Univoque numbers and the Thue-Morse sequence

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Joint work with
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Greedy expansions

Base $\lambda > 1$, $x \in [0, 1]$.  

Greedy algorithm of Rényi:

$x_0 := \lfloor \lambda x \rfloor$

$r_0 := \lambda x - x_0$

$x_n := \lfloor \lambda r_{n-1} \rfloor$

$r_n := \lambda r_{n-1} - x_n$

$$x = \sum_{n \geq 0} x_n \lambda^{-(n+1)}$$

Digits $x_n \in A_\lambda = \{0, 1, \ldots, \lceil \lambda \rceil - 1\}$

$d_\lambda(x) = x_0 x_1 x_2 \cdots$

is the greedy $\lambda$-expansion of $x$.

It is the greatest representation in the lexicographic order.
Univoque numbers

\( \lambda > 1 \) is univoque if there exists a unique sequence of integers \((a_n)_{n \geq 0}\), with \(0 \leq a_n < \lambda\), such that

\[
1 = \sum_{n \geq 0} a_n \lambda^{-(n+1)}
\]

2 is univoque, as \(1 = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots = .111\ldots\)

\(\frac{1+\sqrt{5}}{2}\) is not univoque since \(1 = .11 = .(10)^k11 = .(10)^\infty = .01\infty\)
Γ := \{A ∈ \{0, 1\}^\mathbb{N}, \ ∀k \geq 0, \ \overline{A} \leq \sigma^k A \leq A\}

Γ_{strict} := \{A ∈ \{0, 1\}^\mathbb{N}, \ ∀k \geq 1, \ \overline{A} < \sigma^k A < A\}

\sigma \text{ is the shift on sequences}
\overline{a} := (1 - a); \ \overline{A} := (1 - a_n)_{n \geq 0}

\lambda ∈ (1, 2) \text{ is univoque iff } d_\lambda(1) ∈ Γ_{strict}
Smallest univoque number

(Komornik and Loreti 1998) There exists a smallest univoque number $\kappa \approx 1.787231$ and $d_\kappa(1) = (t_n)_{n \geq 0}$, where $(t_n)_{n \geq 0} = 11010011 \ldots$ is the shifted Thue-Morse sequence.

Thue-Morse sequence: $0 \rightarrow 01; \ 1 \rightarrow 10$

The Komornik-Loreti constant $\kappa$ is transcendental (Allouche and Cosnard 2000).

$d_\kappa(1)$ is the smallest element of $\Gamma_{\text{strict}}$, i.e. the smallest nonperiodic sequence of $\Gamma$.

The smallest element of $\Gamma$ is the periodic sequence $(10)^\infty$ and it is a representation of 1 in base $\frac{1+\sqrt{5}}{2}$.

Generalization to bigger alphabets?
Admissible sequences

$b$ a positive integer, if $t \in \{0, 1, \ldots, b\}$, $\bar{t} = b - t$.
A sequence $A = (a_n)_{n \geq 0}$ on $\{0, 1, \ldots, b\}$ is admissible if

\[
\forall k \geq 0 \text{ such that } a_k < b, \quad \sigma^{k+1}A < A,
\]
\[
\forall k \geq 0 \text{ such that } a_k > 0, \quad \sigma^{k+1}A > \bar{A}.
\]

Theorem (Komornik and Loreti 2002)

There is a bijection from the set of univoque numbers in $(1, b + 1)$
to the set of admissible sequences on $\{0, 1, \ldots, b\}$:

\[
\lambda \in (1, b + 1) \mapsto (a_n)_{n \geq 0} \in \{0, 1, \ldots, b\}^\mathbb{N}
\]

such that

\[
1 = \sum_{n \geq 0} a_n \lambda^{-(n+1)}
\]
Two possible generalizations to greater alphabets:

- to look at the smallest (if any) *admissible sequence* on the alphabet \(\{0, 1, \ldots, b\}\) (Komornik and Loreti 2002)

- to look at the smallest (if any) *univoque number* in \((b, b + 1)\) (de Vries and Komornik 2007).

