M.F.C.S.'99
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On-Line Addition in Real Base

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Beta-Representations

Base β integer > 1, digit set $A = \{0, \ldots, \beta - 1\}$.
A^* = Finite sequences $\longleftrightarrow$ positive integers.
$A^{\mathbb{N}}$ = Infinite sequences $\longleftrightarrow$ positive real numbers.

Base β a real number > 1. A *β-representation* of a number x of $[0, 1]$ is an infinite sequence $(d_k)_{k \geq 1}$ such that $\sum_{k \geq 1} d_k \beta^{-k} = x$.

Any real number $x \in [0, 1]$ can be represented in base β by a greedy algorithm [Rényi 57]:
$x_1 = \lfloor \beta x \rfloor$ and $r_1 = \{\beta x\}$.
For $k \geq 2$, $x_k = \lfloor \beta r_{k-1} \rfloor$ and $r_k = \{\beta r_{k-1}\}$.
Thus

$$x = \sum_{k \geq 1} x_k \beta^{-k}$$

where x_k are in the *canonical* alphabet
$A = \{0, \ldots, \lfloor \beta \rfloor\}$ if $\beta \notin \mathbb{N}$,
$A = \{0, \ldots, \beta - 1\}$ if $\beta \in \mathbb{N}$.

The sequence $(x_k)_{k\geq 1}$ of $A^{\mathbb{N}}$ obtained by the greedy algorithm is called the β-*expansion* of x.

When β is not an integer, a number x may have several different β-representations on A: this system is naturally redundant.

EXAMPLE **1** . $\beta = \frac{1+\sqrt{5}}{2}$, $A = \{0, 1\}$.

$$\begin{aligned} 3-\sqrt{5} \quad &=_\beta \quad 10010^\omega \\ &=_\beta \quad 01110^\omega \\ &=_\beta \quad 100(01)^\omega \end{aligned}$$

Applications:

- coding: digital data recording and transmission
- quasicrystal

in particular for $\beta = \frac{1+\sqrt{5}}{2}$.

On-line Arithmetic

To pipe-line additions/subtractions, multiplications and divisions, computations are to be done Most Sigificant Digit First, *i.e.* from left to right.

Similarly for the processing of real numbers (infinite expansions).

Additional requirement: deterministic processing and, after a certain delay δ of latency, for one input digit there is one output digit.
To generate the jth digit of the result, it is necessary and sufficient to have the first $(j + \delta)$ digits of the input available.

[Ercegovac and Trivedi, 77]

Remark **1** . *Addition base 2 on* $\{0,1\}$ *is not on-line computable.*

$01^n10^\omega + 00^n10^\omega = 10^\omega$ and
$01^n10^\omega + 0^\omega = 01^n10^\omega$.
The least significant digits have to be known to be able to output the most significant digit of the result.

Remark **2** . *Addition (multiplication) base 2 is on-line computable only if the representation of numbers is redundant.*

Remark **3** . *Addition base 2 on* $\{-1,0,1\}$ *is on-line computable* [Avizienis 61], [Chow and Robertson 78].

In fact it is computable in constant time in parallel.
Used for the construction of fast multipliers.

Remark **4** . *Multiplication base 2 on* $\{-1,0,1\}$ *is on-line computable* [Ercegovac and Trivedi, 77].

Digit Set Conversion

$D \subset \mathbb{R}$ a digit set.

Digit set conversion in base β from D to A is a function $\chi : D^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ such that for each sequence $(d_k)_{k\geq 1} \in D^{\mathbb{N}}$ where $x = \sum_{k\geq 1} d_k \beta^{-k}$ belongs to $[0, 1]$, there exists a sequence $(a_k)_{k\geq 1} \in A^{\mathbb{N}}$ such that $x = \sum_{k\geq 1} a_k \beta^{-k}$.

