Univoque numbers and the Thue-Morse sequence

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Greedy expansions

Base $\lambda > 1$, $x \in [0, 1]$.
Greedy algorithm of Rényi:
$x_0 := \lfloor \lambda x \rfloor$
$r_0 := \lambda x - x_0$
$x_n := \lfloor \lambda r_{n-1} \rfloor$
$r_n := \lambda r_{n-1} - x_n$

$$x = \sum_{n \geqslant 0} x_n \lambda^{-(n+1)}$$

Digits $x_n \in A_\lambda = \{0, 1, \dots, \lceil \lambda \rceil - 1\}$

$$d_\lambda(x) = x_0 x_1 x_2 \cdots$$

is the greedy λ-expansion of x.
It is the greatest representation in the lexicographic order.

Univoque numbers

$\lambda > 1$ is univoque if there exists a unique sequence of integers $(a_n)_{n \geqslant 0}$, with $0 \leqslant a_n < \lambda$, such that

$$1 = \sum_{n \geqslant 0} a_n \lambda^{-(n+1)}$$

2 is univoque, as $1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = .111\cdots$

$\frac{1+\sqrt{5}}{2}$ is not univoque since $1 = .11 = .(10)^k 11 = .(10)^\infty = .01^\infty$

$$\Gamma := \{A \in \{0,1\}^{\mathbb{N}},\ \forall k \geqslant 0,\ \overline{A} \leqslant \sigma^k A \leqslant A\}$$

$$\Gamma_{strict} := \{A \in \{0,1\}^{\mathbb{N}},\ \forall k \geqslant 1,\ \overline{A} < \sigma^k A < A\}$$

σ is the shift on sequences
$\overline{a} := (1-a)$; $\overline{A} := (1-a_n)_{n\geqslant 0}$

$\lambda \in (1,2)$ is univoque iff $d_\lambda(1) \in \Gamma_{strict}$

Smallest univoque number

(Komornik and Loreti 1998) There exists a smallest univoque number $\kappa \approx 1.787231$ and $d_\kappa(1) = (t_n)_{n\geqslant 0}$, where $(t_n)_{n\geqslant 0} = 11010011\ldots$ is the shifted Thue-Morse sequence

Thue-Morse sequence: $0 \to 0\ 1;\ \ 1 \to 1\ 0$

The Komornik-Loreti constant κ is transcendental (Allouche and Cosnard 2000).

$d_\kappa(1)$ is the smallest element of Γ_{strict}, i.e. the smallest nonperiodic sequence of Γ.

The smallest element of Γ is the periodic sequence $(10)^\infty$ and it is a representation of 1 in base $\frac{1+\sqrt{5}}{2}$.

Generalization to bigger alphabets?

Admissible sequences

b a positive integer, if $t \in \{0, 1, \dots, b\}$, $\overline{t} = b - t$.
A sequence $A = (a_n)_{n \geqslant 0}$ on $\{0, 1, \dots, b\}$ is admissible if

$$\begin{array}{lrcl} \forall k \geqslant 0 \textit{ such that } a_k < b, & \sigma^{k+1} A & < & A, \\ \forall k \geqslant 0 \textit{ such that } a_k > 0, & \sigma^{k+1} A & > & \overline{A}. \end{array}$$

Theorem (Komornik and Loreti 2002)

There is a bijection from the set of univoque numbers in $(1, b+1)$ to the set of admissible sequences on $\{0, 1, \dots, b\}$:

$$\lambda \in (1, b+1) \mapsto (a_n)_{n \geqslant 0} \in \{0, 1, \dots, b\}^{\mathbb{N}}$$

such that

$$1 = \sum_{n \geqslant 0} a_n \lambda^{-(n+1)}$$

Two possible generalizations to greater alphabets:

- to look at the smallest (if any) admissible sequence on the alphabet $\{0, 1, \ldots, b\}$ (Komornik and Loreti 2002)
- to look at the smallest (if any) univoque number in $(b, b+1)$ (de Vries and Komornik 2007).

