Automata and numeration systems

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Symbolic dynamical systems

A a finite alphabet. A symbolic dynamical system (or subshift) is a closed shift invariant subset of $A^{\mathbb{N}}$.

A subshift S of $A^{\mathbb{N}}$ is of finite type if it is defined by the interdiction of a finite set of factors.

A subshift S of $A^{\mathbb{N}}$ is sofic if $L(S) \subseteq A^*$, the language of S, is rational, or, equivalently if S is recognised by a finite Büchi automaton.

A subshift S of $A^{\mathbb{N}}$ is coded if there exists a prefix code $Y \subset A^*$ such that $L(S) = F(Y^*)$.

Symbolic dynamical systems and the lexicographic order

A is a totally ordered alphabet. $u = u_1 u_2 \cdots, v = v_1 v_2 \cdots$ in $A^{\mathbb{N}}$,
$u <_{lex} v$ if $u_1 \cdots u_{k-1} = v_1 \cdots v_{k-1}$ and $u_k < v_k$.

v in $A^{\mathbb{N}}$, $v_{[n]} = v_1 v_2 \cdots v_n$. $v_{[0]} = \varepsilon$.

Shift: $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$.

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Shift: $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$.

$$S_v = \{u \in A^{\mathbb{N}} \mid \forall k \geqslant 0,\ \sigma^k(u) \leqslant_{lex} v\},$$
$$D_v = \{u \in A^{\mathbb{N}} \mid \forall k \geqslant 0,\ \sigma^k(u) <_{lex} v\},$$
$$Y_v = \{v_{[n]} a \in A^* \mid \forall n \geqslant 0, \forall a \in A,\ a <_{lex} v_{n+1}\}.$$

A word $v = v_1 v_2 \cdots$ in $A^{\mathbb{N}}$ is said to be a lexicographically shift maximal word (lsmax-word for short) if for every $k \geqslant 0$, $\sigma^k(v) \leqslant_{lex} v$.

Proposition

If v in $A^{\mathbb{N}}$ is an lsmax-word, then S_v is a subshift coded by Y_v.

Let $\mathcal{S}_v$ be the (infinite) automaton:

- states are the $v_{[n]}$ for all n in $\mathbb{N}$
- transitions are $v_{[n]} \xrightarrow{v_{n+1}} v_{[n+1]}$ and $v_{[n]} \xrightarrow{a} v_{[0]}$ for every $a < v_{n+1}$.

All states are final and $v_{[0]}$ is initial.

$\mathcal{S}_v$ recognises $\mathrm{Pref}(\mathrm{Y}_\mathrm{v}^*)$, which is equal to $F(Y_v^*)$. As a Büchi automaton, $\mathcal{S}_v$ recognises S_v.

Let $\mathcal{D}_v$ be the automaton obtained from $\mathcal{S}_v$ by taking $v_{[0]}$ as unique final state. As a Büchi automaton, $\mathcal{D}_v$ recognises D_v.

Proposition

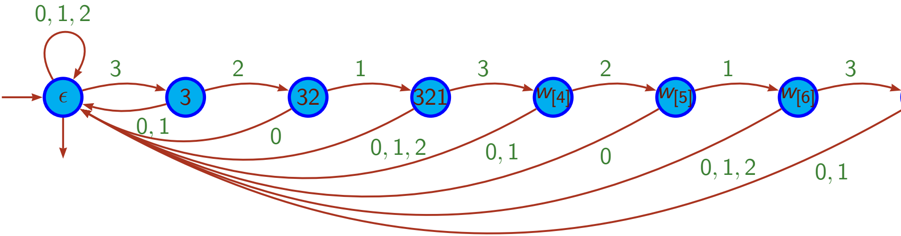
Let v be an lsmax-word in $A^{\mathbb{N}}$.

1. *The following conditions are equivalent*
   - *the subshift S_v is sofic*
   - *the set D_v is recognised by a finite Büchi automaton*
   - *v is eventually periodic.*
2. *The subshift S_v is of finite type if, and only if, v is purely periodic.*

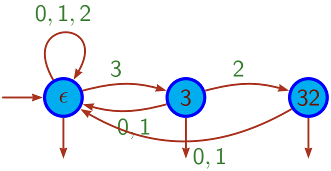
Similar results hold true for a lexicographically shift minimal word and the subshift defined accordingly.

