Conversion between two multiplicatively dependent linear numeration systems

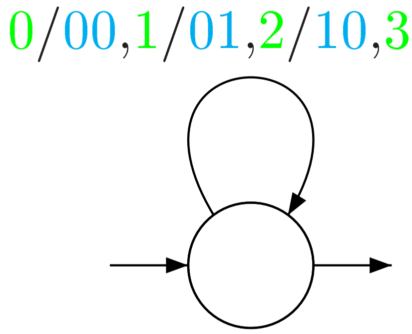
Christiane Frougny

L.I.A.F.A. and Université Paris 8

http://www.liafa.jussieu.fr/~cf/

Conversion

Conversion base 4 - base 2 is computable by a finite automaton



Conversion base 3 - base 2 is not computable by a finite automaton

p and q are multiplicatively dependent if exist positive integers k and ℓ such that $p^k = q^\ell$.

$S \subset \mathbb{N}$ is p-recognizable if the set of representations in base p of its elements is recognizable by a finite automaton.

THEOREM **1** Cobham (1969) *The only sets that are both p- and q-recognizable, when p and q are two multiplicatively independent integers > 1, are unions of arithmetic progressions, and thus are k-recognizable for any integer $k > 1$.*

U-representations

$U = (u_n)_{n \geq 0}$ a strictly increasing sequence of integers with $u_0 = 1$.

U-representation of $N \geq 0$ is a finite sequence of integers $(d_i)_{k \geq i \geq 0}$ such that $N = \sum_{i=0}^{k} d_i u_i$.

$(N)_U = d_k \cdots d_0$

Normal or greedy U-representation of N :
$q(m,p)$ and $r(m,p)$ = quotient and remainder of the Euclidean division of m by p.
$u_k \leq N < u_{k+1}$.
$d_k = q(N, u_k)$ and $r_k = r(N, u_k)$,
$d_i = q(r_{i+1}, u_i)$ and $r_i = r(r_{i+1}, u_i)$.
$N = d_k u_k + \cdots + d_0 u_0$. Normal U-representation of N = $\langle N \rangle_U = d_k \cdots d_0$

$G(U)$= set of greedy or normal U-representations of all the non-negative integers.

U linearly recurrent then the digits of a normal U-representation are in a canonical finite alphabet A_U associated with U.

D a finite alphabet of integers.
Normalization in U on D^*

$$\nu_{U,D^*} : D^* \to A_U^*$$

$\nu_{U,D^*}(d_k \cdots d_0) = \langle N \rangle_U$ normal U-representation of $N = \sum_{i=0}^{k} d_i u_i$.

Conversion

U and V two sequences of integers
D a finite alphabet of integers.

Conversion from V to U on D^*

$$\chi : D^* \to A_U^*$$

$(N)_V = d_k \cdots d_0 \in D^*$, $\chi(d_k \cdots d_0) = \langle N \rangle_U$
normal U-representation of $N = \sum_{i=0}^{k} d_i v_i$.

Previous results

Pisot number : an algebraic integer > 1 such that if α is algebraic conjugate then $|\alpha| < 1$

U a linearly recurrent sequence of integers such that its characteristic polynomial is exactly the minimal polynomial of a Pisot number.
Then $G(U)$ is recognizable by a finite automaton, and, for every D, normalization ν_{U,D^*} is computable by a finite automaton.

Addition and multiplication by a fixed positive integer constant are particular cases of normalization, and thus are computable by a finite automaton in the system U.

Beta-expansion of 1

$r_0 = 1$ and for $i \geq 1$, $t_i = \lfloor \beta r_{i-1} \rfloor$, $r_i = \{\beta r_{i-1}\}$.

Then $1 = \sum_{i \geq 1} t_i \beta^{-i}$, where t_i is in the canonical alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if β is not an integer, or $A_\beta = \{0, \ldots, \beta - 1\}$ if β is an integer.

$$d_\beta(1) = (t_i)_{i \geq 1}$$

If β is a Pisot number then $d_\beta(1)$ is finite (ends by infinitely many 0's) or eventually periodic (Bertrand 1977).

Example $\beta =$ golden mean, $d_\beta(1) = 11$.

$\beta = \frac{3+\sqrt{5}}{2}$, $d_\beta(1) = 21^\omega$.

Fibonacci and Lucas

Fibonacci numeration system

$$F = \{1, 2, 3, 5, 8, 13, \ldots\}.$$

$A_F = \{0, 1\}$.
Words containing a factor 11 are forbidden :
$G(F) = 1\{0, 1\}^* \setminus \{0, 1\}^* 11\{0, 1\}^* \cup \varepsilon$.

