Quasicrystals and beta-numeration

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Crystals and quasicrystals

Crystals: solids in dimension 2 or 3, with atoms arranged periodically.
Symmetry of order n.
n must satisfy

$$\rho = 2\cos\frac{2\pi}{n} \in \mathbb{Z}$$

hence $n = 2, 3, 4, 6$.

Quasicrystal Alloy aluminium-manganese with order 5 symmmetry Shechtman et al. 1984
Quasi-periodicity.

Geometrical modelization

$\Lambda \subset \mathbb{R}^d$ is uniformly discrete if exists $r > 0$ such that every ball of radius r contains at most a point of Λ.
Λ is relatively dense if exists $R > 0$ such that every ball of radius R contains at least a point of Λ.

If both conditions are satisfied, Λ is said to be a Delaunay set.

Model set (Y. Meyer 1970, 1972)

Cut and projection scheme

$$\mathbb{R}^d \quad \xleftarrow{\pi_1} \mathbb{R}^d \times G \xrightarrow{\pi_2} \quad G$$
$$D$$

G loc. compact abelian group (internal space)
$\mathbb{R}^d$ physical space
D lattice *i.e.* discret sub-group of $\mathbb{R}^d \times G$ such that $(\mathbb{R}^d \times G)/D$ is compact
$\pi_1|_D$ 1-to-1
$\pi_2(D)$ dense in G.
Let $* = \pi_2 \circ (\pi_1|_D)^{-1}$

$$* : M = \pi_1(D) \longrightarrow G$$

$\Lambda \subset \mathbb{R}^d$ is a model set if there exist a cut and projection scheme and a relatively compact set $\Omega \subset G$ of non-empty interior such that

$$\Lambda = \{x \in M \mid x^* \in \Omega\}$$

Example: The Fibonacci chain

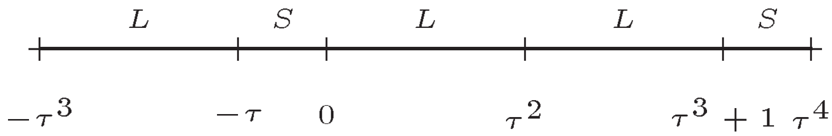
Let $\tau = \frac{1+\sqrt{5}}{2}$.

$$\begin{array}{ccccc} \mathbb{R} & \xleftarrow{\pi_1} & \mathbb{R}\times\mathbb{R} & \xrightarrow{\pi_2} & \mathbb{R} \\ & & \mathbb{Z}^2 & & \end{array}$$

$\pi_1|_{\mathbb{Z}^2} \sim \mathbb{Z}[\tau] = \{a + b\tau \mid a, b \in \mathbb{Z}\}$

Fibonacci chain

$$\begin{aligned} \mathcal{F} &= \{x = a + b\tau \mid x' = a - \frac{b}{\tau} \in \Omega = [0,1)\} \\ &= \{\dots, -\tau^3, -\tau, 0, \tau^2, \tau^3 + 1, \tau^4, \dots\} \end{aligned}$$



Tiling of $\mathbb{R}$ with 2 tiles L et S

$$\begin{aligned} L &\mapsto LLS \\ S &\mapsto LS \end{aligned}$$

with $|L| = \tau^2$ et $|S| = \tau$

$$\begin{array}{rcl} S & | & L \\ LS & | & LLS \\ LLSLS & | & LLSLLSLS \end{array}$$

Meyer set

$\Lambda \subset \mathbb{R}^d$ is a Meyer set if it is Delaunay and if there exists a finite set F such that

$$\Lambda - \Lambda \subset \Lambda + F$$

THEOREM **1** . *A model set is a Meyer set. Conversely if Λ is a Meyer set, there exist a finite set F and a model set Λ_0 such that $\Lambda \subset \Lambda_0 + F$.*

PROPOSITION **1** . *Λ is a Meyer set if and only if Λ and $\Lambda - \Lambda$ are Delaunay.*

Pisot number algebraic integer such that every conjugate is < 1 in modulus.
Salem number algebraic integer such that every conjugate is ≤ 1 in modulus.

