On-line odometers for two-sided symbolic dynamical systems

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The golden ratio shift

Set V of biinfinite sequences of 0's and 1's with no factor 11. It is a symbolic dynamical system of finite type.

Perturbation: adding 1 in position 0.

$$\begin{aligned}
a &= \cdots 0000101.01010000 \cdots \\
a+1 &= \cdots 0000102.01010000 \cdots \\
&\sim \cdots 0000110.02010000 \cdots \\
&\sim \cdots 0001000.10020000 \cdots \\
&\sim \cdots 0001000.10100100 \cdots
\end{aligned}$$

Addition of 1 = odometer: τ

$\tau(a) = b = \cdots 0001000.10100100 \cdots \in V$

$011 \sim 100$ and $0200 \sim 1001$ in base $\beta = (1+\sqrt{5})/2$

$$
\begin{aligned}
a' &= \cdots 010101010101.010101010101\cdots \\
a'+1 &= \cdots 010101010102.010101010101\cdots \\
&\sim \cdots 010101010110.020101010101\cdots \\
&\sim \cdots 010101011000.100201010101\cdots \\
&\sim \cdots 010101100000.101002010101\cdots \\
&\cdots \\
&\sim \cdots 000000000000.101010101010\cdots
\end{aligned}
$$

$$
\tau(a') = b' = \cdots 000000000000.101010101010\cdots
$$

Beta-expansions

β real number > 1, $x \in [0, 1]$

$r_0 = x$ and for $k \geq 1$, $x_k = \lfloor \beta r_{k-1} \rfloor$ and $r_k = \{\beta r_{k-1}\}$.

Thus $x = \sum_{k\geq 1} x_k \beta^{-k}$, x_k in $A_\beta = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$, and $A_\beta = \{0, \ldots, \beta - 1\}$ otherwise.

$d_\beta(x) = (x_k)_{k\geq 1}$ is the beta-expansion of x.

$d_\beta(1) = (t_i)_{i\geq 1}$ and set $d^*_\beta(1) = d_\beta(1)$ if $d_\beta(1)$ is infinite and $d^*_\beta(1) = (t_1 \cdots t_{m-1}(t_m - 1))^\omega$ if $d_\beta(1) = t_1 \cdots t_{m-1} t_m$ is finite.

D_β = set of beta-expansions of numbers of $[0, 1[$. $S_\beta = \overline{D_\beta}$ is the beta-shift. Theorem of Parry

$$D_\beta = \{s \in A_\beta^{\mathbb{N}} \mid \forall p \geq 0 \ \sigma^p(s) \prec d^*_\beta(1)\}$$

$$S_\beta = \{s \in A_\beta^{\mathbb{N}} \mid \forall p \geq 0 \ \sigma^p(s) \preceq d^*_\beta(1)\}$$

σ is the shift on $A_\beta^{\mathbb{N}}$.

Two-sided beta-shift

$$V_\beta = \{s \in A_\beta^{\mathbb{Z}} \mid \forall k \in \mathbb{Z}\ (s_n)_{n \leq k} \in S_\beta\}$$

$$V_\beta^+ = \{\cdots s_3 s_2 s_1 s_0 \mid s \in V_\beta\}$$

$$V_\beta^- = \{s_{-1} s_{-2} s_{-3} \cdots \mid s \in V_\beta\}$$

$L_\beta =$ set of factors of D_β

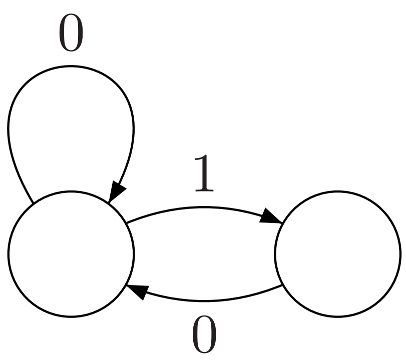
When $d_\beta(1) = t_1 \cdots t_m$, the beta-shift is a system of finite type, and to determine if a word belongs to L_β can be checked on factors of length m.

Example: Golden ratio shift

$\beta = (1 + \sqrt{5})/2$

$d_\beta(1) = 11$, $d^*_\beta(1) = (10)^\omega$

V_β = set of labels of biinfinite paths in the automaton



On-line finite automaton with delay δ

Sequential transducer on $A \times (B \cup \varepsilon)$: input deterministic

On-line with delay δ: labels a/ε during a time δ, then labels a/b.