Lucas numeration system

$$L = \{1, 3, 4, 7, 11, 18, \ldots\}.$$

$A_L = \{0, 1, 2\}$.
$G(L) = G(F) \cup (G(F) \setminus \varepsilon)\{02\} \cup \{2\}$.

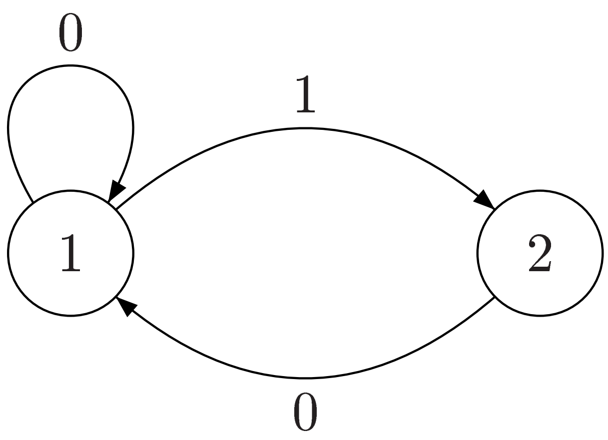
N	Fibonacci	Lucas
1	1	1
2	10	2
3	100	10
4	101	100
5	1000	101
6	1001	102
7	1010	1000
8	10000	1001
9	10001	1002
10	10010	1010
11	10100	10000

Characteristic polynomial $P(X) = X^2 - X - 1$.

φ= the golden mean, φ'=algebraic conjugate.

For $n \geq 1$, $L_n = \varphi^n + \varphi'^n$.

The associated dynamical system is the golden mean shift : set of bi-infinite sequences on $\{0, 1\}$ having no factor 11.



Conversion between Lucas and Fibonacci

PROPOSITION 1 *The conversion from a Lucas representation of an integer to the normal Fibonacci representation of that integer is computable by a finite automaton.*