THEOREM **2** . *If $\Lambda \subset \mathbb{R}^d$ is a Meyer set and if $\beta > 1$ is a real number such that $\beta\Lambda \subset \Lambda$ then β is a Pisot or a Salem number.*
Conversely for each d and for each Pisot or Salem number β, there exists a Meyer set $\Lambda \subset \mathbb{R}^d$ such that $\beta\Lambda \subset \Lambda$.

Beta-numeration (Rényi, Parry)

$\beta > 1$, $x > 0$

Beta-expansion of x:

$\beta^k \leq x < \beta^{k+1}$

$x_k = \lfloor x/\beta^k \rfloor$ et $r_k = \{x/\beta^k\}$.

For $i < k$, let $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$.

$$\langle x \rangle_\beta = x_k x_{k-1} \dots x_1 x_0 \cdot x_{-1} x_{-2} \dots$$

$x_i \in A$ where $A = \{0, \dots, \beta - 1\}$ if β integer or $A = \{0, \dots, \lfloor \beta \rfloor\}$ if β is not an integer.

Let $T_\beta(x) = \beta x \bmod 1$ and $d_\beta(1) = (t_i)_{i \geq 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$.

If β is Pisot then $d_\beta(1)$ is eventually periodic (A. Bertrand)

If $d_\beta(1)$ is eventually periodic, β is called a beta-number.

Example

Let $x = \frac{\sqrt{5}}{2} = \tau + \tau^{-1}$

$$\langle x \rangle_\tau = 10.1$$

$$d_\tau(1) = 11$$

Beta-integers

The set of beta-integers

$$\begin{aligned} \mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \end{aligned}$$

Then

$$\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta \ , \ \mathbb{Z}_\beta = -\mathbb{Z}_\beta$$

THEOREM **3** . *If β is a Pisot number then $\mathbb{Z}_\beta$ is a Meyer set.*

Tiling and substitution associated with a beta-number

$\mathbb{R}^+$ admits a self-similar tiling with a finite number of tiles of length $T_\beta^i(1)$, for $i \geq 0$ (Thurston).

• $d_\beta(1) = t_1 \cdots t_m$
m tiles of length 1, $\beta - t_1$, $\ldots$,
$\beta^{m-1} - t_1\beta^{m-2} - \cdots - t_{m-1}$.
Substitution on $\mathbb{A} = \{a_0, \cdots, a_{m-1}\}$

$$\sigma_\beta : \left\{ \begin{array}{ccc} a_0 & \mapsto & a_0^{t_1} a_1 \\ a_1 & \mapsto & a_0^{t_2} a_2 \\ & \cdots & \\ a_{m-2} & \mapsto & a_0^{t_{m-1}} a_{m-1} \\ a_{m-1} & \mapsto & a_0^{t_m}. \end{array} \right.$$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$

$m+p$ tiles of length 1, $\beta - t_1$, … ,
$\beta^{m+p-1} - t_1 \beta^{m+p-2} - \cdots - t_{m+p-1}$.
Substitution on $\mathbb{A} = \{a_0, \cdots, a_{m+p-1}\}$

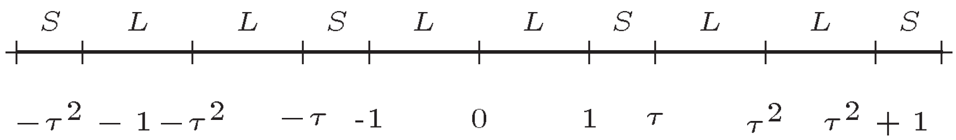
$$\sigma_\beta : \begin{cases} a_0 & \mapsto & a_0^{t_1} a_1 \\ a_1 & \mapsto & a_0^{t_2} a_2 \\ & \cdots & \\ a_{m+p-2} & \mapsto & a_0^{t_{m+p-1}} a_{m+p-1} \\ a_{m+p-1} & \mapsto & a_0^{t_{m+p}} a_m. \end{cases}$$

