On the successor function

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Numeration
Nancy
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Pierre Liardet

Groupe d'Etude sur la Numération 1999

Peano

The successor function is a primitive recursive function Succ such that $\text{Succ}(n) = n + 1$ for each natural number n.

Peano axioms define the natural numbers beyond 0:
1 is defined to be Succ(0)

Addition on natural numbers is defined recursively by:
$m + 0 = m$
$m + \text{Succ}(n) = \text{Succ}(m) + n$

Odometer

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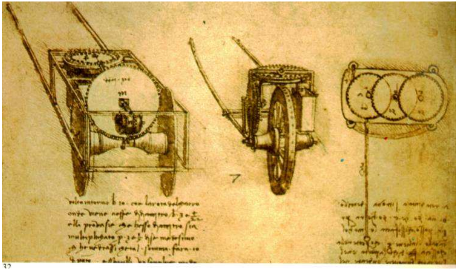
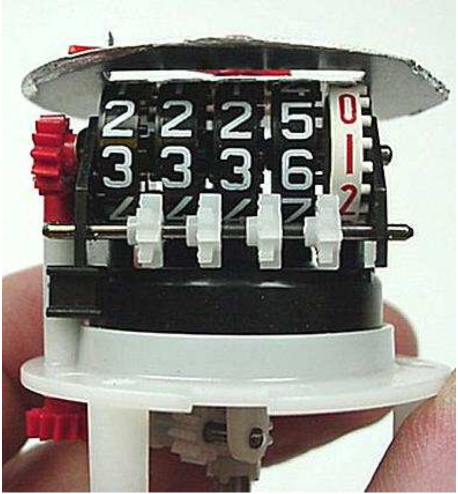
Leonardo da Vinci 1519: odometer of Vitruvius

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Adding machine

Machine arithmétique Pascal 1642 : Pascaline

The first calculator to have a controlled carry mechanism which allowed for an effective propagation of multiple carries.
French currency system used livres, sols and deniers with 20 sols to a livre and 12 deniers to a sol.
Length was measured in toises, pieds, pouces and lignes with 6 pieds to a toise, 12 pouces to a pied and 12 lignes to a pouce.
Computation in base 6, 10, 12 and 20.

To reset the machine, set all the wheels to their maximum, and then add 1 to the rightmost wheel.
In base 10, $999999 + 1 = 000000$.

Subtractions are performed like additions using 9's complement arithmetic.

Adding machine and adic transformation

An adic transformation is a generalisation of the adding machine in the ring of p-adic integers to a more general Markov compactum.

Vershik (1985 and later): adic transformation based on Bratteli diagrams: it acts as a successor map on a Markov compactum defined as a lexicographically ordered set of infinite paths in an infinite labeled graph whose transitions are provided by an infinite sequence of transition matrices.

In the stationary case (the transition matrices coincide, the infinite graph is a tree whose levels all have the same structure), (generalised) adic transformations correspond to substitutions and stationary odometers.

Solomyak 1991, 1992: spectral theory, beta-expansions.

Herman, Putnam, Skau 1992: every minimal Cantor dynamical system is isomorphic to a Bratteli-Vershik dynamical system.

Durand, Host, Skau 1999: algorithmic proof.

Sidorov: arithmetic dynamics 2002.

Durand chapter in CANT 2010.

Odometers

Grabner, Liardet, Tichy 1995: continuity
Barat, Downarowicz, Iwanik, Liardet 2000: metrical approach
Barat, Downarowicz, Liardet 2002: combinatorial and topological point of view
Berthé, Rigo 2007: Abstract numeration systems on regular languages.

Part I: the carry propagation

When does the amortised carry propagation exist?

Carry propagation

Example (Fibonacci numeration system)

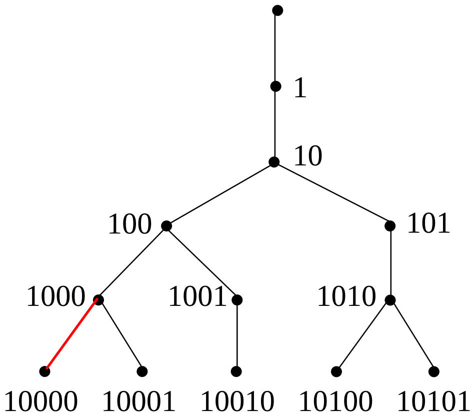
Defined by $(F_n)_{n \geqslant 0}$ where $F_0 = 1$, $F_1 = 2$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geqslant 0$. Set of greedy expansions of the natural integers is $F = 1\{0,1\}^* \setminus \{0,1\}^* 11 \{0,1\}^* \cup \{\varepsilon\}$.