An old work of Allouche 1983 gives a general tool.
The generalized $\Gamma$ and $\Gamma_{strict}$ sets

(Allouche 1983)

$b$ a positive integer

$A = \{\alpha_0, \alpha_1, \ldots, \alpha_b\}$ in increasing order

$\alpha_j = \alpha_{b-j}$

$$\Gamma(A) := \{A = (a_n)_{n \geq 0} \in A^\mathbb{N}, \ a_0 = \alpha_b, \ \forall k \geq 0, \ \overline{A} \leq \sigma^k A \leq A\}$$

$$\Gamma_{strict}(A) := \{A = (a_n)_{n \geq 0} \in A^\mathbb{N}, \ a_0 = \alpha_b, \ \forall k \geq 1, \ \overline{A} < \sigma^k A < A\}.$$ 

A sequence belongs to $\Gamma_{strict}(A)$ if and only if it belongs to $\Gamma(A)$ and is nonperiodic.

**Proposition**

Let $A = (a_n)_{n \geq 0}$ be a sequence in $\{0, 1, \ldots, b\}^\mathbb{N}$, such that $a_0 = t \in [0, b]$. Suppose that $A \neq b^\infty$. Then $A$ is admissible if and only if $2t > b$ and $A \in \Gamma_{strict}(\{b - t, b - t + 1, \ldots, t\})$.

Note that $\overline{j} = b - j$. 
\( A = \{\alpha_0, \alpha_1, \ldots, \alpha_b\} \)

\( A = (a_n)_{n \geq 0} \) a periodic sequence of *smallest* period \( T \), and such that \( a_{T-1} = \alpha_j < \alpha_b \):

\[
A = (a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_j)^\infty
\]

Define

\[
\Phi(A) := (a_0 \ a_1 \ \ldots \ a_{T-2} \ \alpha_{j+1} \ \overline{a_0} \ \overline{a_1} \ \ldots \ \overline{a_{T-2}} \ \alpha_{b-j-1})^\infty.
\]
Proposition

The smallest element of $\Gamma(\{b - t, b - t + 1, \ldots, t\})$ (where $2t > b$) is the 2-periodic sequence $P := (t (b - t))^\infty = (t (b - t) t (b - t) t \ldots)$.

Theorem (Allouche 1983)

The smallest element of $\Gamma_{\text{strict}}(\{b - t, b - t + 1, \ldots, t\})$ is the sequence $M$ defined by

$$M := \lim_{s \to \infty} \Phi^s(P),$$

that actually takes the (not necessarily distinct) values $b - t, b - t + 1, t - 1, t$. Furthermore, this sequence $M = (m_n)_{n \geq 0} = t \ b - t + 1 \ b - t \ t \ b - t \ t - 1 \ldots$ can be recursively defined by

$$\forall k \geq 0, \ m_{2^{2k-1}} = t,$$
$$\forall k \geq 0, \ m_{2^{2k+1}-1} = b + 1 - t,$$
$$\forall k \geq 0, \ \forall j \in [0, 2^{k+1} - 2], \ m_{2^{k+1} + j} = \overline{m_j}. $$
“Universal” morphism

Morphism $\Theta$ on the alphabet \{e_0, e_1, e_2, e_3\}

$$
e_3 \rightarrow e_3 e_1, \quad e_2 \rightarrow e_3 e_0, \quad e_1 \rightarrow e_0 e_3, \quad e_0 \rightarrow e_0 e_2$$

Infinite fixed point beginning in $e_3$

$$\Theta^\infty(e_3) = \lim_{k \to \infty} \Theta^k(e_3) = e_3 \ e_1 \ e_0 \ e_3 \ e_0 \ e_2 \ e_3 \ e_1 \ e_0 \ e_2 \ldots.$$
Theorem
Let \((\varepsilon_n)_{n \geq 0}\) be the Thue-Morse sequence, defined by \(\varepsilon_0 = 0\) and for all \(k \geq 0\), \(\varepsilon_{2k} = \varepsilon_k\) and \(\varepsilon_{2k+1} = 1 - \varepsilon_k\). Then \(M = (m_n)_{n \geq 0}\) the smallest element of \(\Gamma_{\text{strict}}(\{b - t, b - t + 1, \ldots, t\})\) satisfies

\[
\forall n \geq 0, \ m_n = \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1.
\]