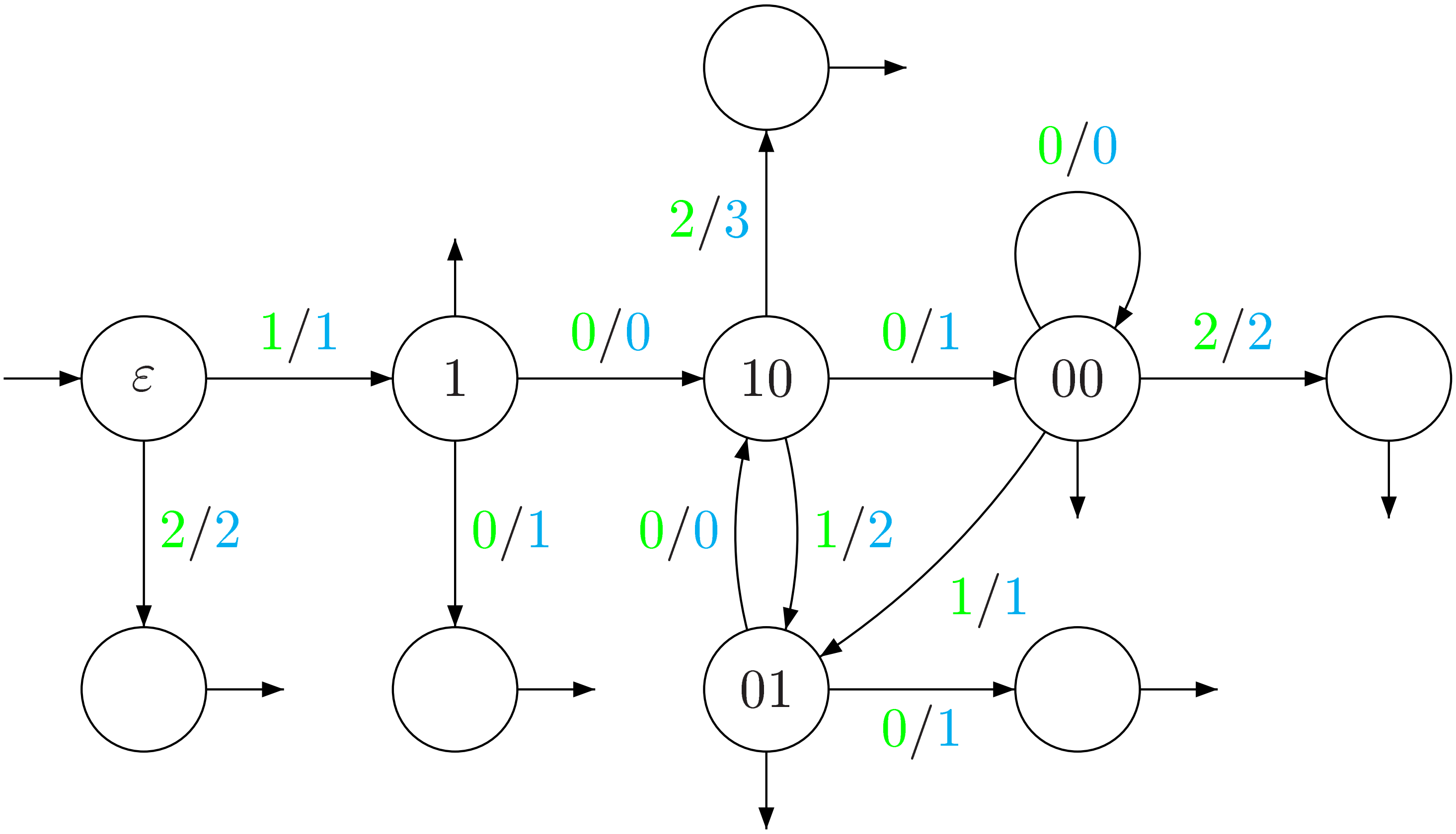
$\forall n \geq 3,$

$$L_n = F_{n-1} + F_{n-3}$$

$(N)_L = d_k \cdots d_1$
$N = d_k L_k + \cdots + d_1 L_1$
$N = d_k F_{k-1} + d_{k-1} F_{k-2} + (d_{k-2} + d_k) F_{k-3} + \cdots + (d_3 + d_5) F_2 + (d_2 + d_4) F_1 + (d_1 + d_2 + d_3) F_0$
The word
$d_k d_{k-1} (d_{k-2}+d_k) \cdots (d_3+d_5)(d_2+d_4)(d_1+d_2+d_3)$
is a Fibonacci representation of N on an alphabet of digits D.

The conversion from $d_k \cdots d_1$, $k \geq 4$, onto $d_k d_{k-1} (d_{k-2}+d_k) \cdots (d_3+d_5)(d_2+d_4)(d_1+d_2+d_3)$ on D^* is computable by a finite automaton.

Then normalize in the Fibonacci system on D^* by a finite automaton.



Conversion from normal Lucas representations to Fibonacci representations on $\{0, \ldots, 3\}$

A technical result

Two linearly recurrent sequences $U = (u_n)_{n \geq 0}$ and $V = (v_n)_{n \geq 0}$

PROPOSITION **2** *If there exist r rational constants λ_i's for $1 \leq i \leq r$ and $K \geq 0$ such that for every $n \geq K$, $v_n = \lambda_1 u_{n+r-1} + \cdots + \lambda_r u_n$, and if the normalization in the system U is computable by a finite automaton on any alphabet, then the conversion from a V-representation of an integer to the normal U-representation of that integer is computable by a finite automaton.*

$\lambda_i = p_i / q$

$(N)_V = b_j \cdots b_0$, then $qN = b_j q v_j + \cdots + b_0 q v_0$.

For $n \geq K$, $qv_n = p_1 u_{n+r-1} + \cdots + p_r u_n$, thus $qN = d_{j+r-1} u_{j+r-1} + \cdots + d_0 u_0$, $d_i \in D$.

ν_{U,D^*} is computable by a finite automaton.

$\nu_{U,D^*}(d_{j+r-1} \cdots d_0) = < qN >_U$ and $< qN >_U \mapsto < N >_U$ is the inverse of the multiplication by q, so is computable by a finite automaton.

Common characteristic polynomial

The Fibonacci and the Lucas numeration systems are examples of different numeration systems having the same characteristic polynomial, but different initial conditions.

Theorem **2** *Let P be the minimal polynomial of a Pisot number of degree m. Let U and V be two sequences with common characteristic polynomial P and different initial conditions. The conversion from a V-representation of a positive integer to the normal U-representation of that integer is computable by a finite automaton.*

P minimal polynomial of a Pisot number thus normalization in U is computable by a finite automaton on any alphabet.

$\{u_n, u_{n+1}, \ldots, u_{n+m-1} \mid n \geq 0\}$ is free.

U and V have same characteristic polynomial, so exist constants $\lambda_i \in \mathbb{Q}$ such that,
$v_n = \lambda_1 u_{n+m-1} + \cdots + \lambda_m u_n$.