	N	cp
ε	0	1
1	1	2
10	2	3
100	3	1
101	4	4

	N	cp
1000	5	1
1001	6	2
1010	7	5
10000	8	1
10001	9	2

	N	cp
10010	10	3
10100	11	1
10101	12	6
100000	13	1
100001	14	

Fibonacci words of length 5



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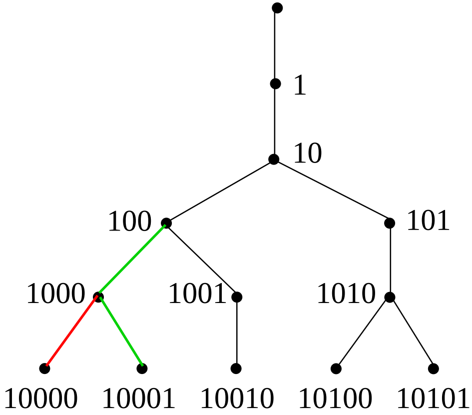

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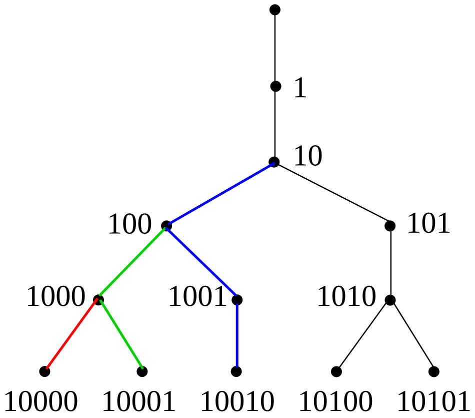

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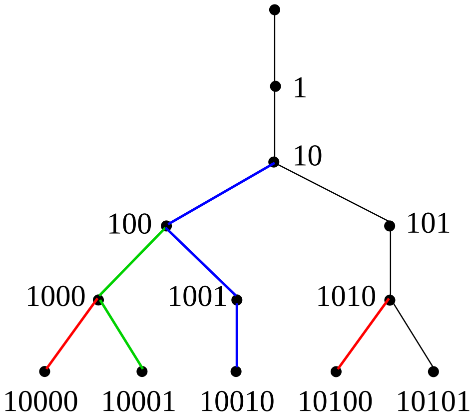

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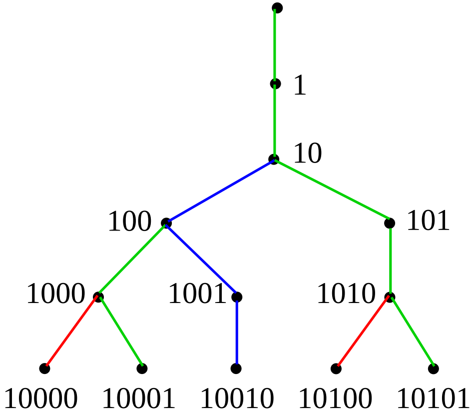

General framework

$(A, <)$ is a finite (totally) ordered alphabet.
Radix order: w and z two words in A^*; $w \prec z$ if
$|w| < |z|$, or
$|w| = |z|$ and $w = pas$, $z = pbt$, $a < b$.

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The successor function on $L \subseteq A^*$ ordered by radix order is the function $\mathrm{Succ}_L : A^* \to A^*$ that maps a word of L onto its successor in the radix order in L.
L is prefix-closed if every prefix of a word of L is in L.
The trie $\mathcal{T}_L$ of $L \subseteq A^*$ is a tree:

edges are labeled by letters from A

nodes are labeled by prefixes of words of L.

if w is a prefix of a word of L and a is a letter, there is an edge $w \xrightarrow{a} wa$ if wa is a prefix of a word of L.

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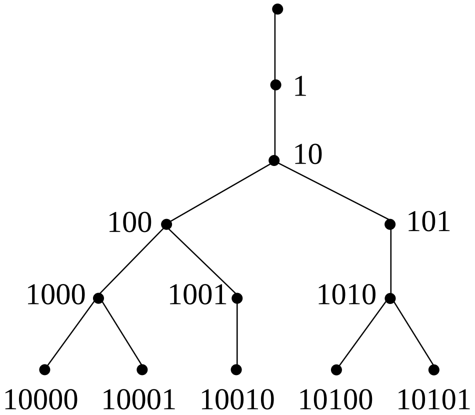
L is (right) extendable if every branch in the trie $\mathcal{T}_L$ is infinite, *i.e.*, for all $w \in L$, there exists $u \neq \varepsilon$ such that $wu \in L$. A language L is called pce if it is prefix-closed and right extendable.