- if \(2t \geq b + 3\), then \(M = \Theta^\infty(e_3)\) with \(e_0 := b - t\), \(e_1 := b - t + 1\), \(e_2 := t - 1\), \(e_3 := t\) (\(2t \geq b + 3\) implies that these 4 numbers are distinct);

- if \(2t = b + 2\) (thus \(b - t + 1 = t - 1\)), then \(M\) is the pointwise image of \(\Theta^\infty(e_3)\) by the map \(g\) where \(g(e_3) := t\), \(g(e_2) = g(e_1) := t - 1\), \(g(e_0) := b - t\);

- if \(2t = b + 1\) (thus \(b - t = t - 1\) and \(b - t + 1 = t\)), then \(M\) is the pointwise image of \(\Theta^\infty(e_3)\) by the map \(h\) where \(h(e_3) = h(e_1) := t\), \(h(e_2) = h(e_0) := t - 1\).
Square-free sequences on three letters

Istrail sequence: 1 0 2 1 2 0 1 0 2 ... fixed point of the (non-uniform) morphism

\[0 \rightarrow 12, \ 1 \rightarrow 102, \ 2 \rightarrow 0\]

is square-free (Istrail 1977).
The Istrail sequence is the image of \(\Theta' \infty(1)\) where

\[\Theta': 0 \rightarrow 12, \ 1 \rightarrow 13, \ 2 \rightarrow 20, \ 3 \rightarrow 21\]

by the map \(0 \rightarrow 0, \ 1 \rightarrow 1, \ 2 \rightarrow 2, \ 3 \rightarrow 0\).
It cannot be the fixed point of a uniform morphism (Berstel 1979).

\(\Theta'\) is an avatar of \(\Theta: 3 \rightarrow 31, \ 2 \rightarrow 30, \ 1 \rightarrow 03, \ 0 \rightarrow 02\), obtained by renaming letters as follows: \(0 \rightarrow 2, \ 1 \rightarrow 3, \ 2 \rightarrow 0, \ 3 \rightarrow 1\).
The sequence \((m_n)_{n \geq 0}\), in the case where \(2t = b + 2\), is the fixed point of the non-uniform morphism

\[(t - 1) \rightarrow t \ (b - t), \ t \rightarrow t \ (t - 1) \ (b - t), \ (b - t) \rightarrow (t - 1)\]

i.e., is an avatar of Istrail’s square-free sequence. From Berstel this sequence on three letters cannot be the fixed point of a uniform morphism.

The square-free **Braunholtz sequence** (Braunholtz 1963) is exactly our sequence \((m_n)_{n \geq 0}\) when \(t = b = 2\), i.e., the sequence

\[2\ 1\ 0\ 2\ 0\ 1\ 2\ 1\ 0\ 1\ 2\ 0\ldots\]
Let \((\varepsilon_n)_{n \geq 0}\) be the Thue-Morse sequence, defined by \(\varepsilon_0 = 0\) and for all \(k \geq 0\), \(\varepsilon_{2k} = \varepsilon_k\) and \(\varepsilon_{2k+1} = 1 - \varepsilon_k\).

**Corollary (Komornik and Loreti 2002)**

Let \(b\) be an integer \(\geq 1\). The smallest admissible sequence with values in \(\{0, 1, \ldots, b\}\) is the sequence

- \((z + \varepsilon_{n+1})_{n \geq 0}\) if \(b = 2z + 1\)
- \((z + \varepsilon_{n+1} - \varepsilon_n)_{n \geq 0}\) if \(b = 2z\).
Small univoque numbers in \((b, b+1)\)

\(\lambda \in (b, b+1)\) and \(d_\lambda(1) = (a_n)_{n \geq 0}\) is equivalent to \(a_0 = b\).
So we study the admissible sequences with values in \(\{0, 1, \ldots, b\}\) that begin in \(b\), i.e., the set \(\Gamma_{\text{strict}}(\{0, 1, \ldots, b\})\).