Example: The symmetrized Fibonacci half-chain

$$
\begin{aligned}
\mathbb{Z}_\tau &= \mathbb{Z}_\tau^+ \cup (-\mathbb{Z}_\tau^+) \\
&= \{0,\ 1,\ \tau,\ \tau^2,\ \tau^2+1,\ \ldots\} \\
&\cup\ \{-1,\ -\tau,\ -\tau^2,\ -\tau^2-1,\ \ldots\}
\end{aligned}
$$

generated by the Fibonacci substitution

$$
\begin{aligned}
L &\mapsto LS \\
S &\mapsto L
\end{aligned}
$$



$\mathbb{Z}_\tau$ is a Meyer set which is not a model set.

Numeration systems associated with a beta-number

- $d_\beta(1) = t_1 \cdots t_m$

$$u_{n+m} = t_1 u_{n+m-1} + \cdots + t_m u_n$$

$u_0 = 1,\ u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1,\ 1 \leq i \leq m-1.$

- $d_\beta(1) = t_1 \cdots t_m (t_{m+1} \cdots t_{m+p})^\omega$

$$\begin{aligned} u_{n+m+p} &= t_1 u_{n+m+p-1} + \cdots + t_{m+p} u_n \\ &+ u_{n+m} - t_1 u_{n+m-1} - \cdots - t_m u_n \end{aligned}$$

$u_0 = 1,\ u_i = t_1 u_{i-1} + \cdots + t_i u_0 + 1,\ 1 \leq i \leq m+p-1.$

$U_\beta = (u_n)_{n \geq 0}$ defines the numeration system associated with β

Let

$$\Phi : f_k\beta^k + \cdots + f_1\beta + f_0 \mapsto f_k u_k + \cdots + f_1 u_1 + f_0$$

PROPOSITION **2** . *Let β be a beta-number. If $\langle x \rangle_\beta = x_k \cdots x_0$, then $N = \Phi(x)$ has for U_β-representation $\langle N \rangle_{U_\beta} = x_k \cdots x_0$.*

The set $\mathbb{Z}_\beta$ is ordered: let b_n be the n-th β-integer. Then

$$\Phi(b_n) = n.$$

Example The numeration system associated with τ is the Fibonacci numeration system with $u_0 = 1$ and $u_1 = 2$.

$\mathbb{Z}$	Fibonacci	$\mathbb{Z}_\tau$
1	1	1
2	10	τ
3	100	τ^2
4	101	$\tau^2 + 1$
5	1000	τ^3
6	1001	$\tau^3 + 1$

Cyclotomic Pisot numbers

If n is not crystallographic, $\rho = 2\cos\frac{\pi}{n}$ is an algebraic integer of degree $m \leq \lfloor n-1 \rfloor/2$.

Cyclotomic Pisot number β such that

$$\mathbb{Z}[\rho] = \mathbb{Z}[\beta]$$

Then $m = \varphi(n)/2$ and $\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta$ is a ring invariant under rotation of order n, with $\zeta = \exp(2i\pi/n)$.

Real quasicrystals

- $n = 5$ or $n = 10$: $\beta = \rho = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$, $M_\beta(X) = X^2 - X - 1$
- $n = 8$: $\beta = 1 + \rho = 1 + \sqrt{2} = 1 + 2\cos\frac{\pi}{4}$, $M_\beta(X) = X^2 - 2X - 1$
- $n = 12$: $\beta = 2 + \rho = 2 + \sqrt{3} = 2 + 2\cos\frac{\pi}{6}$, $M_\beta(X) = X^2 - 4X + 1$.

Quadratic Pisot units

Other cyclotomic Pisot units.

• $n = 7$ or $n = 14$: $\beta = 1 + \rho = 1 + 2\cos\frac{\pi}{7}$,
$M_\beta(X) = X^3 - 2X^2 - X + 1$
• $n = 9$ or $n = 18$: $\beta = 1 + \rho = 1 + 2\cos\frac{\pi}{9}$,
$M_\beta(X) = X^3 - 3X^2 + 1$.