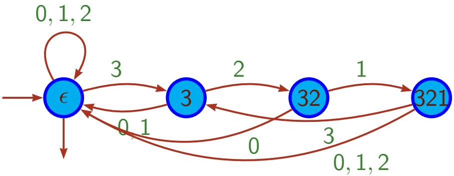
Example: $w = (321)^\omega$.

Infinite automaton for D_w



Finite automata for S_w and D_w





Symbolic dynamical systems and the alternate order

$u = u_1 u_2 \cdots, v = v_1 v_2 \cdots$ in $A^{\mathbb{N}}$, $u \prec_{alt} v$ if
$u_1 \cdots u_{k-1} = v_1 \cdots v_{k-1}$ and $(-1)^k (u_k - v_k) < 0$.

Symbolic dynamical systems and the alternate order

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$u_1 \cdots u_{k-1} = v_1 \cdots v_{k-1}$ and $(-1)^k (u_k - v_k) < 0$.
A word $v = v_1 v_2 \cdots$ in $A^{\mathbb{N}}$ is said to be an alternately shift maximal word (asmax-word for short)
if $v_1 = \min A$ and for every $k \geqslant 0$, $\sigma^k(v) \preceq_{alt} v$.

Symbolic dynamical systems and the alternate order

$u = u_1 u_2 \cdots, v = v_1 v_2 \cdots$ in $A^{\mathbb{N}}$, $u \prec_{alt} v$ if
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A word $v = v_1 v_2 \cdots$ in $A^{\mathbb{N}}$ is said to be an alternately shift maximal word (asmax-word for short)
if $v_1 = \min A$ and for every $k \geqslant 0$, $\sigma^k(v) \preceq_{alt} v$.
$S_v^{(a)} = \{u \in A^{\mathbb{N}} \mid \forall k \geqslant 0,\ \sigma^k(u) \preceq_{alt} v\}$,
$D_v^{(a)} = \{u \in A^{\mathbb{N}} \mid \forall k \geqslant 0,\ \sigma^k(u) \prec_{alt} v\}$.

Proposition

Let v be an asmax-word in $A^{\mathbb{N}}$.

1. *The following conditions are equivalent*
   - *the subshift $S_v^{(a)}$ is sofic*
   - *the set $D_v^{(a)}$ is recognised by a finite Büchi automaton*
   - *v is eventually periodic.*
2. *The subshift $S_v^{(a)}$ is of finite type if, and only if, v is purely periodic.*

Similarly for an alternately shift minimal word.

Representation in real base α, $|\alpha| > 1$

Definition (Hejda, Masáková and Pelantová 2012)

Let $\alpha \in \mathbb{R}$, $|\alpha| > 1$, finite alphabet $A \subset \mathbb{R}$ and J bounded interval containing 0. Let $D : J \to A$ such that $T(x) = \alpha x - D(x)$ maps J to J. The α-representation is a mapping $d_{\alpha,J,D} : J \to A^{\mathbb{N}}$ s.t.

$$d_{\alpha,J,D}(x) = x_1 x_2 \cdots \quad \text{with } x_j = D(T^{j-1}(x)).$$

$$x = \sum_{j \geqslant 1} x_j \alpha^{-j}$$

Proposition

x and y in J, $d_{\alpha,J,D}(x) = x_1 x_2 \cdots$ and $d_{\alpha,J,D}(y) = y_1 y_2 \cdots$.

