lambda \cdot B}{\Lambda \cdot R} = \frac{x \lambda^l}{\Lambda \cdot R} > 0,
\]

where \(R\) is a strictly positive right eigenvector for \(M\). Hence \(M^l B = [d_i]_{i=1}^k > 0\) for \(l\) sufficiently large, and by virtue of (1),

\[
x = \sum_{i=1}^k d_i \lambda^{-s-k+i} \in \mathbb{Z}_+[\lambda^{-1}] .
\]

**Corollary.** The following are equivalent:

(i) \(\beta\) satisfies condition \((F_3)\);

(ii) \(\beta\) satisfies condition \((F_2)\) and \(\beta\) is a simple \(\beta\)-number \((d(1, \beta) \text{ is finite})\).
Proof. If $\beta$ satisfies condition (F$_1$) then $\beta - [\beta] \in \text{Fin}(\beta)$, $\sigma(d(1, \beta)) = d(\beta - [\beta])$ hence $d(1, \beta)$ is finite. Conversely, if $d(1, \beta) = a_1a_2 \ldots a_m$, then $\beta$ is the positive root of the polynomial $x^m - a_1x^{m-1} - \cdots - a_m$, which has a primitive companion matrix because $a_1 = [\beta] \equiv 1$. Thus by Lemma 2, $(\mathbb{Z}[\beta^{-1}])_* = \mathbb{Z}_+[\beta^{-1}])$, and conditions (F$_2$) and (F$_3$) are identical.

The following examples show that there are Pisot numbers such that $d(1, \beta)$ is finite but $\mathbb{Z}_+ \not\subset \text{Fin}(\beta)$.

**Example 1.** Let $\beta$ be the dominant root of the polynomial $X^3 - 3X^2 + 2X - 2$. Then $\beta$ is a Pisot number, $d(1, \beta) = 2102$, and the $\beta$-expansion of 6 is $6 = 20.210(00112)_\beta$.

**Example 2.** Let $\beta$ be the positive root of the polynomial $X^4 - 2X^3 - X - 1$. Then $\beta$ is a unitary Pisot number, $d(1, \beta) = 2011$, and the $\beta$-expansion of 3 is $3 = 10.111(00012)_\beta$.

4. Systems of finite type

**Theorem 2.** Let $\beta$ be the positive root of the polynomial $M(X) = X^n - a_1X^{n-1} - a_2X^{n-2} - \cdots - a_m$, $a_i \in \mathbb{Z}_+$, and $a_1 \geq a_2 \geq \cdots \geq a_m > 0$. Then $\beta$ is a Pisot number, $d(1, \beta) = a_1a_2 \cdots a_m$, and $\text{Fin}(\beta) = (\mathbb{Z}[\beta^{-1}])_*^\infty$.

The special case of the Theorem when $M(X) = X^2 - X - 1$, and $\beta = (1 + \sqrt{5})/2$ is the golden ratio, is known and seems to belong to folklore. Note also that the case of $M(X) = X^2 - nX - 1$ is implicitly contained in [S, 3.4].

**Proof.** Since the sequence $a_1a_2 \ldots a_m$ is lexicographically greater than its shifts, one has from [P] that $d(1, \beta) = a_1a_2 \ldots a_m$. From [Br, Th. 2] it follows that $\beta$ is a Pisot number with the minimal polynomial $M(X)$. Using Lemma 2 and dividing by $\beta^k$ if necessary, we see that one needs to prove only that $\mathbb{Z}_+[\beta^{-1}] \cap [0, 1[ \subset \text{Fin}(\beta)$. Let $x \in \mathbb{Z}_+[\beta^{-1}]$, $0 < x < 1$. Then $x$ has a $\beta$-representation which ends in infinitely many zeros. Of course, it does not have to be the $\beta$-expansion of $x$. Let us say that a sequence is $\beta$-admissible if it corresponds to a $\beta$-expansion of some number. We are going to apply an algorithm which modifies a finite $\beta$-representation of $x$, aiming at a $\beta$-admissible sequence.

First we formulate the algorithm, and then prove that the process stops after finitely many passes. The following notations will be used: if $w = w_1w_2 \ldots$ is a sequence, then $s \oplus w$ denotes the sequence $s, s_2, \ldots, s_{k-1}(s_k + w_1)(s_{k+1} + w_2) \ldots$. The factor $s, s_2, \ldots, s_{n+k}$ of $s$ will be denoted by $s[i,i+k]$.