$\chi((d_k)_{k\geq 1})$ belongs to $A^{\mathbb{N}}$, but *need not* be the β-expansion of x as computed by the greedy algorithm (it is a nonnormalized β-representation).

Addition in base β:
$v = \sum_{k \geq 1} v_k \beta^{-k}$ and $y = \sum_{k \geq 1} y_k \beta^{-k}$ with v_k and y_k in $A = \{0, \ldots, \lfloor \beta \rfloor\}$, such that $v + y \in [0, 1]$.
$z_k = v_k + y_k \in B = \{0, \ldots, 2\lfloor \beta \rfloor\}$, and $v + y = \sum_{k \geq 1} z_k \beta^{-k}$.

Nonnormalized addition = conversion from B to A.

Nonnormalized multiplication by a fixed integer $m \geq 1$ = conversion from $\{0, \ldots, m\lfloor \beta \rfloor\}$ to A.

On-Line Digit Set Conversion in Real Base β

$A = \{0, \ldots, \lfloor \beta \rfloor\}$ canonical alphabet associated with β,
$D = \{0, \ldots, d\}$ with $d \geq \lfloor \beta \rfloor$.

THEOREM **1** . *There exists a digit set conversion $\chi : D^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ in base β which is on-line computable with delay δ, where δ is the smallest positive integer such that*

$$\beta^{\delta+1} + d \leq \beta^{\delta}(\lfloor \beta \rfloor + 1) \qquad (*)$$

Let k be the smallest integer ≥ 0 such that $d/\beta^k(\beta - 1) \leq 1$. To avoid overflow, all input words begin with a run of k zeroes.

On-Line Algorithm

Input: a sequence $(d_j)_{j\geq 1} \in D^{\mathbb{N}}$ such that $d_1 = \cdots = d_k = 0$.

Output: a sequence $(a_j)_{j\geq 1} \in A^{\mathbb{N}}$ such that $\sum_{j\geq 1} a_j \beta^{-j} = \sum_{j\geq 1} d_j \beta^{-j}$.

begin

$\quad q_0 \leftarrow 0$

$\quad$ **for** $i \leftarrow 1$ to δ **do**

$\quad\quad q_i \leftarrow \beta q_{i-1} + d_i$

$\quad j \leftarrow 1$

$\quad$ **while** $j \geq 1$ **do**

$\quad\quad z_{\delta+j} \leftarrow \beta q_{\delta+j-1} + d_{\delta+j}$

$\quad\quad$ **if** $z_{\delta+j} < \beta^{\delta+1}$

$\quad\quad\quad$ **then** $a_j \leftarrow \lfloor z_{\delta+j}/\beta^{\delta} \rfloor$

$\quad\quad\quad$ **else** $a_j \leftarrow \lfloor \beta \rfloor$

$\quad\quad q_{\delta+j} \leftarrow z_{\delta+j} - \beta^{\delta} a_j$

$\quad\quad j \leftarrow j + 1$

end

Proof of the algorithm.

Claim: For all $j \geq 1$, $0 \leq q_j < \beta^\delta$ and $a_j \in A$.

1. For $1 \leq i \leq \delta$,

$$q_i = \beta^{i-1} d_1 + \cdots + d_i$$

Then $q_i \leq q_\delta < \beta^\delta$ by hypothesis on the input $(d_j)_{j \geq 1}$.

2. Suppose that for some $j \geq 1$

$$0 \leq q_{\delta+j-1} < \beta^\delta \tag{H}$$

• If $z_{\delta+j} < \beta^{\delta+1}$ then $a_j = \lfloor z_{\delta+j}/\beta^\delta \rfloor$. Thus $0 \leq a_j < \beta$. Then $q_{\delta+j} = \beta^\delta \{z_{\delta+j}/\beta^\delta\} < \beta^\delta$.
• If $z_{\delta+j} \geq \beta^{\delta+1}$ then $a_j = \lfloor \beta \rfloor$ and $q_{\delta+j} = z_{\delta+j} - \lfloor \beta \rfloor \beta^\delta$.
Then $q_{\delta+j} \geq \beta^{\delta+1} - \lfloor \beta \rfloor \beta^\delta \geq 0$.
On the other hand,
$q_{\delta+j} = \beta q_{\delta+j-1} + d_{\delta+j} - \lfloor \beta \rfloor \beta^\delta < \beta^{\delta+1} + d - \lfloor \beta \rfloor \beta^\delta$
by hypothesis (H), thus by condition (*),
$q_{\delta+j} < \beta^\delta$.