Distance ρ on $A^{\mathbb{N}}$: let $v = (v_i)_{i \in \mathbb{N}}$ and $w = (w_i)_{i \in \mathbb{N}}$ in $A^{\mathbb{N}}$, $\rho(v, w) = 2^{-r}$ where $r = \min\{i \geq 0 \mid v_i \neq w_i\}$ if $v \neq w$, and $\rho(v, w) = 0$ otherwise.

$A^{\mathbb{N}}$ is a compact metric space. A function realized by an on-line finite automaton is continuous.

Right on-line automaton: same but words are processed from right to left.

Bidirectional on-line automaton with delay δ: composed of a left on-line finite automaton with delay δ and of a right on-line finite automaton with delay δ. Starts from the middle, and then goes synchronously in both directions.

Distance ρ on $A^{\mathbb{Z}}$: let $v = (v_i)_{i \in \mathbb{Z}}$ and $w = (w_i)_{i \in \mathbb{Z}}$ in $A^{\mathbb{Z}}$, $\rho(v, w) = 2^{-r}$ if $v \neq w$ and $r = \max\{i \mid v_i \cdots v_{-i} = w_i \cdots w_{-i}\}$, and $\rho(v, w) = 0$ if $v = w$. $A^{\mathbb{Z}}$ is a compact metric space.

A function realized by a bidirectional on-line finite automaton is continuous.

Previous results

The one-sided odometer, defined on V_β^+, is continuous iff $d_\beta(1)$ is finite, Grabner, Liardet, Tichy.

If $d_\beta(1) = t_1 \cdots t_m$ is finite, then the one-sided odometer is computable by a right on-line finite automaton with delay $m-1$, F.

A Pisot number is an algebraic integer > 1 such that all its conjugates have modulus < 1.

If β is a Pisot number then on V_β^- addition is computable by a finite automaton (not on-line), and non-normalized addition is computable by a (left) on-line finite automaton, F.

Result

β such that $d_\beta(1) = t_1 \cdots t_m$ with $t_1 \geq t_2 \geq \cdots t_m \geq 1$. Then β is a Pisot number.

Theorem *The two-sided odometer associated with β is computable by a bidirectional on-line finite automaton with delay m.*

Remark There exist other Pisot numbers with $d_\beta(1) = t_1 \cdots t_m$ such that the two-sided odometer is computable by an on-line finite automaton, but the delay is $> m$.

For $d_\beta(1) = 201$ the two-sided odometer is computable by an on-line finite automaton with delay 4.

Idea of the proof

3 parts: a central initial part, a right on-line finite automaton $\mathcal{P}$ working on V_β^+ and a left on-line finite automaton $\mathcal{N}$ working on V_β^-.

Input: $a = (a_k)_{k \in \mathbb{Z}} \in V_\beta$
Output: $b = (b_k)_{k \in \mathbb{Z}} \in V_\beta$ such that $b = \tau(a)$

$u \sim v$ if v is obtained from u using the rewriting rules

$0t_1 \cdots t_m \to 10^m$

$0t_1 \cdots t_{j-1}(t_j + 1)0^m \to$

$10^j(t_1 - t_{j+1}) \cdots (t_{m-j} - t_m)t_{m-j+1} \cdots t_m$

$1 \leq j \leq m - 1$

1 - Central part

Central initial state

$c = a_{m-1} \cdots a_1(a_0+1) \centerdot a_{-1} \cdots a_{-m}$.

Delay $= m$.

Edges

$$p_0 \xleftarrow{a_m/b_0} c \xrightarrow{a_{-m-1}/b_{-1}} n_1$$

with $p_0 = (s,t)$ and $n_1 = (u,v)$, where s and u are words of length m such that

$sb_0 \centerdot b_{-1}u \sim a_m c a_{-m-1}$.

t and v represent the past of s and u respectively.

p_0 is an initial state of $\mathcal{P}$ and n_1 is an initial state of $\mathcal{N}$.

2 - Right on-line finite automaton $\mathcal{P}$

R_n = set of admissible words of length n.

Four classes of states

$C_1 = \{p_k = (a_{m+k} \cdots a_{k+1}, a_{m+k} \cdots a_{k+2}) \in R_m \times R_{m-1} \mid k \geq 0\}$

$C_2 = \{p_k = (a_{m+k} \cdots a_{j+k-1}(a_{j+k}+1)0^{j-1}, a_{m+k} \cdots a_{j+k} t_1 \cdots t_{j-2}) \in R_m \times R_{m-1},\ 2 \leq j \leq m \mid k \geq 0\}$

$C_3 = \{p_k = (a_{m+k} \cdots a_{k+2}(a_{k+1}+1), a_{m+k} \cdots a_{k+2}) \in R_m \times R_{m-1} \mid k \geq 0\}$

$C_4 = \{(t_1 \cdots t_m, t_1 \cdots t_{m-1})\}$.