Fibonacci trie



Carry propagation of a language

$$\Delta(w,z) = \begin{cases} \max(|w|,|z|) & \text{if } |w| \neq |z|, \\ |w| - |lclf(w,z)| & \text{if } |w| = |z| \end{cases}$$

where $lclf(w,z)$ is the longest common left factor of w and z.
L ordered by radix order, the carry propagation at a word w of L is:

$$\mathrm{cp}_L(w) = \Delta(w, \mathrm{Succ}_L(w))$$

By abuse, we also write, for every integer i,

$$\mathrm{cp}_L(i) = \mathrm{cp}_L(\langle i \rangle_L) = \Delta(\langle i \rangle_L, \langle i+1 \rangle_L)$$

The set of words of L of each length that are maximal in the radix order is denoted as $\mathrm{maxlg}(L)$.
Let u and v be two words in a language L.
$\mathbf{d}_L(u, v) =$ the length of the shortest path from u to v in the trie $\mathcal{T}_L$.

Proposition

Let L be a pce language and w a word in L. Then

1. *if* $w \notin \mathrm{maxlg}(L)$ *then* $|\mathrm{Succ}_L(w)| = |w|$ *and*

$$\mathrm{cp}_L(w) = \frac{1}{2}\,\mathbf{d}_L(w, \mathrm{Succ}_L(w))$$

2. *if* $w \in \mathrm{maxlg}(L)$ *then* $|\mathrm{Succ}_L(w)| = |w| + 1$ *and*

$$\mathrm{cp}_L(w) = |w| + 1 = k_L + \frac{1}{2}\left(\mathbf{d}_L(w, \mathrm{Succ}_L(w)) + 1\right).$$

The carry propagation of a language L is the amortised carry propagation at the words of L, that is, the limit, if it exists, of the mean of the carry propagation at the first N words of the language:

$$\mathrm{CP}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cp}_L(i)$$

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$$\boxed{\mathbf{u}_L(\ell) = \#(L \cap A^{\ell}) \qquad \mathbf{v}_L(\ell) = \#(L \cap A^{\leqslant \ell})}$$

$$\mathrm{acp}_L(\ell) = \sum_{\substack{w\in L\\ |w|=\ell}} \mathrm{cp}_L(w) = \sum_{i=\mathbf{v}_L(\ell-1)}^{\mathbf{v}_L(\ell)-1} \mathrm{cp}_L(i)$$

Proposition

If L is a pce language, then, $\forall \ell$, $\mathrm{acp}_L(\ell) = \mathbf{v}_L(\ell)$.

Example (Fibonacci)

$\ell = 4$, $\mathbf{v}_F(4) = 8$

$$1000 \mapsto 1001 \mapsto 1010 \mapsto 10000$$

The length-filtered carry propagation of L is the limit, if it exists, of the mean of the carry propagation at the first $\mathbf{v}_L(\ell)$ words of L:

$$\mathrm{FCP}_L = \lim_{n\to\infty} \frac{1}{\mathbf{v}_L(\ell)} \sum_{\substack{w\in L\\ |w|\leqslant\ell}} \mathrm{cp}_L(w) = \lim_{n\to\infty} \frac{1}{\mathbf{v}_L(\ell)} \sum_{i=0}^{\ell} \mathrm{acp}_L(i)$$

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$$\boxed{\mathrm{FCP}_L = \lim_{\ell\to\infty} \frac{1}{\mathbf{v}_L(\ell)} \sum_{i=0}^{\ell} \mathbf{v}_L(i)}$$

Remark

If CP_L *exists then* FCP_L *exists and* $\mathsf{CP}_L = \mathsf{FCP}_L$*.*

Remark

Let L *be a pce language such that* $\mathbf{u}_L(\ell) = P(\ell)$ *for some polynomial* P *of degree* d *with rational coefficients. Then* $\mathbf{v}_L(\ell)$ *is a polynomial of degree* $d+1$ *and,*

$$\sum_{i=0}^{\ell} \mathsf{acp}_L(i) = \sum_{i=0}^{\ell} \mathbf{v}_L(i)$$

is a polynomial of degree $d+2$*. Therefore,*

$$\lim_{\ell\to\infty} \frac{1}{\mathbf{v}_L(\ell)} \sum_{i=0}^{\ell} \mathsf{acp}_L(i) = +\infty$$

and FCP_L *does not exist.*

Example (Integer base p)

The successor function changes the least digit of every number, plus another one every p numbers, plus again another one every p^2 numbers, and so on...
Hence the carry propagation is equal to

$$1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots = \frac{p}{p-1}.$$

Local growth rate

L is a pce language with exponential growth ($\mathbf{u}_L(\ell)$ the number of words of length ℓ grows exponentially).