- *If $\alpha > 1$ and D is non-decreasing then*

$$x < y \iff x_1 x_2 \cdots <_{lex} y_1 y_2 \cdots .$$

- *If $\alpha < -1$ and D is non-increasing then*

$$x < y \iff x_1 x_2 \cdots \prec_{alt} y_1 y_2 \cdots .$$

β-expansions, $\beta > 1$

Rényi 1957

$J = [0,1)$, $A = \{0, 1, \dots, \lceil\beta\rceil - 1\}$

$$D : [0,1) \to A \text{ with } D(x) = \lfloor \beta x \rfloor$$

$$T : [0,1) \to [0,1) \text{ with } T(x) = \beta x - D(x)$$

Greedy algorithm

$r_0 := x$; $j := 1$;
`for` $j \geqslant 1$ `do`
  $x_j := \lfloor \beta r_{j-1} \rfloor$; $r_j := \beta r_{j-1} - x_j$
  $j := j + 1$

The greedy expansion $g_\beta(x) = x_1 x_2 \cdots$ is the maximal representation of x (for the lexicographic order).

$$x < y \iff g_\beta(x) <_{lex} g_\beta(y).$$

If s is the greedy β-expansion of some $x \in [0,1)$ it is said to be β-admissible. The set of β-admissible sequences is D_β, and the β-shift S_β is the closure of D_β.

The greedy algorithm applied to 1 gives an expansion which plays an important role. Set $\mathrm{d}_\beta(1) = (e_n)_{n\geqslant 1}$ and define

$$\mathrm{d}^*_\beta(1) := \begin{cases} \mathrm{d}_\beta(1) & \text{if } \mathrm{d}_\beta(1) \text{ is infinite} \\ (e_1 \cdots e_{m-1}(e_m - 1))^\omega & \text{if } \mathrm{d}_\beta(1) = e_1 \cdots e_{m-1}e_m \text{ is finite.} \end{cases}$$

$\mathrm{d}^*_\beta(1)$ is called the quasi-greedy β-expansion of 1.

Theorem (Parry 1960)

Let $s = (s_n)_{n\geqslant 1}$ be a sequence in $A^{\mathbb{N}}$. Then

- *$s \in D_\beta$ if, and only if,*

$$\forall k \geqslant 0, \quad 0^\omega \leqslant_{lex} \sigma^k(s) <_{lex} \mathrm{d}^*_\beta(1)$$

- *$s \in S_\beta$ if, and only if,*

$$\forall k \geqslant 0, \quad 0^\omega \leqslant_{lex} \sigma^k(s) \leqslant_{lex} \mathrm{d}^*_\beta(1)$$

- *s is the greedy β-expansion of 1 for some (unique) $\beta > 1$ if, and only if,*

$$\forall k \geqslant 1, \quad 0^\omega < \sigma^k(s) <_{lex} s.$$

Remark: The quasi-greedy β-expansion of 1 is a lsmax-word.

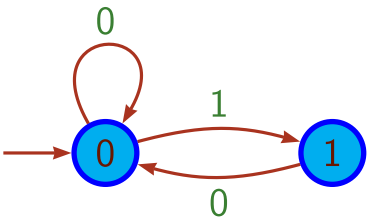
Theorem (Ito and Takahashi 1974, Bertrand-Mathis 1986, Blanchard 1989)

The β-shift S_β is a coded symbolic dynamical system which is

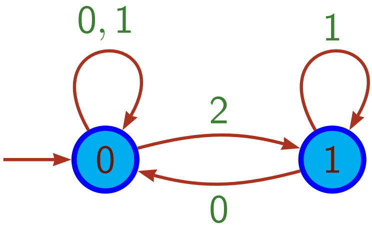
1. *sofic if, and only if,* $\mathrm{d}^*_\beta(1)$ *is eventually periodic,*
2. *of finite type if, and only if,* $\mathrm{d}^*_\beta(1)$ *is purely periodic, i.e.,* $\mathrm{d}_\beta(1)$ *is finite.*

Numbers β such that $\mathrm{d}_\beta(1)$ is eventually periodic (resp. finite) are called Parry numbers (resp. simple Parry numbers).

Example The golden mean shift: $d_\beta(1) = 11$ and $d^*_\beta(1) = (10)^\omega$.
11 is forbidden. System of finite type. Local automaton.



Example The β-shift for $\beta = \frac{3+\sqrt{5}}{2}$: $d_\beta(1) = d^*_\beta(1) = 21^\omega$. Sofic system not of finite type. Non-local automaton.



There is an important case where the β-expansion of 1 is eventually periodic.