Initial states $\{p_0 \in \cup_{1 \leq i \leq 4} C_i\}$.

Shift edges

$p_k = (u_k, v_k) \xrightarrow{a_{m+k+1}/b_{k+1}} p_{k+1} = (u_{k+1}, v_{k+1})$ with $p_k \in C_1 \cup C_2 \cup C_3$, $a_{m+k+1} v_k \in R_m$, and $a_{m+k+1} u_k = u_{k+1} b_{k+1}$

Reduction edge

$p_k = (t_1 \cdots t_m, t_1 \cdots t_{m-1}) \xrightarrow{a_{m+k+1}/0} p_{k+1} = ((a_{m+k+1}+1)0^{m-1}, a_{m+k+1} t_1 \cdots t_{m-2})$

3 - Left on-line finite automaton $\mathcal{N}$

If $x = x_1 \cdots x_m$ let $\mu_\beta(x) = x_1\beta^{-1} + \cdots + x_m\beta^{-m}$.
States $S = \{n_k = (u_k, v_k) \mid k \geq 1,\ |u_k| = m,\ 0 \leq \mu_\beta(u_k) < 1,\ v_k \in R_{m-1}\}$.
Initial states $I = \{n_1 = (u_1, v_1) \mid |u_1| = m,\ 0 \leq \mu_\beta(u_1) \leq 1 - \beta^{-m},\ v_1 = a_{-3} \cdots a_{-m-1}\}$.

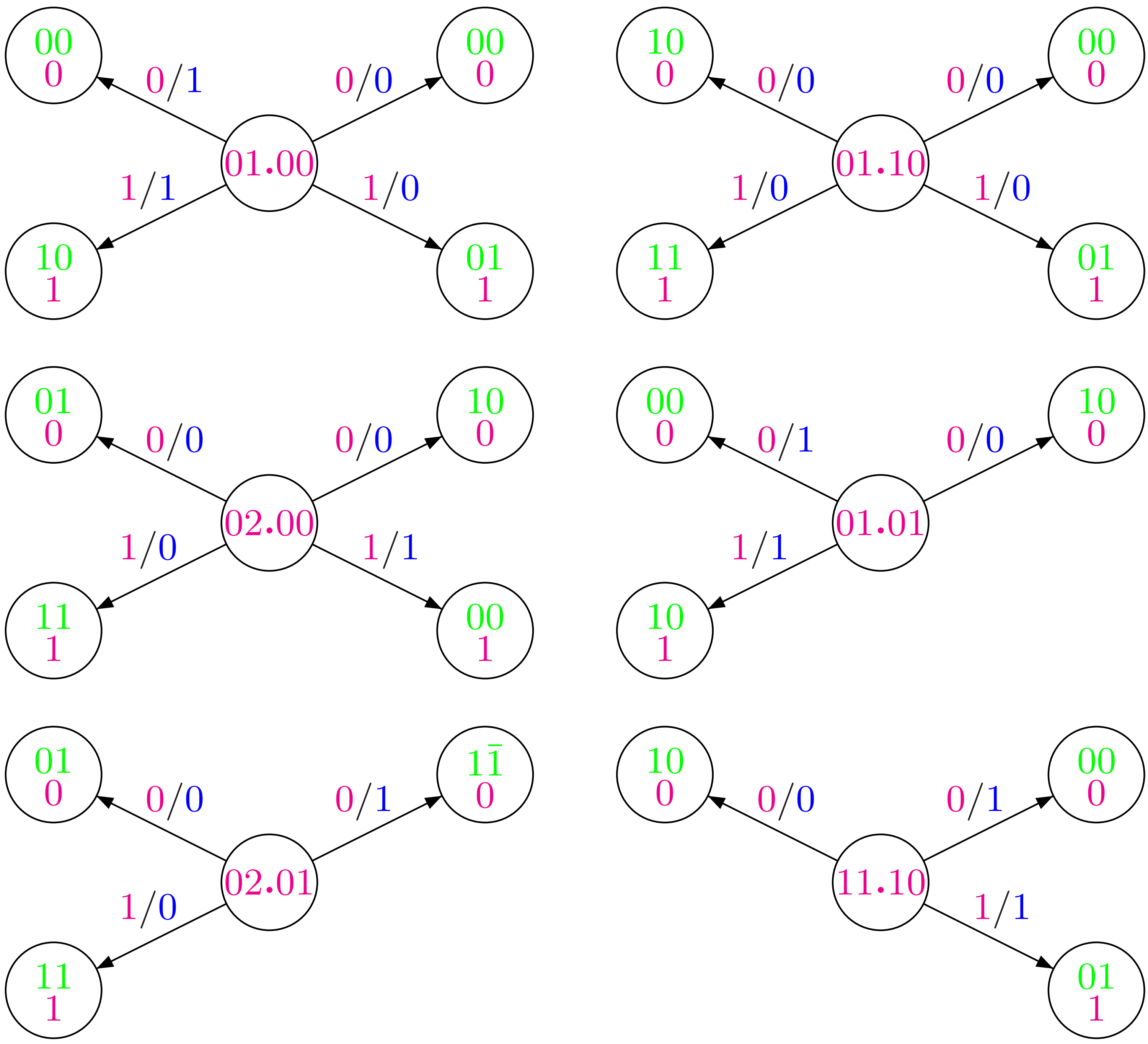
β Pisot implies that S is finite.