It follows from Theorem 1 that a sequence $s = s_1s_2 \ldots s_0^w$ is $\beta$-admissible if and only if for all $k$ holds

$$s[k+1, k+m] < s_k a_1 \ldots a_m.$$

**Algorithm.** Applied to a sequence $s = s_1s_2 \ldots s_0^w$, $s_0 \in \mathbb{Z}_+$, such that $x = \sum_{i=0}^{\infty} s_i \beta^{-i} < 1$.

**Step 1.** If there is a factor $s[k+1, k+m] \geq a_1a_2 \ldots a_m$ (termwise), let

$$s' = s \oplus \frac{1}{\beta}(-a_1) \cdots (-a_m).$$

Repeat this as long as possible, then go to Step 2.
Step 2. Find the least $k$ such that
\[ s[k+1, k+m] >_{1,n} a_1, a_2 \ldots a_m. \] (2)
If there is no such $k$, the sequence $s$ is $\beta$-admissible, and the process stops (since Step 1 has been performed before, equality in (2) is impossible). Observe that $k \geq 1$, since $1 > x > s_i \beta^{-1} \Rightarrow s_i \leq [\beta] = a_1$.

It follows from (2) that there exists $l \geq k+1$ such that
\[ s[k+1, l-1] = a_1 a_2 \ldots a_{i-k-1}, \quad s_l > a_{i-k}. \]
Since we have already performed all possible operations in Step 1, $l \leq k + m - 1$. Let
\[ s' = s \oplus \left( \left[ 1(\ldots (-a_{l-1}) \ldots (-a_{l+m}) \right] \right) \oplus [(-1)a_1 \ldots a_m]. \]

More explicitly, we have
\[ s'_l = \begin{cases} 
  s_l, & \text{for } i \leq k - 1, \text{ and } i \geq l + m + 1; \\
  s_l + 1, & \text{for } i = k; \\
  0, & \text{for } k + 1 \leq i \leq l - 1; \\
  s_l - a_{l-k} - 1, & \text{for } i = l; \\
  s_l - a_{l-k} + a_{l-b} & \text{for } l + 1 \leq i \leq k + m \\
  s_l + a_{l-i} & \text{for } k + m + 1 \leq i \leq l + m.
\end{cases} \] (3)

Go to Step 1.

It is clear that in both steps of the Algorithm $s'$ is non-negative. The fact that $s'$ is a representation of $x$ in base $\beta$ follows from the equation $\beta' = a_1 \beta^{-1} + a_2 \beta^{-2} + \cdots + a_m \beta^{-m}$. Thus, it remains to check that the process stops after finitely many passes of the algorithm.

Observe that after each operation in Step 1, the sum of all digits decreases: $\Sigma_i s'_i = \Sigma_i s_i - (a_i + \cdots + a_m - 1)$. It follows that the operation in Step 1 can be applied only finitely many times.

Let us analyze what is going on in Step 2. First we note that the operation in Step 2 does not change the sum of the digits: $\Sigma_i s'_i = \Sigma_i s_i$. Let $l$ be the index in Step 2. The quantity
\[ \text{Adm}_x s = \sum_{i=1}^{l-1} s_i \beta^{-i}, \]
will be called the $\beta$-admissible part of $x$ with respect to $s$. Note that $l$ is the maximal index such that the sequence $s_1 s_2 \ldots s_{l-1} 0^m$ is $\beta$-admissible. We have to consider two cases.

Case 1. The sequence $s'_1 s'_2 \ldots s'_{l-1} 0^m$ is $\beta$-admissible.

\[ \text{Adm}_x s' \geq \sum_{i=1}^{l-1} s'_i \beta^{-i} = \text{Adm}_x s + \beta^{-k} - a_1 \beta^{-k-1} - \cdots - a_{l-1} \beta^{-l-1} \geq \text{Adm}_x s + \beta^{-l}. \]

Case 2. The sequence $s'_1 s'_2 \ldots s'_{l-1} 0^m$ is not $\beta$-admissible.
Then from (3) and the choice of $k$ and $l$ in Step 2 it follows that $s[k-m+1, k] = a_l \ldots a_{m-1} a_m$, and the operation in Step 1 will be applied in the next pass of the algorithm.