An old work of Allouche 1983 gives a general tool.

The generalized Γ and Γ_{strict} sets

(Allouche 1983)
b a positive integer
$\mathcal{A} = \{\alpha_0, \alpha_1, \dots, \alpha_b\}$ in increasing order
$\overline{\alpha_j} = \alpha_{b-j}$

$$\Gamma(\mathcal{A}) := \{A = (a_n)_{n \geqslant 0} \in \mathcal{A}^{\mathbb{N}},\ a_0 = \alpha_b,\ \forall k \geqslant 0,\ \overline{A} \leqslant \sigma^k A \leqslant A\}$$

$$\Gamma_{strict}(\mathcal{A}) := \{A = (a_n)_{n \geqslant 0} \in \mathcal{A}^{\mathbb{N}},\ a_0 = \alpha_b,\ \forall k \geqslant 1,\ \overline{A} < \sigma^k A < A\}.$$

A sequence belongs to $\Gamma_{strict}(\mathcal{A})$ if and only if it belongs to $\Gamma(\mathcal{A})$ and is nonperiodic.

Proposition

Let $A = (a_n)_{n \geqslant 0}$ be a sequence in $\{0, 1, \dots, b\}^{\mathbb{N}}$, such that $a_0 = t \in [0, b]$. Suppose that $A \neq b^{\infty}$. Then A is admissible if and only if $2t > b$ and $A \in \Gamma_{strict}(\{b - t, b - t + 1, \dots, t\})$.

Note that $\overline{j} = b - j$.

$\mathcal{A} = \{\alpha_0, \alpha_1, \ldots, \alpha_b\}$
$A = (a_n)_{n \geqslant 0}$ a periodic sequence of *smallest* period T, and such that $a_{T-1} = \alpha_j < \alpha_b$:

$$A = (a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_j)^\infty$$

Define

$$\Phi(A) := (a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_{j+1} \ \overline{a_0} \ \overline{a_1} \ \ldots \ \overline{a_{T-2}} \ \alpha_{b-j-1})^\infty.$$

Proposition

The smallest element of $\Gamma(\{b-t, b-t+1, \dots, t\})$ *(where* $2t > b$*) is the 2-periodic sequence* $P := (t\ (b-t))^\infty = (t\ (b-t)\ t\ (b-t)\ t\ \dots)$.

Theorem (Allouche 1983)

The smallest element of $\Gamma_{strict}(\{b-t, b-t+1, \dots, t\})$ *is the sequence* M *defined by*

$$M := \lim_{s\to\infty} \Phi^s(P),$$

that actually takes the (not necessarily distinct) values $b-t$, $b-t+1$, $t-1$, t. *Furthermore, this sequence* $M = (m_n)_{n\geqslant 0} = t \quad b-t+1 \quad b-t \quad t \quad b-t \quad t-1 \ \dots$ *can be recursively defined by*

$$\begin{aligned}
&\forall k \geqslant 0,\ \ m_{2^{2k}-1} = t, \\
&\forall k \geqslant 0,\ \ m_{2^{2k+1}-1} = b+1-t, \\
&\forall k \geqslant 0,\ \ \forall j \in [0, 2^{k+1}-2],\ \ m_{2^{k+1}+j} = \overline{m_j}.
\end{aligned}$$

“Universal” morphism

Morphism Θ on the alphabet $\{e_0, e_1, e_2, e_3\}$

$$e_3 \longrightarrow e_3 e_1, \;\; e_2 \longrightarrow e_3 e_0, \;\; e_1 \longrightarrow e_0 e_3, \;\; e_0 \longrightarrow e_0 e_2$$

Infinite fixed point beginning in e_3

$$\Theta^\infty(e_3) = \lim_{k\to\infty} \Theta^k(e_3) = e_3 \; e_1 \; e_0 \; e_3 \; e_0 \; e_2 \; e_3 \; e_1 \; e_0 \; e_2 \ldots .$$