A Pisot number is an algebraic integer > 1 such that all its Galois conjugates have modulus < 1. The natural integers and the golden mean are Pisot numbers.

Theorem (Schmidt 1980)

If β is a Pisot number, then every number of $\mathbb{Q}(\beta) \cap [0, 1]$ has an eventually periodic β-expansion.

For some Pisot numbers, for instance the golden mean, every element of $\mathbb{Z}(\beta) \cap \mathbb{R}_+$ has a finite β-expansion.

Lazy β-expansions

Lazy algorithm

$r_0 := x$; $j := 1$;
`for` $j \geqslant 1$ `do`
    $x_j := \max(0, \lceil \beta r_{j-1} - \frac{\lfloor \beta \rfloor}{\beta - 1} \rceil)$; $r_j := \beta r_{j-1} - x_j$
    $j := j + 1$

The lazy expansion $\ell_\beta(x) = x_1 x_2 \cdots$, where $x_j \in A = \{0, 1, \ldots, \lceil \beta \rceil - 1\}$, is the minimal representation of x (for the lexicographic order).

$$x < y \iff \ell_\beta(x) <_{lex} \ell_\beta(y).$$

Let $s = (s_n)_{n \geqslant 1}$ be in $A^{\mathbb{N}}$. Denote by $\overline{s_n} := \lfloor \beta \rfloor - s_n$ the complement of s_n, and by extension $\bar{s} := (\overline{s_n})_{n \geqslant 1}$.

$$s = g_\beta(x) \iff \bar{s} = \ell_\beta(\frac{\lfloor \beta \rfloor}{\beta - 1} - x).$$

Theorem (Erdős, Joó and Komornik 1990, Dajani and Kraaikamp 2002)

Let $s = (s_n)_{n \geqslant 1}$ be a sequence in $A^{\mathbb{N}}$. Then

- *s is the lazy β-expansion of some $x \in [0, 1)$ if and only if*

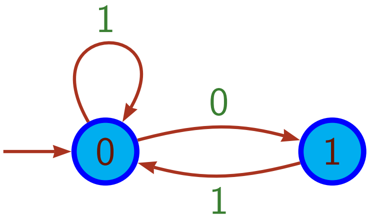
$$\forall k \geqslant 0, \quad 0^\omega \leqslant_{lex} \sigma^k(\bar{s}) <_{lex} \mathrm{d}^*_\beta(1)$$

- *s is the lazy β-expansion of 1 for some $\beta > 1$ if and only if*

$$\forall k \geqslant 1, \quad 0^\omega < \sigma^k(\bar{s}) <_{lex} s.$$

The (greedy) β-shift and the lazy β-shift have the same structure.

Example The lazy golden mean shift: 00 is forbidden. System of finite type. Local automaton.



Univoque numbers

$\beta > 1$ is said to be univoque if there exists a unique sequence of integers $(s_n)_{n\geqslant 1}$, with $0 \leqslant s_n < \beta$, such that $1 = \sum_{n\geqslant 1} s_n\beta^{-n}$.

Definition (Allouche 1983)

- A sequence $s = (s_n)_{n\geqslant 1}$ in $\{0,1\}^{\mathbb{N}}$ is self-bracketed if for every $k \geqslant 1$
$$\bar{s} \leqslant_{lex} \sigma^k(s) \leqslant_{lex} s$$
- If all the inequalities above are strict, the sequence s is said to be strictly self-bracketed. If one of the inequalities is an equality, then s is said to be periodic self-bracketed.

Theorem (Erdős, Joó, Komornik 1990)

A sequence in $\{0,1\}^{\mathbb{N}}$ is the unique β-expansion of 1 for a univoque number β in $(1,2)$ if and only if it is strictly self-bracketed.

Theorem (Komornik and Loreti 1998)

There exists a smallest univoque real number $\kappa \in (1, 2)$.
$\kappa \approx 1.787231$, and $d_\kappa(1) = (t_n)_{n \geqslant 1}$, where $(t_n)_{n \geqslant 1} = 11010011\ldots$ is obtained by shifting the Thue-Morse sequence.

Theorem (Allouche and Cosnard 2000)

The Komornik-Loreti constant κ is transcendental.