Local growth rate of L is the limit, if it exists

$$\gamma_L = \lim_{\ell \to +\infty} \frac{\mathbf{u}_L(\ell+1)}{\mathbf{u}_L(\ell)}.$$

Proposition

L is a pce language with exponential growth. Then

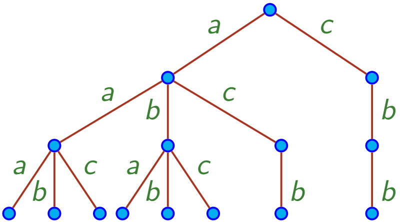
$$\mathrm{FCP}_L \textit{ exists} \iff \gamma_L \textit{ exists.}$$

In this case

$$\mathrm{FCP}_L = \frac{\gamma_L}{\gamma_L - 1}.$$

γ_L exists $\not\Rightarrow$ CP_L exists

$A = \{a, b, c\}$ and H is defined by: $H_\ell = H \cap A^\ell$. H'_ℓ (resp. H''_ℓ) the first (resp. last) $2^{\ell-1}$ words of length ℓ in H_ℓ.
Set $H_1 = \{a, c\}$. For all $\ell > 0$, $H_{\ell+1} = \{H'_\ell\}A \cup \{H''_\ell\}b$. Thus $\mathbf{u}_H(\ell) = 2^\ell$ and $\mathbf{v}_H(\ell) = 2^{\ell+1} - 1$.



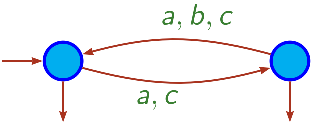
Since $\gamma_H = 2$, $\mathrm{FCP}_H = 2$.
Let $J_\ell = H \cap A^{\leqslant \ell} \cup H'_{\ell+1}$. The amortised carry propagation of the subsequence J_ℓ is equal to $\frac{11}{6} \neq 2$.
Note that H is not recognisable by a finite automaton.

The carry propagation of rational languages

Rational language = language recognised by a finite automaton

A rational language such that the local growth rate does not exist

$A = \{a, b, c\}$ and $L = (\{a, c\}\{a, b, c\})^* \{a, c, \varepsilon\}$.
We get $\mathbf{u}_L(0) = 1$, $\mathbf{u}_L(2\ell + 1) = 2\,\mathbf{u}_L(2\ell)$ and $\mathbf{u}_L(2\ell + 2) = 3\,\mathbf{u}_L(2\ell + 1)$, hence γ_L does not exist, although the global growth rate is $\eta_L = \limsup_{\ell \to +\infty} \sqrt[\ell]{\mathbf{u}_L(\ell)} = \sqrt{6}$.



Remark

If the local growth rate γ_L exists then $\gamma_L = \eta_L$.

Proposition

Let L be a pce language with exponential growth. Then, FCP_L *exists if and only if* γ_L *exists and in this case* $\mathrm{FCP}_L = \frac{\gamma_L}{\gamma_L - 1}$.

Corollary

If CP_L *exists then* γ_L *exists and* $\mathrm{CP}_L = \frac{\gamma_L}{\gamma_L - 1}$.

DEV languages

A rational language L has a dominating eigenvalue, and we say that L is DEV, if there is, among the eigenvalues of the adjacency matrix of its trim minimal automaton, a unique (real) eigenvalue λ such that, for all other eigenvalues λ_i, $\lambda > |\lambda_i|$.

Proposition

Let L be a DEV language and let λ be its the dominating eigenvalue. Then γ_L exists and $\gamma_L = \lambda$.

Theorem

Let L be a pce DEV language and let λ be its dominating eigenvalue. Then CP_L exists and $\mathrm{CP}_L = \frac{\lambda}{\lambda-1}$.

Corollary

In the Fibonacci numeration system the carry propagation is equal to $\frac{\varphi}{\varphi-1}$ where φ is the Golden Ratio.

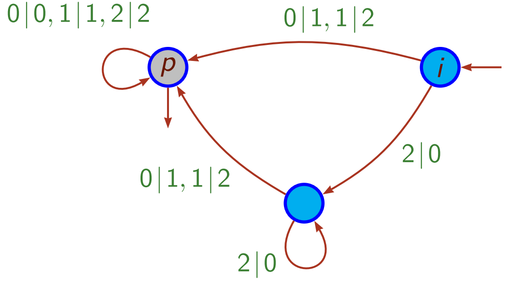
Part II: the concrete complexity

How to compute the successor function of a rational pce language and evaluate the complexity of this computation?