**Conclusion of the proof.** Let $s^{(0)} = s$, and let $s^{(i)}$ be the representations of $x$ obtained by repeated application of the Algorithm. If the process stops at $s^{(N)}$, then it is $\beta$-admissible, and we get a finite $\beta$-expansion of $x$ because $s^{(N)} = 0$ for $i$ sufficiently large. Suppose that the process goes on indefinitely. Since the sum of the digits remains unchanged in Step 2 and decreases in each operation of Step 1, the operation in Step 1 can be applied only finitely many times. Choose $K$ so that for $j > K$, only the operation in Step 2 was applied. Then for $j > K$ we always find ourselves in Case 1. Let $l_j$ be the index $l$ for the sequence $s^{(j)}$, 

$$l_j = \max \{ r : s^{(j)}_r s^{(j)}_{r+1} \ldots s^{(j)}_{l_j} = 0^m \}$$

is $\beta$-admissible.

We have for $j > K$,

$$x \in \text{Adm}_x, s^{(j+1)} \simeq \text{Adm}_x, s^{(j)} + \beta^{-1}.$$ 

Therefore, $l_j \to \infty$ as $j \to \infty$. Looking at (3) we see that $s^{(j+1)} = s^{(j)}$ for $i < l_j - 2m$, so there exists the limiting sequence $\tilde{s}$ such that $\tilde{s}_i = s^{(j)}_i$ for all $j$ sufficiently large. Since $\sum s^{(j)}_i \leq \sum s_i = S$, we conclude that

$$x = \lim_{j \to \infty} \sum_{i=1}^{l_j} s^{(j)}_i \beta^{-i} = \sum_{i=1}^{\infty} \tilde{s}_i \beta^{-i},$$

and also $\sum \tilde{s}_i \leq S$. But $\tilde{s}_i \in \mathbb{Z}_+$, so there exists $K_1$ such that $\tilde{s}_i = 0$ for $i > K_1$. Let $j_0$ be such that $s^{(j)}_i = \tilde{s}_i$ for $i \leq K_1$, $j \geq j_0$. Then we have

$$x = \sum_{i=1}^{K_1} \tilde{s}_i \beta^{-i} = \sum_{i=1}^{\infty} s^{(j)}_i \beta^{-i}, \quad j \geq j_0,$$

whence $s^{(j)}_i = \tilde{s}_i$, $j \geq j_0$, for all $i$. But this means that $s^{(j+1)} = s^{(j)}$, so no operation was applied. This is a contradiction. We have shown that the process stops, and the proof of Theorem 2 is now complete. 

\[ \square \]

5. Softic systems

**Theorem 3.** Let $\beta > 1$ be a real number such that $d(1, \beta) = t_1 t_2 \ldots t_m (t_{m+1})^m$, with $t_1 \geq t_2 \geq \cdots \geq t_m > t_{m+1} > 0$. Then $\beta$ is a Pisot number, and $\mathbb{Z}_+ [\beta^{-1}] \subset \text{Fin} (\beta)$.

**Proof.** From the equality $d(1, \beta) = t_1 t_2 \ldots t_m (t_{m+1})^m$, one deduces that $\beta$ is a real root of the polynomial

$$M(X) = X^{m+1} - (t_1 + 1) X^m + (t_1 - t_2) X^{m-1} + \cdots + (t_{m-1} - t_m) X + (t_m - t_{m+1}).$$

From a result of Perron (see [Br, II]) one has that $\beta$ is a Pisot number and that $M$ is the minimal polynomial for $\beta$.

Let $x \in \mathbb{Z}_+ [\beta^{-1}]$. As in the proof of Theorem 2, one can assume that $x < 1$. Then $x$ has a $\beta$-representation $x = \sum_{i=1}^{l_1} s_i \beta^{-i}$, $s_i \in \mathbb{Z}_+$. We shall use an algorithm which reduces a finite $\beta$-representation to a finite $\beta$-expansion. Set $t = d(1, \beta) = (t_i)_{i=1}^{\infty}$, keeping in mind that $t_i = t_{m+1}$ for $i \geq m + 1$. 