**Corollary**

*There exists a smallest univoque number in \((b, b+1)\). It is the solution of the equation \(1 = \sum_{n \geq 0} d_n \lambda^{-(n+1)}\), where the sequence \((d_n)_{n \geq 0}\) is given by \(d_n = \varepsilon_{n+1} - (b-1)\varepsilon_n + b - 1\)

\[
(d_n)_{n \geq 0} = b \ 1 \ 0 \ b \ 0 \ b - 1 \ b \ 1 \ 0 \ b - 1 \ b \ 0 \ldots
\] (1)

The sequence (1) corresponding to the smallest univoque number in \((b, b+1)\) was obtained by de Vries and Komornik in 2007 by a different method.
Transcendence results

Theorem

$t \in [0, b]$ an integer with $2t \geq b + 1$,

$m_n = \varepsilon_{n+1} - (2t - b - 1)\varepsilon_n + t - 1$ for all $n \geq 0$.

Then the univoque number $\lambda$ belonging to $(1, b + 1)$ defined by

$1 = \sum_{n\geq0} m_n \lambda^{-(n+1)}$ is transcendental.
Proof.
Define \( r_n := (-1)^{\varepsilon_n} = 1 - 2\varepsilon_n \).
For \( X < 1 \) let \( F(X) = \sum_{n \geq 0} r_n X^n \), then \( F(X) = \prod_{k \geq 0} (1 - X^{2^k}) \).

\[
2m_n = 2\varepsilon_{n+1} - 2(2t - b - 1)\varepsilon_n + 2t - 2 = b - r_{n+1} + (2t - b - 1)r_n
\]
implies

\[
2X \sum_{n \geq 0} m_n X^n = ((2t - b - 1)X - 1)F(X) + 1 + \frac{bX}{1-X}.
\]

Taking \( X = 1/\lambda \) where \( 1 = \sum_{n \geq 0} m_n \lambda^{-(n+1)} \), we get

\[
2 = ((2t - b - 1)(1/\lambda) - 1)F(1/\lambda) + 1 + \frac{b}{\lambda - 1}.
\]

If \( \lambda \) were algebraic, then \( F(1/\lambda) \) would be algebraic. But, since \( 1/\lambda \) would be an algebraic number in \((0, 1)\), the quantity \( F(1/\lambda) \) would be transcendental from a result of Mahler, a contradiction. \( \square \)
In particular the univoque number corresponding to the smallest admissible sequence with values in \( \{0, 1, \ldots, b\} \) is transcendental (Komornik and Loreti 2002).

Also the smallest univoque number in \((b, b + 1)\) is transcendental.
Univoque Pisot numbers in \((b, b + 1)\)

A Pisot number is an algebraic integer \(> 1\) with all its Galois conjugates \(< 1\) in modulus.
There exists a smallest univoque Pisot number \(\approx 1.8800\), of degree 14 (Allouche, Frougny, Hare 2007).

The number corresponding to the smallest element of \(\Gamma(\{b - t, b - t + 1, \ldots, t\})\) (where \(2t > b\)) is the larger root of the polynomial \(X^2 - tX - (b - t + 1)\), hence a quadratic Pisot number \(\beta\).
If \(t = b\), then \(d_\beta(1) = b1\), and \(\beta = \frac{b + \sqrt{b^2 + 4}}{2}\) is the smallest element of \(\Gamma \cap (b, b + 1)\).
For any $b \geq 2$, the real number $\beta$ such $d_\beta(1) = b1^\infty$ is a univoque Pisot number in $(b, b+1)$.

It is a limit point from above of univoque Pisot numbers with expansion $b1^n2^\infty$.

$b+1$ is a limit point from below of univoque Pisot numbers with expansion $b^n(b-1)^\infty$.

**Open question** Smallest univoque Pisot number in $(b, b+1)$?