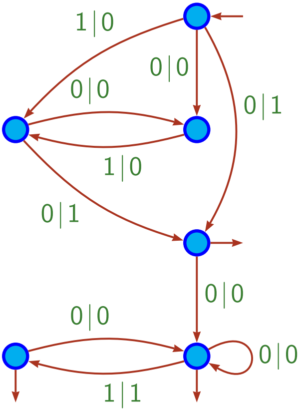
Theorem (F. 1997)

The successor function of a rational language can be realised by a right letter-to-letter finite transducer.

Base 3



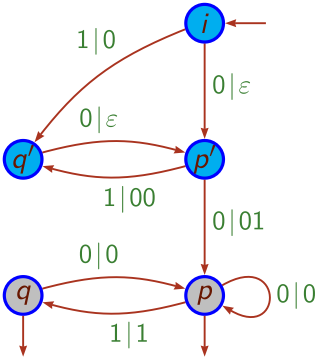
Fibonacci numeration system



Transducer is not (right) sequential, i.e. not input deterministic.

Sequentiality

In the Fibonacci numeration system the successor function is right sequential



Gives an algorithm

The numeration system based on the sequence of Fibonacci numbers of even rank $(G_n)_{n \geqslant 0} = \{1, 3, 8, 21, \ldots\}$

Set of greedy expansions of the natural integers is
$G = \{w \in \{0,1,2\}^* \mid w$ does not contain $21^*2\}$.

The successor function in G is not realisable by a (right) sequential finite transducer.

$$\begin{array}{rcr} 211111111 & \mapsto & 1000000000 \\ 011111111 & \mapsto & 011111112 \end{array}$$

The successor function in G is not continuous (see Grabner, Liardet, Tichy).

A function realisable by a right sequential finite transducer is called a co-sequential function.

A function which is a finite union of (co-)sequential functions with pairwise disjoint domains is called a piecewise (co-)sequential function.

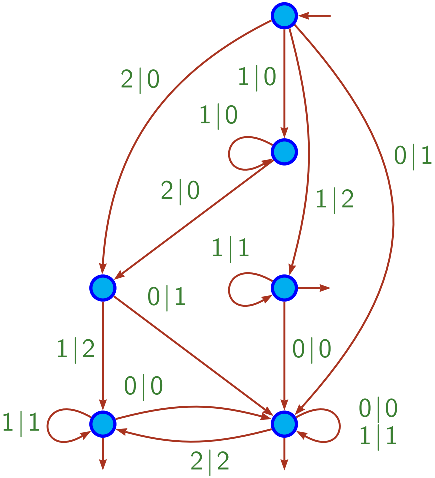
Theorem (Angrand and Sakarovitch 2010)

The successor function of a rational language is piecewise co-sequential.

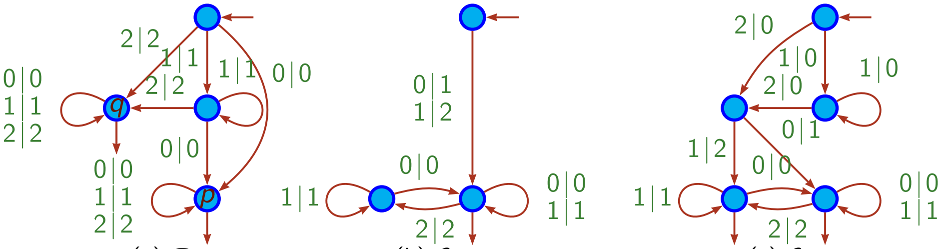
Theorem (Angrand and Sakarovitch 2010)

A piecewise (co)-sequential function is realised by a cascade of (right) finite transducers.

The successor function in G (Fibonacci numbers of even rank)



A cascade for the successor function in G (Fibonacci numbers of even rank)



(a) $\mathcal{R}$ (b) $\mathcal{E}_p$ (c) $\mathcal{E}_q$

Complexity of computations

Base 3 (sequential letter-to-letter)

$w = \mathbf{102}022222 \mapsto \mathrm{Succ}_3(\mathrm{w}) = \mathbf{102}100000$

$\mathrm{cp(w)} = 6$. No additional information neded.