Theorem

Let $(\varepsilon_n)_{n\geqslant 0}$ be the Thue-Morse sequence, defined by $\varepsilon_0 = 0$ and for all $k \geqslant 0$, $\varepsilon_{2k} = \varepsilon_k$ and $\varepsilon_{2k+1} = 1 - \varepsilon_k$. Then $M = (m_n)_{n\geqslant 0}$ the smallest element of $\Gamma_{strict}(\{b-t, b-t+1, \dots, t\})$ satisfies

$$\forall n \geqslant 0,\ \ m_n = \varepsilon_{n+1} - (2t-b-1)\varepsilon_n + t - 1.$$

- *if $2t \geqslant b+3$, then $M = \Theta^\infty(e_3)$ with $e_0 := b-t$, $e_1 := b-t+1$, $e_2 := t-1$, $e_3 := t$ ($2t \geqslant b+3$ implies that these 4 numbers are distinct);*
- *if $2t = b+2$ (thus $b-t+1 = t-1$), then M is the pointwise image of $\Theta^\infty(e_3)$ by the map g where $g(e_3) := t$, $g(e_2) = g(e_1) := t-1$, $g(e_0) := b-t$;*
- *if $2t = b+1$ (thus $b-t = t-1$ and $b-t+1 = t$), then M is the pointwise image of $\Theta^\infty(e_3)$ by the map h where $h(e_3) = h(e_1) := t$, $h(e_2) = h(e_0) := t-1$.*

Square-free sequences on three letters

Istrail sequence: 1 0 2 1 2 0 1 0 2 . . . fixed point of the (non-uniform) morphism

$$0 \to 12, \; 1 \to 102, \; 2 \to 0$$

is square-free (Istrail 1977).
The Istrail sequence is the image of $\Theta'^{\infty}(1)$ where

$$\Theta' : \; 0 \to 12, \; 1 \to 13, \; 2 \to 20, \; 3 \to 21$$

by the map $0 \to 0$, $1 \to 1$, $2 \to 2$, $3 \to 0$.
It cannot be the fixed point of a uniform morphism (Berstel 1979).

Θ' is an avatar of Θ: $3 \to 31$, $2 \to 30$, $1 \to 03$, $0 \to 02$, obtained by renaming letters as follows: $0 \to 2$, $1 \to 3$, $2 \to 0$, $3 \to 1$.

The sequence $(m_n)_{n\geqslant 0}$, in the case where $2t = b+2$, is the fixed point of the non-uniform morphism

$$(t-1) \to t\ (b-t),\ \ t \to t\ (t-1)\ (b-t),\ \ (b-t) \to (t-1)$$

i.e., is an avatar of Istrail's square-free sequence.
From Berstel this sequence on three letters cannot be the fixed point of a uniform morphism.

The square-free Braunholtz sequence (Braunholtz 1963) is exactly our sequence $(m_n)_{n\geqslant 0}$ when $t = b = 2$, i.e., the sequence
$2\ 1\ 0\ 2\ 0\ 1\ 2\ 1\ 0\ 1\ 2\ 0 \ldots$

Small admissible sequences with values in $\{0, 1, \ldots, b\}$

Let $(\varepsilon_n)_{n\geqslant 0}$ be the Thue-Morse sequence, defined by $\varepsilon_0 = 0$ and for all $k \geqslant 0$, $\varepsilon_{2k} = \varepsilon_k$ and $\varepsilon_{2k+1} = 1 - \varepsilon_k$.

Corollary (Komornik and Loreti 2002)

Let b be an integer $\geqslant 1$. The smallest admissible sequence with values in $\{0, 1, \ldots, b\}$ is the sequence

- $(z + \varepsilon_{n+1})_{n\geqslant 0}$ *if* $b = 2z + 1$
- $(z + \varepsilon_{n+1} - \varepsilon_n)_{n\geqslant 0}$ *if* $b = 2z$.