Theorem (Allouche, F. and Hare 2007)

There exists a smallest univoque Pisot number, of degree 14.

$(-\beta)$-expansions, $\beta > 1$

Ito and Sadahiro 2009

$J = [-\frac{\beta}{\beta+1}, \frac{1}{\beta+1})$, $A = \{0, 1, \ldots, \lfloor \beta \rfloor\}$

$$D : J \to A \text{ with } D(x) = \lfloor -\beta x + \frac{\beta}{\beta+1} \rfloor$$

$$T : J \to J \text{ with } T(x) = -\beta x - D(x)$$

For every $x \in J$ denote $\mathrm{d}_{-\beta}(x)$ the $(-\beta)$-expansion of x. Then $\mathrm{d}_{-\beta}(x) = (x_i)_{i \geqslant 1}$ if and only if $x_i = \lfloor -\beta T_{-\beta}^{i-1}(x) + \frac{\beta}{\beta+1} \rfloor$, and $x = \sum_{i \geqslant 1} x_i (-\beta)^{-i}$.

$$x < y \iff \mathrm{d}_{-\beta}(x) \prec_{alt} \mathrm{d}_{-\beta}(y).$$

A word $(x_i)_{i\geqslant 1}$ is $(-\beta)$-admissible if there exists a real number $x \in J$ such that $\mathrm{d}_{-\beta}(x) = (x_i)_{i\geqslant 1}$.
The $(-\beta)$-shift $S_{-\beta}$ is the closure of the set of $(-\beta)$-admissible words.
Define the sequence $\mathrm{d}^*_{-\beta}(\frac{1}{\beta+1})$ as follows:

- if $\mathrm{d}_{-\beta}(-\frac{\beta}{\beta+1}) = d_1 d_2 \cdots$ is not a periodic sequence with odd period,

$$\mathrm{d}^*_{-\beta}(\frac{1}{\beta+1}) = \mathrm{d}_{-\beta}(\frac{1}{\beta+1}) = 0 d_1 d_2 \cdots$$

- otherwise if $\mathrm{d}_{-\beta}(-\frac{\beta}{\beta+1}) = (d_1 \cdots d_{2p+1})^\omega$,

$$\mathrm{d}^*_{-\beta}(\frac{1}{\beta+1}) = (0 d_1 \cdots d_{2p}(d_{2p+1} - 1))^\omega.$$

Theorem (Ito and Sadahiro 2009)

Let $s = (s_n)_{n \geqslant 1}$ be a sequence in $A^{\mathbb{N}}$. Then

- *s is $(-\beta)$-admissible if and only if*

$$\forall k \geqslant 0, \quad \mathrm{d}_{-\beta}\left(-\frac{\beta}{\beta+1}\right) \preceq_{alt} \sigma^k(s) \prec_{alt} \mathrm{d}^*_{-\beta}\left(\frac{1}{\beta+1}\right).$$

- *s is an element of the $(-\beta)$-shift if and only if*

$$\forall k \geqslant 0, \quad \mathrm{d}_{-\beta}\left(-\frac{\beta}{\beta+1}\right) \preceq_{alt} \sigma^k(s) \preceq_{alt} \mathrm{d}^*_{-\beta}\left(\frac{1}{\beta+1}\right).$$

Remark: $\mathrm{d}_{-\beta}(-\frac{\beta}{\beta+1})$ is an asmin-word and $\mathrm{d}^*_{-\beta}(\frac{1}{\beta+1})$ is an asmax-word.

Theorem (Ito and Sadahiro 2009, F. and Lai 2009)

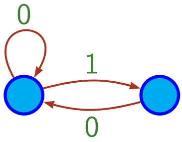
The $(-\beta)$-shift $S_{-\beta}$ is a symbolic dynamical system which is

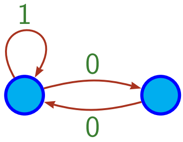
1. *sofic if, and only if,* $\mathrm{d}_{-\beta}(-\frac{\beta}{\beta+1})$ *is eventually periodic,*
2. *of finite type if, and only if,* $\mathrm{d}_{-\beta}(-\frac{\beta}{\beta+1})$ *is purely periodic.*

Theorem (F. and Lai 2009)

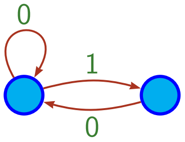
If β is a Pisot number, then every number of $\mathbb{Q}(\beta) \cap [0, 1]$ has an eventually periodic $(-\beta)$-expansion.