Then, for all $j \geq 1$,

$$\frac{d_1}{\beta} + \cdots + \frac{d_{\delta+j}}{\beta^{\delta+j}} = \frac{a_1}{\beta} + \cdots + \frac{a_j}{\beta^j} + \frac{q_{\delta+j}}{\beta^{\delta+j}}.$$

Since $q_{\delta+j} < \beta^\delta$, when $j \to \infty$,

$$\sum_{j \geq 1} d_j \beta^{-j} = \sum_{j \geq 1} a_j \beta^{-j}$$

with digits a_j in A, thus $\chi((d_j)_{j\geq 1}) = (a_j)_{j\geq 1}$.

Continuity

Distance ρ on $D^{\mathbb{N}}$:
let $v = (v_i)_{i\geq 1}$ and $w = (w_i)_{i\geq 1}$ be in $D^{\mathbb{N}}$,
$\rho(v, w) = 2^{-r}$ where $r = \min\{i \mid v_i \neq w_i\}$.

The set $D^{\mathbb{N}}$ is a compact metric space. This topology is equivalent to the product topology.

PROPOSITION **1** . *Let D and A be two finite alphabets. Any function $\varphi : D^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ which is on-line computable with delay δ is 2^{δ}-lipschitzian, and is thus uniformously continuous.*

COROLLARY **1** . *Let D be a set of nonnegative digits containing A. The digit set conversion $\chi : D^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ in base β defined in Theorem 1 is uniformously continuous.*

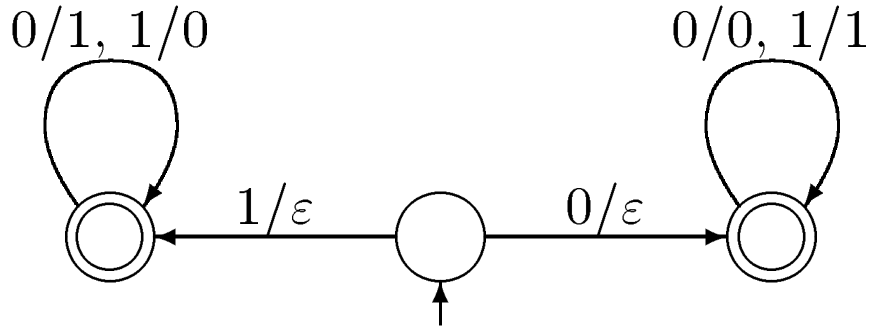
On-Line Finite State Automaton

On-line finite state automaton with delay δ is a sequential transducer:

- input deterministic (from left to right)
- transient part: it reads without writing during a time δ
- synchronous part: after the delay, for one input letter there is one output letter.

Introduced by J.M. Muller [94].

EXAMPLE **2** . On-line finite state automaton with delay 1, computes the tent function on $[0, 1]$ in base 2.



EXAMPLE **3** . Addition base 2 on $\{-1, 0, 1\}$ is computable by an on-line finite automaton.

The Pisot Case

An *algebraic integer* is a root of a monic polynomial with integral coefficients. A *Pisot number* is an algebraic integer > 1 such that all its algebraic conjugates are < 1 in modulus.

EXAMPLE **4** . The integers and the golden mean $\frac{1+\sqrt{5}}{2}$ are Pisot numbers.