Edges

$n_k = (u_k, v_k) \xrightarrow{a_{-m-k-1}/b_{-k-1}} n_{k+1} = (u_{k+1}, v_{k+1})$
if $v_k a_{-m-k-1} \in R_m$, and b_{-k-1} is chosen as follows:
If $\mu_\beta(u_k a_{-m-k-1}) < 1$ take
$b_{-k-1} = \lfloor \beta\mu_\beta(u_k a_{-m-k-1}) \rfloor$,
$y_{k+1} = \{\beta\mu_\beta(u_k a_{-m-k-1})\}$ and u_{k+1} of length m such that $\mu_\beta(u_{k+1}) = y_{k+1}$.
If $\mu_\beta(u_k a_{-m-k-1}) \geq 1$ take $b_{-k-1} = t_1$ and u_{k+1} of length m such that $u_k a_{-m-k-1} \sim t_1 u_{k+1}$.

So $\mu_\beta(u_k a_{-m-k-1}) = \mu_\beta(b_{-k-1} u_{k+1})$ holds for each k, with n_{k+1} in S and b_{-k-1} in A_β.

Example: Golden ratio



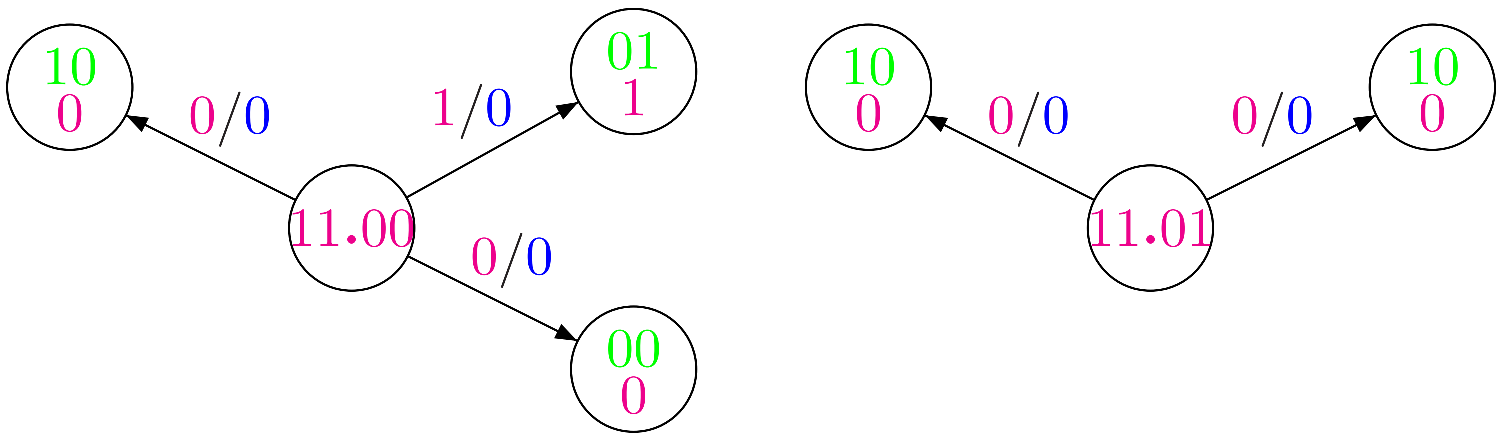


Figure 1: Central part of the two-sided odometer with delay 2 associated with the golden ratio