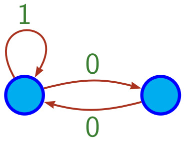
Example Let $\varphi = \frac{1+\sqrt{5}}{2}$. Then $d_{-\varphi}(-\frac{\varphi}{\varphi+1}) = 10^\omega$ the $(-\varphi)$-shift is a sofic system which is not of finite type.
Finite automata for the φ-shift (left) and for the $(-\varphi)$-shift (right)



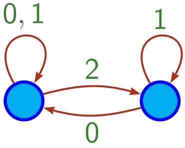


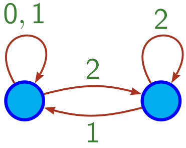
Example Let $\varphi = \frac{1+\sqrt{5}}{2}$. Then $\mathsf{d}_{-\varphi}(-\frac{\varphi}{\varphi+1}) = 10^\omega$ the $(-\varphi)$-shift is a sofic system which is not of finite type.
Finite automata for the φ-shift (left) and for the $(-\varphi)$-shift (right)





Example $\beta = \frac{3+\sqrt{5}}{2}$. $\mathsf{d}_{-\beta}(-\frac{\beta}{\beta+1}) = (21)^\omega$ and the $(-\beta)$-shift is of finite type: the set of minimal forbidden factors is $\{20\}$.
Finite automata for the β-shift (left) and for the $(-\beta)$-shift (right)





Entropy

The topological entropy of a subshift S is

$$h(S) = \lim_{n\to\infty} \frac{1}{n} \log(B_n(S))$$

where $B_n(S)$ is the number of factors of S of length n.
When S is sofic, the entropy of S is equal to the logarithm of the spectral radius of the adjacency matrix of the finite automaton recognising $L(S)$.

Theorem (Takahashi 1980, F. and Lai 2009)

The entropy of the β-shift and of the $(-\beta)$-shift are equal to $\log \beta$.

Theorem (Steiner 2013)

A sequence $s = (s_n)_{n \geqslant 1}$ in $A^{\mathbb{N}}$ is the $(-\beta)$-expansion of $-\frac{\beta}{\beta+1}$ (for some unique β) if, and only if,

1. $\forall k \geqslant 2, \quad s \preceq_{alt} \sigma^k(s)$,
2. $s \prec_{alt} u = 100111001001001110\cdots$, *where* $u = \psi^\omega(1)$ *with* $\psi(1) = 100$ *and* $\psi(0) = 1$,
3. $s \notin \{s_1 \cdots s_k,\ s_1 \cdots s_{k-1}(s_k - 1)0\}^\omega \setminus (s_1 \cdots s_k)^\omega$ *for all* $k \geqslant 1$ *with* $(s_1 \cdots s_k)^\omega \prec_{alt} u$,
4. $s \notin \{s_1 \cdots s_k 0,\ s_1 \cdots s_{k-1}(s_k + 1)\}^\omega$ *for all* $k \geqslant 1$ *with* $(s_1 \cdots s_{k-1}(s_k + 1))^\omega \prec_{alt} u$.

Maximal and minimal $(-\beta)$-expansions

Hejda, Masáková and Pelantová 2012

Proposition

s is the maximal $(-\beta)$-expansion (for the alternate order) of x if, and only if, $\bar{s}$ is the minimal $(-\beta)$-expansion of $-\frac{\lfloor\beta\rfloor}{\beta+1} - x$.

Remark

The Ito-Sadahiro transformation does not give the maximal $(-\beta)$-expansion (for the alternate order).