For $n = 11$, $n = 13$, $n = 15$ such cyclotomic Pisot units exist (D. Boyd)

The group of beta-integers

β is a quadratic Pisot unit

β is the root > 1 of $X^2 - aX - 1$, $a \geq 1$

• $d_\beta(1) = a1$

• every number > 0 of $\mathbb{Z}[\beta]$ has a finite β-expansion

• let $\mathbb{A} = \{L, S\}$. The substitution σ_β is defined by

$$\sigma_\beta : \begin{cases} L & \mapsto & L^a S \\ S & \mapsto & L. \end{cases}$$

Addition

Proposition **3** . *Let β be the root > 1 of $X^2 - aX - 1$, $a \geq 1$. Then*

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm(1 - \frac{1}{\beta})\} \subset \frac{\mathbb{Z}_\beta}{\beta^2}$$

If x, $y \in \mathbb{Z}_\beta$ exist unique $z \in \mathbb{Z}_\beta$ and $\eta \in \{0, \pm 1\}$ such that

$$x + y = z + \eta(1 - \frac{1}{\beta})$$

Set

$$x \oplus y = z$$

Then

$$b_m \oplus b_n = b_{m+n}$$

Proposition **4** . *$\mathbb{Z}_\beta$ with addition $\oplus$ is a group isomorphic to $\mathbb{Z}$.*

Multiplication

PROPOSITION **5** . *Let β be the root > 1 of $X^2 - aX - 1$, with $a \geq 1$. Then*

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \frac{\mathbb{Z}_\beta}{\beta^2}$$

It is not possible to define a multiplication law $\otimes$ in $\mathbb{Z}_\beta$ such that $\mathbb{Z}_\beta$ and $\mathbb{Z}$ are isomorphic rings. Nevertheless it is possible to define a commutative non-associative law

$$\begin{array}{rcl} \otimes : \mathbb{Z}_\beta \times \mathbb{Z}_\beta & \to & \mathbb{Z}_\beta \\ (b_m, b_n) & \mapsto & b_{mn - a\rho_S(m)\rho_S(n)} \end{array}$$

where $\rho_S(m)$ is the number of S in the prefix of length m of the infinite word $\sigma_\beta(L)$.

PROPOSITION **6** .

1) $b_m b_n - b_m \otimes b_n \in \{0, \pm 1, \ldots, \pm a\}(1 - \frac{1}{\beta})$
2) If $b_m b_n \in \mathbb{Z}_\beta$ *then* $b_m \otimes b_n = b_m b_n$.

Remark For τ, $x \oplus y$ (resp. $x \otimes y$) is the nearest τ-integer to $x + y$ (resp. xy).

β is the root > 1 of $X^2 - aX + 1$, $a \geq 3$

• $d_\beta(1) = (a-1)(a-2)^\omega$
• every number > 0 of $\mathbb{Z}[\beta]$ has an eventually periodic β-expansion which is finite for those in $\mathbb{N}[\beta]$
• the substitution σ_β is

$$\sigma_\beta : \begin{cases} L & \mapsto & L^{a-1}S \\ S & \mapsto & L^{a-2}S. \end{cases}$$

Addition

Proposition **7** . *Let β the root > 1 of $X^2 - aX + 1$, $a \geq 3$. Then*

$$\mathbb{Z}_\beta^+ + \mathbb{Z}_\beta^+ \subset \frac{\mathbb{Z}_\beta^+}{\beta}$$

$$\mathbb{Z}_\beta^+ - Zb^+ \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}$$

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + \{0, \pm\frac{1}{\beta}\}$$

If x, $y \in \mathbb{Z}_\beta$ then exist unique $z \in \mathbb{Z}_\beta$ and $\eta \in \{0, \pm 1\}$ such that

$$x + y = z + \frac{\eta}{\beta}$$

Set

$$x \oplus y = z$$

Then

$$b_m \oplus b_n = b_{m+n}$$

PROPOSITION **8** . *$\mathbb{Z}_\beta$ with addition $\oplus$ is a group isomorphic to $\mathbb{Z}$.*

Multiplication

PROPOSITION **9** . *Let β the root > 1 of $X^2 - aX + 1$, $a \geq 3$. Then*

$$\mathbb{Z}_\beta \times \mathbb{Z}_\beta \subset \frac{\mathbb{Z}_\beta}{\beta}$$

We define a commutative non-associative law

$$\begin{array}{rcl} \otimes : \mathbb{Z}_\beta \times \mathbb{Z}_\beta & \to & \mathbb{Z}_\beta \\ (b_m, b_n) & \mapsto & b_{mn - \rho_S(m)\rho_S(n)} \end{array}$$

PROPOSITION **10** .