Two numeration systems associated with a Pisot number

β a Pisot number.

Fibonacci-like numeration system U_β

1. $d_\beta(1) = t_1 \cdots t_N$ finite.

$$u_n = t_1 u_{n-1} + \cdots + t_N u_{n-N} \quad \text{for} \quad n \geq N$$

$u_0 = 1$, $u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1$ for $1 \leq i \leq N-1$
Characteristic polynomial of U_β=

$$K(X) = X^N - t_1 X^{N-1} - \cdots - t_N.$$

2. $d_\beta(1) = t_1 \cdots t_N (t_{N+1} \cdots t_{N+p})^\omega$ eventually periodic with N and p minimal.

$u_n = t_1 u_{n-1} + \cdots + t_{N+p} u_{n-N-p} + u_{n-p} - t_1 u_{n-p-1} - \cdots - t_N u_{n-N-p}$ for $n \geq N+p$, $u_0 = 1$, $u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1$ for $1 \leq i \leq N+p-1$

Characteristic polynomial of U_β is

$$K(X) = X^{N+p} - \sum_{i=1}^{N+p} t_i X^{N+p-i} - X^N + \sum_{i=1}^{N} t_i X^{N-i}.$$

$K(X)$ is the beta-polynomial of β.

U_β is the canonical numeration system associated with β, the natural one from the point of view of symbolic dynamical systems.

Lucas-like numeration system V_β

β a Pisot number of degree m
$\beta_1 = \beta$, β_2, ..., β_m the roots of the minimal polynomial $P(X) = X^m - a_1 X^{m-1} - \cdots a_m$ of β.

$$v_0 = 1, \text{ and for } n \geq 1, \; v_n = \beta_1^n + \cdots + \beta_m^n.$$

Characteristic polynomial of V_β is $P(X)$.

$\beta = \varphi$ the golden mean. U_φ is the set of Fibonacci numbers, and V_φ is the set of Lucas numbers (for $n \geq 1$).

β integer, then $U_\beta = V_\beta =$ standard β-ary numeration system.

PROPOSITION **3** *Let β be a Pisot number such that its beta-polynomial $K(X)$ is equal to its minimal polynomial. Let U with characteristic polynomial equal to $K(X)$ (in particular U_β). The conversion from V_β to U (and conversely) is computable by a finite automaton.*

Multiplicatively dependent numeration systems

β Pisot number of degree m then β^k Pisot number of degree m.

Two Pisot numbers β and γ are multiplicatively dependent if there exist two positive integers k and ℓ such that $\beta^k = \gamma^\ell$.

Then β and γ have the same degree m.

THEOREM **3** *Let β and γ be two multiplicatively dependent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. Then the conversion from the Y-numeration system to the U-numeration system is computable by a finite automaton.*

$\delta = \beta^k = \gamma^\ell$.
$V_\beta = (v_n)_{n \geq 0}$ with $v_0 = 1$ and $v_n = \beta_1^n + \cdots + \beta_m^n$ for $n \geq 1$.
The conjugates of δ are of the form $\delta_i = \beta_i^k$ for $2 \leq i \leq m$.
$W = (w_n)_{n \geq 0}$ with $w_n = \delta_1^n + \cdots + \delta_m^n$ for $n \geq 1$.

W is the Lucas-like numeration system associated with δ.

For $n \geq 1$, $w_n = v_{kn}$.

$(N)_W = d_k \cdots d_0$ so
$(N)_{V_\beta} = d_k 0^{k-1} d_{k-1} 0^{k-1} \cdots d_1 0^{k-1} d_0$, and thus the conversion from W to V_β is computable by a finite automaton.

The same is true for the conversion from W-representations to V_γ-representations.

The conversion from Y to V_γ, and that from V_β to U are computable by a finite automaton, and the result follows.

$$U \longrightarrow V_\beta \longleftarrow W \longrightarrow V_\gamma \longleftarrow Y$$

A set S of natural integers is said to be U-recognizable if the set $\{< n >_U |\ n \in S\}$ is recognizable by a finite automaton.

COROLLARY 1 *Let β and γ be two multiplicatively dependent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. Then a set which is U-recognizable is Y-recognizable as well.*

THEOREM 4 Bès (2000) *Let β and γ be two multiplicatively independent Pisot numbers. Let U and Y be two linear sequences with characteristic polynomial equal to the minimal polynomial of β and γ respectively. The only sets of integers that are both U-recognizable and Y-recognizable are unions of arithmetic progressions.*