Small univoque numbers in $(b, b+1)$

$\lambda \in (b, b+1)$ and $d_\lambda(1) = (a_n)_{n\geqslant 0}$ is equivalent to $a_0 = b$.
So we study the admissible sequences with values in $\{0, 1, \ldots, b\}$ that begin in b, i.e., the set $\Gamma_{strict}(\{0, 1, \ldots, b\})$.

Corollary

There exists a smallest univoque number in $(b, b+1)$. It is the solution of the equation $1 = \sum_{n\geqslant 0} d_n \lambda^{-(n+1)}$, where the sequence $(d_n)_{n\geqslant 0}$ is given by $d_n = \varepsilon_{n+1} - (b-1)\varepsilon_n + b - 1$

$$(d_n)_{n\geqslant 0} = b\ 1\ 0\ b\ 0\ b-1\ b\ 1\ 0\ b-1\ b\ 0 \ldots \qquad (1)$$

The sequence (1) corresponding to the smallest univoque number in $(b, b+1)$ was obtained by de Vries and Komornik in 2007 by a different method.

Transcendence results

Theorem

$t \in [0, b]$ an integer with $2t \geqslant b + 1$,
$m_n = \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1$ for all $n \geqslant 0$.
Then the univoque number λ belonging to $(1, b + 1)$ defined by $1 = \sum_{n \geqslant 0} m_n \lambda^{-(n+1)}$ is transcendental.

Proof.

Define $r_n := (-1)^{\varepsilon_n} = 1 - 2\varepsilon_n$.
For $X < 1$ let $F(X) = \sum_{n \geqslant 0} r_n X^n$, then $F(X) = \prod_{k \geqslant 0}(1 - X^{2^k})$.

$$2m_n = 2\varepsilon_{n+1} - 2(2t - b - 1)\varepsilon_n + 2t - 2 = b - r_{n+1} + (2t - b - 1)r_n$$

implies

$$2X \sum_{n \geqslant 0} m_n X^n = ((2t - b - 1)X - 1)F(X) + 1 + \frac{bX}{1 - X}.$$

Taking $X = 1/\lambda$ where $1 = \sum_{n \geqslant 0} m_n \lambda^{-(n+1)}$, we get

$$2 = ((2t - b - 1)(1/\lambda) - 1)F(1/\lambda) + 1 + \frac{b}{\lambda - 1}.$$

If λ were algebraic, then $F(1/\lambda)$ would be algebraic. But, since $1/\lambda$ would be an algebraic number in $(0, 1)$, the quantity $F(1/\lambda)$ would be transcendental from a result of Mahler, a contradiction. $\square$

In particular the univoque number corresponding to the smallest admissible sequence with values in $\{0, 1, \ldots, b\}$ is transcendental (Komornik and Loreti 2002).

Also the smallest univoque number in $(b, b+1)$ is transcendental.

Univoque Pisot numbers in $(b, b+1)$

A Pisot number is an algebraic integer > 1 with all its Galois conjugates < 1 in modulus.
There exists a smallest univoque Pisot number ≈ 1.8800, of degree 14 (Allouche, Frougny, Hare 2007).

The number corresponding to the smallest element of $\Gamma(\{b-t, b-t+1, \ldots, t\})$ (where $2t > b$) is the larger root of the polynomial $X^2 - tX - (b-t+1)$, hence a quadratic Pisot number β.
If $t = b$, then $d_\beta(1) = b1$, and $\beta = \frac{b+\sqrt{b^2+4}}{2}$ is the smallest element of $\Gamma \cap (b, b+1)$.

For any $b \geqslant 2$, the real number β such $d_\beta(1) = b1^\infty$ is a univoque Pisot number in $(b, b+1)$.

It is a limit point from above of univoque Pisot numbers with expansion $b1^n2^\infty$.

$b+1$ is a limit point from below of univoque Pisot numbers with expansion $b^n(b-1)^\infty$.

Open question Smallest univoque Pisot number in $(b, b+1)$?