Algorithm. Applied to a sequence \( s = s_1s_2 \ldots s_n0^n, s_i \in \mathbb{Z}_+ \), such that \( x = \sum s_i \beta^{-i} \).

Find the least \( k \) such that
\[
s[k+1, \infty] > \exp \{d(1, \beta)k \} = l.
\]

If not found, the sequence \( s \) is \( \beta \)-admissible, and the process stops. Otherwise, let \( l \geq k+1 \) be such that
\[
s[k+1, l-1] = t_1 \ldots t_{l-k-1}, \quad s_l > t_{l-k}.
\]

Put
\[
s' = s \oplus \left[ (1-t_1)(-t_2) \ldots \right] \oplus \left[ (-1)t_1t_2 \ldots \right],
\]
or more explicitly,
\[
s'_i = \begin{cases} s_i, & \text{for } i \leq k-1, \text{ and } i \geq l+m+1; \\ s_i + 1, & \text{for } i = k; \\ 0, & \text{for } k+1 \leq i \leq l-1; \\ s_i - t_{l-k} - 1, & \text{for } i = l; \\ s_i - t_{l-k} + t_{l-1}, & \text{for } l+1 \leq i \leq l+m. \\ \end{cases}
\]

Set \( s := s' \), and go to the beginning of the Algorithm.

From the choice of \( k \) and \( l \), and the monotonic property of the \( t_i \), it follows that \( s' \) is a non-negative sequence. This sequence is a \( \beta \)-representation of \( x \) because \( t = d(1, \beta) \). Finally, since \( t_i = t_{i+1} \) for \( i \geq m+1 \), we have
\[
\sum_{i=1}^{\infty} s'_i = \sum_{i=1}^{\infty} s_i - (1-k) \sum_{i=1}^{\infty} s_i - 1.
\]

Therefore \( s'_i \) terminates with infinitely many zeros, and the sum of all digits has decreased. So the process starting with \( s^{(0)} = s, s^{(1)} = (s^{(0)})', \) stops at \( s^{(N)} \), \( N \leq \sum s_i \), which provides a finite \( \beta \)-expansion of \( x \).

As a corollary of Theorem 2 and Theorem 3 one has the following

**Proposition 1.** For every Pisot number \( \beta \) of degree 2, \( \mathbb{Z}_+ [\beta^{-1}] \subset \text{Fin} (\beta) \).

**Proof of Proposition 1** comes from the following characterization of Pisot numbers of degree 2.

**Lemma 3.** The only Pisot numbers of degree 2 are the dominant roots of the following polynomials in \( \mathbb{Z}[X] \)
\[
X^2 - aX - b \text{ with } a \geq b \geq 1; \\
X^2 - aX - b \text{ with } a \geq 3 \text{ and } -a + 2 \leq b \leq -1.
\]

**Proof.** Let \( P(X) = X^2 - aX - b \), with \( b \neq 0 \), and \( a \neq 0 \) (else its root cannot be a Perron number). Since one of the roots is positive, the discriminant \( \Delta > 0 \), and let \( x_1 = (a + \sqrt{\Delta})/2 \) and \( x_2 = (a - \sqrt{\Delta})/2 \).

First, we note that \( x_1 \) is a Perron number if and only if \( a > 0 \), thus if \( a \geq 1 \). Second, \( x_1 \) is a Pisot number if and only if \( -2 + a < \Delta < a + 2 \).

We have \( \sqrt{\Delta} < a + 2 \) if and only if \( b < a \).
Next, $-2 + a < \sqrt{\Delta}$ if $a \leq 2$. If $a \geq 3$, then $-2 + a < \sqrt{\Delta}$ if and only if $2 - a \leq b$. \hfill \Box

**Proof of Proposition 1.** The case $a \geq b \geq 1$ has been discussed in Theorem 2. The second case gives $d(1, \beta) = (a - 1)(a + b - 1)^\omega$, which has been treated in Theorem 3. \hfill \Box

Finally, we give a ‘quantitative version’ of the results above, which is needed for the application to substitution dynamical systems. Condition (F2) may be restated as follows: finite $\beta$-expansions can be added producing a finite $\beta$-expansion. Condition (F3) means that subtraction is also allowed within finite expansions. One may ask if there is a bound on the increase of the length of the expansion in these operations. The answer is ‘yes’ if the base is a Pisot number. Let $\text{Fin}_N(\beta)$ be the set of numbers $x$ such that in the $\beta$-expansion $x_{-k} = 0$ for $k > N$.