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Fibonacci (sequential but not letter-to-letter)

$w = \mathbf{10}0010101 \mapsto \mathrm{Succ}_F(w) = \mathbf{10}0100000$

cp(w) = 6. Need to read the blue 0. Concrete complexity of w is cc(w) = 7.

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Fibonacci of even rank (non-sequential: cascade)

$w = \mathbf{101}021111 \mapsto \mathrm{Succ}_G(\mathrm{w}) = \mathbf{101}100000$

$\mathrm{cp}(\mathrm{w}) = 6$. $\mathrm{cc}(\mathrm{w}) = 6$.

$w = \mathbf{101}011111 \mapsto \mathrm{Succ}_G(\mathrm{w}) = \mathbf{101}011112$

$\mathrm{cp}(\mathrm{w}) = 1$. Need to read the blue 01111. $\mathrm{cc}(\mathrm{w}) = 6$.

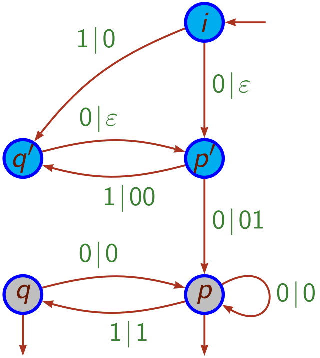
In bold: **recopy**.

Surcharge for computing the successor of w is $\mathrm{sc}(\mathrm{w}) = \mathrm{cc}(\mathrm{w}) - \mathrm{cp}(\mathrm{w})$.

Copy ideal

$\mathcal{T} = (Q, A, B, \delta, \eta, i, T)$ a (right) sequential finite transducer.
The copy ideal $\mathcal{I}$ of $\mathcal{T}$ is the largest subset of Q such that:
(i) it is closed under δ: $\forall s \in \mathcal{I}\ \forall a \in A\ \delta(s, a) \in \mathcal{I}$;
(ii) every state s in $\mathcal{I}$ is final and $T(s) = \varepsilon$;
(iii) every transition inside $\mathcal{I}$ realises the identity:

$$\forall s \in \mathcal{I}\ \forall a \in A\ \eta(s, a) = a.$$



Concrete complexity

Suppose that $\mathrm{Succ_L}$ is realised by a finite right sequential transducer $\mathcal{T}_L$ having some technical "good" properties.

Let w be in L and $i \xrightarrow{w|w'} q$ in $\mathcal{T}_L$.

- If q is in $\mathcal{I}_L$, let p be the first state which belongs to $\mathcal{I}_L$: $i \xrightarrow{u|v} p \xrightarrow{h|h} q$. Put $\mathrm{cc}_L(w) = \max(|u|, |v|)$.
- If q is not in $\mathcal{I}_L$, put $\mathrm{cc}_L(w) = \max(|w|, |w'|)$.

The amortised concrete complexity of Succ_L, is the limit, if it exists,

$$\mathrm{CC}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{cc}_L(i)$$

Definition extends to the case that Succ_L is realised by a cascade of finite right sequential transducers.

The amortised surcharge of Succ_L, is the limit, if it exists,

$$\mathrm{SC}_L = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \mathrm{sc}_L(i) = \mathrm{CC}_L - \mathrm{CP}_L.$$

Proposition

If Succ_L is realised by a right sequential letter-to-letter finite transducer, then

$$\mathrm{CC}_L = \mathrm{CP}_L.$$

Each move in the transducer is determined only by the input letter and produces an output letter, so there is no surcharge.

Numeration on an integer basis and beta numeration

$V = (v_n)_{n \geqslant 0}$ increasing sequence of integers with $v_0 = 1$.
Greedy algorithm (Fraenkel 1985): $N \in \mathbb{N}$ has a V-expansion $a_k \cdots a_0$, with $0 \leqslant a_i < v_{i+1}/v_i$, such that $N = \sum_{i=0}^{k} a_i v_i$. Set of greedy expansions $L(V)$.

Let $\beta > 1$ be a real number. Any real number $x \in [0,1]$ can be represented by a greedy algorithm (Rényi 1957) as $x = \sum_{i=1}^{+\infty} x_i \beta^{-i}$ with $x_i \in A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ for all $i \geqslant 1$.
The greedy sequence $d_\beta(x) = (x_i)_{i \geqslant 1}$ is the β-expansion of x.
When the expansion ends in infinitely many 0's, it is said finite, and the 0's are omitted.
$d_\beta(1) = (t_n)_{n \geqslant 1}$ the β-expansion of 1.