1) $b_m b_n - b_m \otimes b_n \in \{0, 1, \ldots, a-1\} \frac{\mathrm{sgn}(b_m b_n)}{\beta}$
2) If $b_m b_n \in \mathbb{Z}_\beta$ *then* $b_m \otimes b_n = b_m b_n$.

β is a Pisot cyclotomique cubique number, root > 1 of $X^3 - 2X^2 - X + 1$

- $d_\beta(1) = 2(01)^\omega$
- the substitution is defined on $\mathbb{A} = \{L, S, M\}$ by

$$\sigma_\beta : \begin{cases} L & \mapsto & LLS \\ S & \mapsto & M \\ M & \mapsto & LS. \end{cases}$$

Addition

PROPOSITION **11** . *Let*

$$\begin{aligned} F \quad = \quad & \pm\{0,\ -\beta^{-1},\ -2\beta^{-1},\ -3\beta^{-1},\ \beta - 3,\ 2(\beta - 3), \\ & \beta^2 - \beta - 4,\ \beta^2 - 3\beta + 2,\ 2\beta^2 - 5\beta + 1, \\ & 3\beta^2 - 7\beta,\ 2\beta^2 - 3\beta - 5,\ \beta^2 - 4\beta + 5\} \end{aligned}$$

Then

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}_\beta + F$$

If x, $y \in \mathbb{Z}_\beta$ then exist unique $z \in \mathbb{Z}_\beta$ and $f \in F$ such that

$$x + y = z + f$$

Set

$$x \oplus y = z$$

Then

$$b_m \oplus b_n = b_{m+n}$$

PROPOSITION **12** . *$\mathbb{Z}_\beta$ with addition $\oplus$ is a group isomorphic to $\mathbb{Z}$.*

Multiplication

It is possible to define a multiplication law

$$b_m \otimes b_n = b_{mn-\Delta}$$

with

$$\begin{aligned}\Delta &= 26\rho_M(m)\rho_M(n) + 9\rho_M(m)(\rho_S(n) - 2\rho_M(n)) \\ &+ 9\rho_M(n)(\rho_S(m) - 2\rho_M(m)) \\ &+ 2(\rho_S(m) - 2\rho_m(m))(\rho_S(n) - 2\rho_S(n))\end{aligned}$$

Consequences

If β is a quadratic Pisot unit, or the root of $X^3 - 2X^2 - X + 1$, then if (e_i) is a base of $\mathbb{R}^d$

$$\Lambda = \sum_{i=1}^{d} \mathbb{Z}_\beta e_i$$

is a Meyer set and a lattice for the law $\oplus$.

Physical application The Fibonacci chain has a discrete labelling

$$\begin{aligned}
\mathcal{F} &= \{x \in \mathbb{Z}[\tau] \mid x' \in [0,1)\} \\
&= \{x \in \mathbb{Z}_\tau \mid x' \in [0,1)\} \\
&= \{\dots, -\tau^3, -\tau, 0, \tau^2, \tau^3 + 1, \tau^4, \dots\} \\
&= \{\dots, b_{-5}, b_{-2}, b_0, b_3, b_6, b_8, \dots\}
\end{aligned}$$

Sturmians words

An infinite word is Sturmian if the number of factors of length n is equal to $n+1$. It is defined on a 2 letter alphabet $\mathbb{A} = \{L, S\}$.

Equivalently, an infinite word is Sturmian if, for every two factors of length n, the difference between the number of S is bounded by 1.