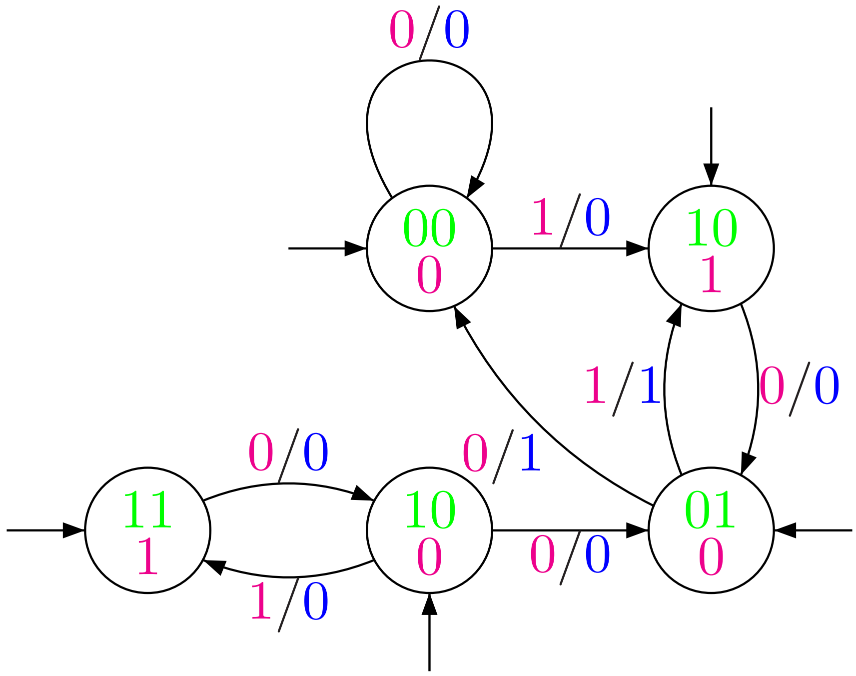


Figure 2: Right on-line finite automaton $\mathcal{P}$ with delay 2 for the two-sided odometer associated with the golden ratio

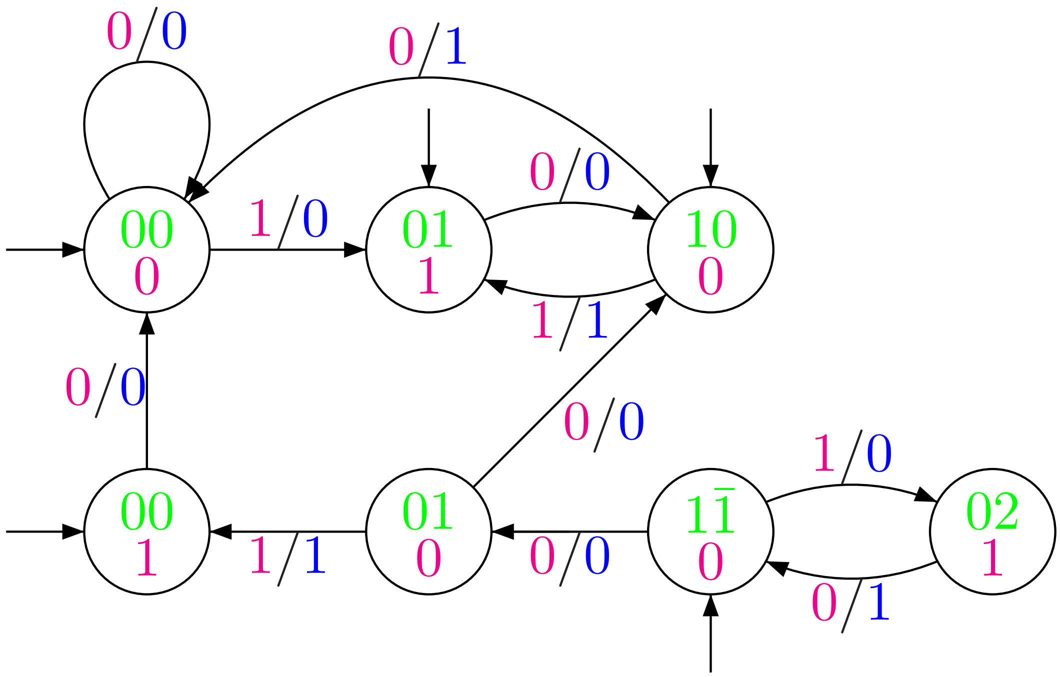


Figure 3: Left on-line finite automaton $\mathcal{N}$ with delay 2 for the two-sided odometer associated with the golden ratio