- If $d_\beta(1)$ is finite, of the form $d_\beta(1) = t_1 \cdots t_m$, $t_m \neq 0$, let $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$
- If the β-expansion of 1 is infinite, set $d^*_\beta(1) = d_\beta(1)$.

Sequentiality

Suppose that $L(V)$ is rational; then V is a linear recurrent sequence. Let P be the characteristic polynomial of V, and assume that P has a dominant root β. Such a number is called a Perron number.

Proposition (F. 1997)

Let V be a linear recurrent sequence with dominant root β, and suppose that $L(V)$ is rational.
Then the successor function on $L(V)$ is right sequential if, and only if,

1. *the β-expansion of 1 is finite, of the form $d_\beta(1) = t_1 \cdots t_m$,*
2. *V is defined by*

$$v_n = t_1 v_{n-1} + \cdots + t_m v_{n-m} \text{ for } n \geqslant n_0 \geqslant m$$

and $1 = v_0 < v_1 < \cdots < v_{n_0-1}$.

Let $\beta > 1$ be a real number.

- If $d_\beta(1) = t_1 \cdots t_m$, then set $v_0 = 1$,
  $v_n = t_1 v_{n-1} + \cdots + t_n v_0 + 1$ for $1 \leqslant n \leqslant m-1$, and
  $v_n = t_1 v_{n-1} + \cdots + t_m v_{n-m}$ for $n \geqslant m$.
- If $d_\beta(1) = (t_n)_{n \geqslant 1}$, then set $v_0 = 1$,
  $v_n = t_1 v_{n-1} + \cdots + t_n v_0 + 1$ for $n \geqslant 1$.

$V_\beta = (v_n)_{n \geqslant 0}$ forms the canonical numeration system associated with β. The set of greedy expansions of the natural integers is denoted L_β.

We have $\lim_{n \to \infty} v_{n+1}/v_n = \beta$ (Bertrand 1989).

Thus the local growth rate of L_β is equal to

$$\gamma_{L_\beta} = \beta.$$

Parry numbers

If the β-expansion of 1 is finite or eventually periodic then β is said to be a Parry number. If the β-expansion of 1 is finite β is said to be a simple Parry number.
When β is a Parry number, the sequence $V_\beta = (v_n)_{n \geqslant 0}$ is linear recurrent, and L_β is rational and in fact is a pce DEV language.

$\varphi = \frac{1+\sqrt{5}}{2}$

The φ-expansion of 1 is equal to 11, thus φ is a simple Parry number. The canonical linear numeration system associated with the golden mean is the Fibonacci numeration system.

$\tau = \frac{3+\sqrt{5}}{2}$

The τ-expansion of 1 is equal to 21^ω, so τ is a non-simple Parry number. The canonical linear numeration system associated with τ is the numeration system based on Fibonacci numbers of even rank.

Theorem

If $\beta > 1$ is a Parry number then the carry propagation of the successor function for the canonical numeration system V_β associated with β is equal to

$$\mathsf{CP}_{L_\beta} = \frac{\beta}{\beta - 1}.$$

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A question

$X^2 - 5X + 5$ has two roots, $\beta = \frac{5+\sqrt{5}}{2} = 3.618$ and $\beta' = \frac{5-\sqrt{5}}{2} = 1.381$.

β is a Perron number which is not a Parry number since $\beta' > 1$. $\mathrm{d}_\beta(1) = 320301021\ldots$ is aperiodic and thus L_β is not recognisable by a finite automaton.

Does the carry propagation of L_β exist?

Parry numeration

Corollary

The function Succ_{L_β} *is realised by a sequential finite right transducer if, and only if,* β *is a simple Parry number.*

Proposition

If β *is a non-integer simple Parry number and the* β*-expansion of* 1 *is of length* m*, the (amortised) concrete complexity of the successor function for the canonical linear numeration system associated with* β *satisfies*

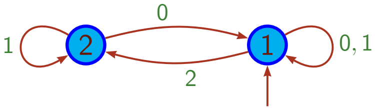
$$\frac{\beta}{\beta-1} < \mathsf{CC}_{L_\beta} \leqslant \frac{\beta}{\beta-1} + (m-1).$$

The upper bound is attained by the Fibonacci numeration system with $\mathsf{CC}_{L_\varphi} = \frac{\varphi}{\varphi-1} + 1$.

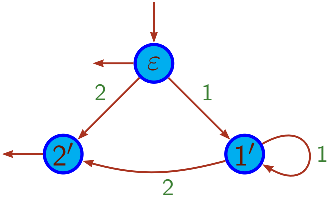
Using some construction of a particular sequential transducer (reps. a cascade of sequential transducers) computing the successor function in the canonical numeration system associated with a simple (resp. non-simple) Parry number we are able to read the surcharge on this transducer (resp. cascade).