Let

$$\sigma_\tau : \begin{cases} L & \mapsto & LS \\ S & \mapsto & L \end{cases}$$

$$\widetilde{\sigma_\tau} : \begin{cases} L & \mapsto & SL \\ S & \mapsto & L \end{cases}$$

$$E : \begin{cases} L & \mapsto & S \\ S & \mapsto & L. \end{cases}$$

PROPOSITION **13** . *If* β *is the root* > 1 *of* $X^2 - aX - 1$, $a \geq 1$, *then* $\sigma_\beta = \sigma_\tau (E\sigma_\tau)^{a-1}$.
If β *is the root* > 1 *of* $X^2 - aX + 1$, $a \geq 3$, *then* $\sigma_\beta = \widetilde{\sigma_\tau}\sigma_\tau (E\sigma_\tau)^{a-3}$.

THEOREM **4** . *If* β *is quadratic Pisot unit, the infinite word generated by* σ_β *is Sturmian.*

Beta-grids

If β is a cyclotomic Pisot number with order n symmetry, and $\zeta = \exp(2i\pi/n)$, then $\mathbb{Z}[\beta] + \mathbb{Z}[\beta]\zeta \equiv \mathbb{Z}[\zeta]$ is a ring invariant under rotation of order n.

Beta-grid

$$\Gamma_q = \mathbb{Z}_\beta + \mathbb{Z}_\beta \zeta^q$$

for $1 \leq q \leq n-1$.

Let

$$\Lambda_q = \bigcup_{j=0}^{n-1} \Gamma_q \zeta^j$$

$$\mathbb{Z}_\beta[\zeta] = \sum_{j=0}^{n-1} \mathbb{Z}_\beta \zeta^j$$

Proposition 14 . *$\mathbb{Z}_\beta[\zeta]$ and Λ_q for $1 \leq q \leq n-1$ are Meyer set.*