**Proposition 2.** Let $\beta$ be a Pisot number. There exists $L = L(\beta)$ having the following property. Let $x, y \in \text{Fin}_N(\beta)$, $x > y$. If $x + y \in \text{Fin}_N(\beta)$ then $x + y \in \text{Fin}_{N+L}(\beta)$, and if $x - y \in \text{Fin}_N(\beta)$ then $x - y \in \text{Fin}_{N+L}(\beta)$.

This property can be deduced from a result of the first author [Fr1] which says that, if $\beta$ is a Pisot number, then addition (and subtraction) in base $\beta$ is computable by a finite 2-automaton with bounded delay, that is a finite 2-automaton such that the distance between the two heads keeps bounded. We give here a direct proof, which is a simple modification of the Schmidt’s argument in [S, p. 271].

**Proof of Proposition 2.** Let $z = x \pm y$. Dividing $x$ and $y$ by $\beta^k$, if necessary, we can assume that $z \in [0, 1]$. The hypotheses of the Proposition imply that

$$z = \sum_{i=1}^{N} A_i \beta^{-i}, \text{ where } A_i \in \mathbb{Z}, |A_i| \leq 2[\beta].$$

Let $z = \varepsilon_1 \varepsilon_2 \ldots \varepsilon_M$ be the $\beta$-expansion. Consider

$$p^{(N)} = \left(z - \sum_{i=1}^{N} \varepsilon_i \beta^{-i}\right)^N = \left(\frac{1}{\beta^{(N)} z}\right)^N.$$

Let $m$ be the degree of $\beta$, and let $\beta_2, \beta_3, \ldots, \beta_m$ denote the conjugate roots of $\beta = \beta_1$. Clearly, $p^{(N)} \in \mathbb{Z}[\beta] \subset \mathbb{Z}[\beta^{-1}]$, so one can write

$$p^{(N)} = \sum_{i=1}^{m} r_i \beta^{-i}, \quad r_i \in \mathbb{Z}.$$

We claim that $|r_i| \leq C(\beta)$, where the constant $C(\beta)$ does not depend on $N$. Let

$$p_j^{(N)} = \sum_{i=1}^{m} r_i \beta_j^{-i}, \quad j = 1, 2, \ldots, m. \quad (4)$$

Note that $\beta$ satisfies the equation:

$$\sum_{i=1}^{m} r_i \beta^{-i} = \sum_{i=1}^{N} (A_i - \varepsilon_i) X^{N-i}.$$

This implies that

$$p_j^{(N)} = \sum_{i=1}^{m} r_i \beta_j^{-i} = \sum_{i=1}^{N} (A_i - \varepsilon_i) X_j^{N-i}, \quad 1 \leq j \leq m.$$
Therefore, for $j \geq 2$,
\[ \rho_j^{(N)} = \max \left( A_j, \max_i (\epsilon_j) \right) (1 - \eta)^{-1} = 3(1 - \eta)^{-1}, \]
where $\eta = \max_{i \geq 2} |\beta_i| < 1$. On the other hand, $|\rho_1^{(N)}| = T_1^{N} \leq 1$. Since the matrix $[\beta_j^{(N)}]_{j,k=1} = \mathbb{N}$ is nonsingular, it follows from (4) that $|r_i| \leq C(\beta)$, $i = 1, \ldots, m$. The claim is proved. Consider the finite set $G = \{ 1 = \sum_{d \in \mathbb{Z}} d \beta^{-d}, d \geq \ell, |d| \leq C(\beta) \}$, and let $L = L(\beta)$ be the maximal length of $\beta$-expansions for numbers $i \in G \cap \text{Fin}(\beta)$. This $L$ is the desired one, since $\rho^{(N)} = G \cap \text{Fin}(\beta)$, and has the $\beta$-expansion $\rho^{(N)} = \epsilon_{N+1} \epsilon_{N+2} \cdots \epsilon_M$.

\[ \Box \]

REFERENCES