$\tau = \frac{3+\sqrt{5}}{2}$, $\mathsf{d}_\tau(1) = 21^\omega$.

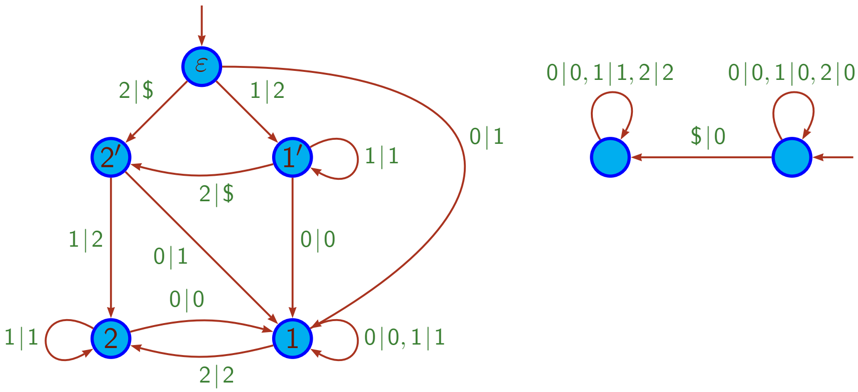
Right automaton $\mathcal{L}$ recognising $0^* L_\tau$:



Right automaton $\mathcal{M}$ recognising the set of maximal words $M = 21^* \cup \{\varepsilon\}$:



Cascade performing $\mathsf{Succ}_{\mathsf{L}_\tau}$:



$w = \mathbf{101}\textcolor{red}{021111} \mapsto \mathbf{1011}\$1112 \mapsto \mathbf{101}\textcolor{red}{100000} = \mathsf{Succ}_\mathsf{G}(\mathsf{w})$
$\mathsf{cp}(\mathsf{w}) = 6.\ \mathsf{sc}(\mathsf{w}) = 6.$
$z = \mathbf{101}\textcolor{blue}{01111}\textcolor{red}{1} \mapsto \mathbf{101}01111\textcolor{red}{2} = \mathsf{Succ}_\mathsf{G}(\mathsf{z})$
$\mathsf{cp}(\mathsf{z}) = 1.\ \mathsf{sc}(\mathsf{z}) = 5.$

Computation of the surcharge:
Let $\mathcal{V}_i(q)$ be the number of words of length i starting from state q.

We have
$\mathsf{SC}_{L_\tau} = \lim_{\ell\to\infty} \frac{1}{v_\ell}\big(\sum_{i=0}^{\ell-1} \mathcal{Z}(1', i)\, \mathcal{V}_{\ell-i-1}(1) + \sum_{i=0}^{\ell-1} \mathcal{Z}(2', i)\, \mathcal{V}_{\ell-i-1}(1) + \sum_{i=0}^{\ell-1} \mathcal{Z}(2', i)\, \mathcal{V}_{\ell-i-1}(2)\big)$

with $\mathcal{Z}(1', i) = i$ for $i > 0$ and $\mathcal{Z}(1', 0) = 0$, and $\mathcal{Z}(2', i) = i + 1$ for $i > 0$ and $\mathcal{Z}(2', 0) = 0$.

Since 1 is the initial state of $\mathcal{L}$, $\mathcal{V}_{\ell-i-1}(1) = v_{\ell-i-1}$.

We need to compute the limit of $\mathcal{V}_{\ell-i-1}(2)/v_\ell$. By standard tools of linear algebra, we have that

$$\mathcal{V}_{\ell-i-1}(2) \sim \tau^{\ell-i-1}\frac{2\tau-1}{\tau^2-1}$$

and

$$v_\ell = \mathcal{V}_\ell(1) \sim \tau^\ell\frac{\tau^2}{\tau^2-1}.$$

Therefore

$$\mathrm{SC}_{L_\tau} = \sum_{i=1}^{\infty} i \frac{1}{\tau^{i+1}} + \sum_{i=1}^{\infty} (i+1) \frac{1}{\tau^{i+1}} + \sum_{i=1}^{\infty} (i+1) \frac{1}{\tau^{i+1}} \frac{2\tau - 1}{\tau^2} = \tau - 1.$$

Thus

$$\boxed{\mathrm{CP}_{L_\tau} = \frac{\tau}{\tau - 1} = \tau - 1 \quad \text{and} \quad \mathrm{CC}_{L_\tau} = 2(\tau - 1).}$$