The τ-grid

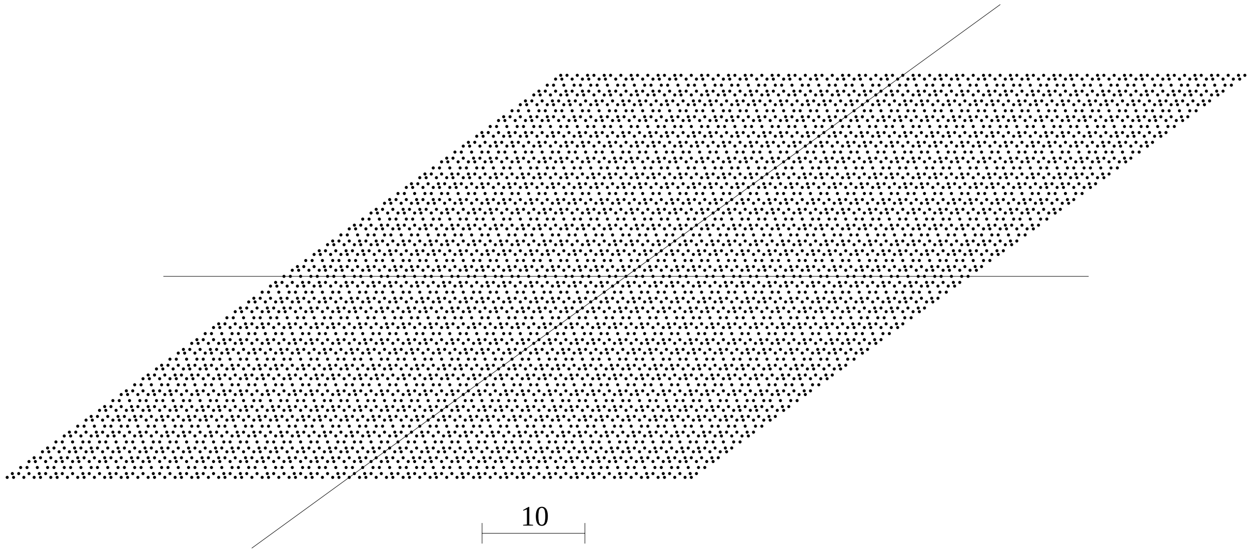


Figure 1: τ-grid Γ_1

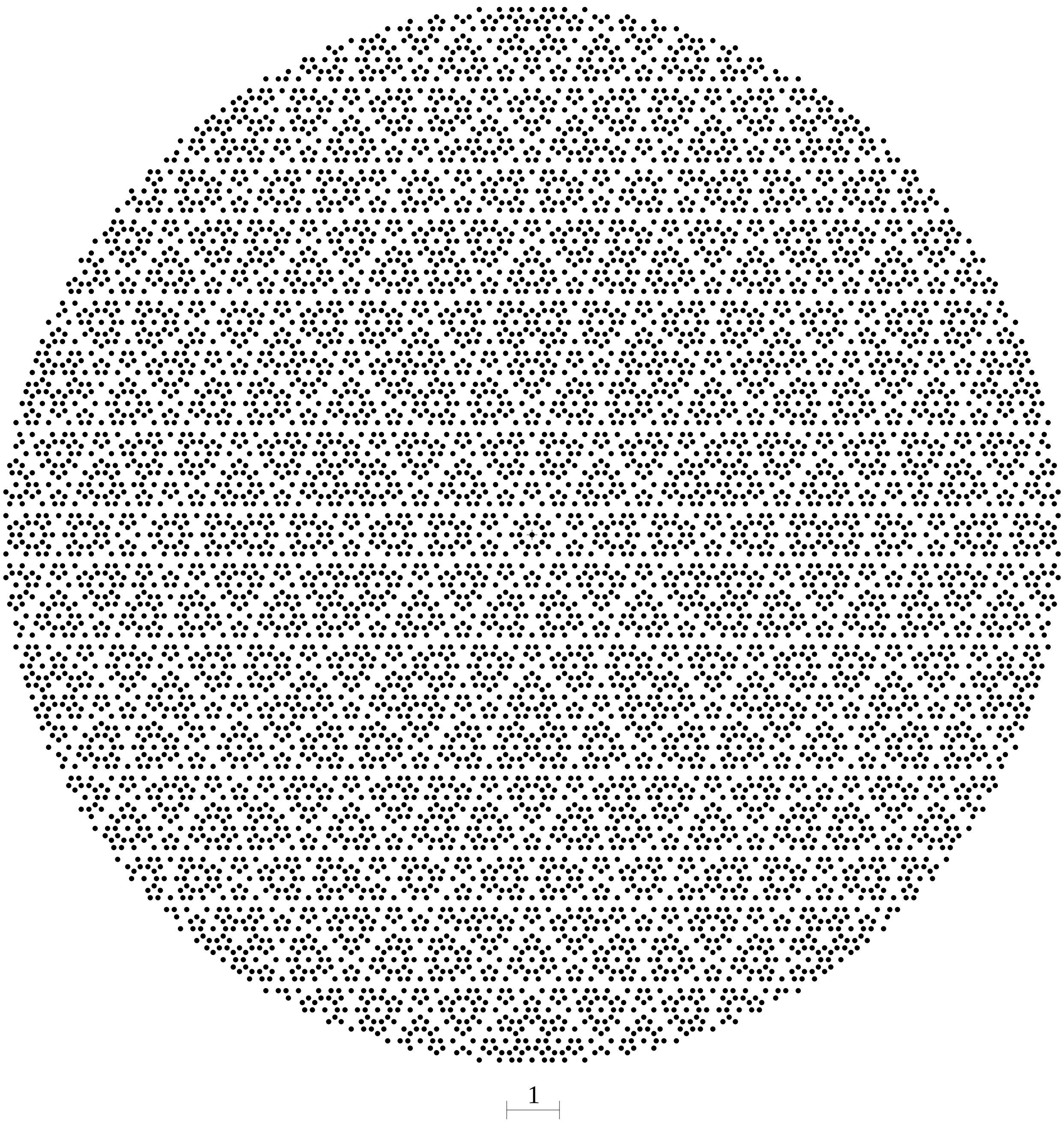


Figure 2: Cyclotomic quasiring of order 5 $\mathbb{Z}_\tau[\exp(i\pi/5)]$

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