Example φ the golden mean. Let $x = -\frac{1}{2}$.
The minimal $(-\varphi)$-expansion of x is $1(001110)^\omega$.
The Ito-Sadahiro $(-\varphi)$-expansion of x is $(100)^\omega$.
The maximal $(-\varphi)$-expansion of x is $(111000)^\omega$.

$$1(001110)^\omega \prec_{alt} (100)^\omega \prec_{alt} (111000)^\omega.$$

There is no transformation of the form $T(x) = -\beta x - D(x)$ which generates for every x the maximal or the minimal $(-\beta)$-expansion of x.

Theorem

Let $\beta > 1$, $\beta \notin \mathbb{N}$, $A = \{0, \ldots, \lfloor \beta \rfloor\}$, $B = \{-b\beta + a \mid a, b \in A\}$.
Let $\pi : B^* \to A^*$ such that $\pi(-b\beta + a) = ba$.
Let $I = [\frac{-\beta \lfloor \beta \rfloor}{\beta^2 - 1}, \frac{\lfloor \beta \rfloor}{\beta^2 - 1}]$.

1.
   - *There exists a transformation $T_G : I \to I$ which generates $G(x)$ the maximal β^2-expansion of x on B;*
   - *$\pi(G(x))$ is the maximal $(-\beta)$-expansion of x on A (for the alternate order).*
2.
   - *There exists a transformation $T_L : I \to I$ which generates $L(x)$ the minimal β^2-expansion of x on B;*
   - *$\pi(L(x))$ is the minimal $(-\beta)$-expansion of x on A (for the alternate order).*

Digit-set conversion and normalisation

Real base α, $|\alpha| > 1$, A finite alphabet allowing representation of elements of an interval J.
C an arbitrary finite alphabet of digits.
A digit-set conversion in base α on C is a partial function $\chi_{\alpha,C} : C^{\mathbb{N}} \to A^{\mathbb{N}}$ such that

$$\chi_{\alpha,C}((c_i)_{i \geqslant 1}) = (a_i)_{i \geqslant 1} \iff \sum_{i \geqslant 1} c_i \alpha^{-i} = \sum_{i \geqslant 1} a_i \alpha^{-i}.$$

The normalisation $\nu_{\alpha,C}$ on C is a digit-set conversion where the result $(a_i)_{i \geqslant 1}$ is α-admissible.

Addition on A is a digit-set conversion $(A + A)^{\mathbb{N}} \to A^{\mathbb{N}}$.

$\beta > 1$, $A = \{0, \ldots, \lfloor \beta \rfloor\}$.

Theorem (F. 1992, Berend and F. 1994, F. and Sakarovitch 1999)

The following are equivalent:

1. *normalisation $\nu_{\beta,C}$ is computable by a finite letter-to-letter transducer on any alphabet C;*
2. *$\nu_{\beta,B}$ is computable by a finite letter-to-letter transducer on $B = \{0, \ldots, \lfloor \beta \rfloor, \lfloor \beta \rfloor + 1\}$;*
3. *β is a Pisot number.*

Example Take β the root > 1 of $X^4 - 2X^3 - 2X^2 - 2$. Then $d_\beta(1) = 2202$ and β is a simple Parry number, but it is not a Pisot number, since there is another root $\alpha \approx -1.134186$. One can show that normalisation on $A = \{0, 1, 2\}$ is not computable by a finite transducer.

Proposition

If $\beta > 1$ is a Pisot number, then normalisation in base $(-\beta)$ on any alphabet C is realisable by a finite transducer.

Proposition

If β is a Pisot number, then conversion from base $(-\beta)$ to base β is realizable by a finite transducer. The result is β-admissible.

These transducers are neither left nor right sequential when β is not an integer.

On-line computations

An on-line algorithm is such that, after a certain delay of latency during which the data are read without writing, a digit of the output is produced for each digit of the input.
Processing most significant digit first. Suitable for real numbers.
Sequentiality and synchronicity.

On-line functions are uniformly continuous for the prefix distance.

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1. Conversion from base β to base $(-\beta)$ is computable by an on-line algorithm (the result is not admissible).
2. If β is a Pisot number, the on-line transducer is finite.

Conclusions

Base β and base $(-\beta)$ are

- quite similar for the nature of the shift, the eventual periodicity of the rationals, and addition,
- quite different for the maximal and minimal representations.