THEOREM **2** . *Let β be a Pisot number, let $A = \{0, \ldots, \lfloor \beta \rfloor\}$, and let $D = \{0, \ldots, d\}$ such that $d \geq \lfloor \beta \rfloor$. There exists an an on-line finite state automaton with delay δ, where δ satisfies* (*)*, which realizes a digit set conversion $\chi : D^{\mathbb{N}} \longrightarrow A^{\mathbb{N}}$ in base β.*

$M(X)$ minimal polynomial of β, of degree m, and let $\beta_1 = \beta$, β_2, $\ldots$, β_m be its roots. For $2 \leq i \leq m$, $|\beta_i| < 1$.
$\Lambda = \mathbb{Z}[X]/(M(X))$ is a discrete lattice of rank m.

Define

$$\Lambda_\delta = \begin{cases} \Lambda \ \text{ if } m \leq \delta \\ \{q(X) = z_{\delta-1}X^{\delta-1} + \cdots + z_{\delta-m}X^{\delta-m} \mid \\ X^{m-\delta}q(X) \in \Lambda\} \text{ otherwise} \end{cases}$$

Norm of $q \in \Lambda_\delta$: $\|q\| = \max_{1 \leq i \leq m} |q(\beta_i)|$.
For $2 \leq i \leq m$ set

$$\gamma_i = \sup\{|c - a\beta_i^\delta| \ \mid c \in D, \ a \in A\}.$$

Construction of an on-line finite state automaton $\mathcal{A} = (Q, D \times (A \cup \varepsilon), E, \{q_0\})$:
The set of synchronous states is equal to $Q_s = \{q(X) \in \Lambda_\delta \mid 0 \leq q(\beta) < \beta^\delta$ and for $2 \leq i \leq m, \ |q(\beta_i)| < \frac{\gamma_i}{1-|\beta_i|}\}$.
For any q in Q_s, $\|q\|$ is bounded, so Q_s is a finite set.

The set of transient states Q_t is defined by
$Q_t = \{q_j(X) = d_1 X^{j-1} + \cdots + d_j \mod M \mid 1 \leq j \leq \delta - 1,\ d_1, \ldots, d_j \in D\} \cup \{q_0\}$.

If $q_j \in Q_t$ then $q_j(\beta) < \beta^\delta$, and for $2 \leq i \leq m$, $|q_j(\beta_i)| < d/(1 - |\beta_i|) \leq \gamma_i/(1 - |\beta_i|)$. Hence transient states satisfy the same bound inequalities as synchronous states.

For $1 \leq i \leq \delta$, transient edges are defined by

$$q_{i-1}(X) \xrightarrow{d_i/\varepsilon} q_i(X)$$

with $q_0(X) = 0$ and $q_i(X) = X q_{i-1}(X) + d_i$.

Synchronous edges: for $j \geq 1$ and $q_{\delta+j-1}(X) \in Q$,

$$q_{\delta+j-1}(X) \stackrel{d_{\delta+j}/a_j}{\longrightarrow} q_{\delta+j}(X)$$

where

$$X q_{\delta+j-1}(X) + d_{\delta+j} = X^\delta a_j + q_{\delta+j}(X) \mod M(X)$$

with a_j in A as in the on-line algorithm of Theorem 1, replacing X by β. Hence for all $j \geq 0$, $0 \leq q_{\delta+j}(\beta) < \beta^\delta$.

For $2 \leq i \leq m$, we get

$$\begin{array}{rcl} |q_{\delta+j}(\beta_i)| & = & |\beta_i q_{\delta+j-1}(\beta_i) + d_{\delta+j} - a_j \beta_i^\delta| \\ & < & |\beta_i| \dfrac{\gamma_i}{1-|\beta_i|} + \gamma_i = \dfrac{\gamma_i}{1-|\beta_i|}. \end{array}$$

Thus, for all $j \geq 0$, $q_{\delta+j} \in Q_s$.

For any infinite path in $\mathcal{A}$

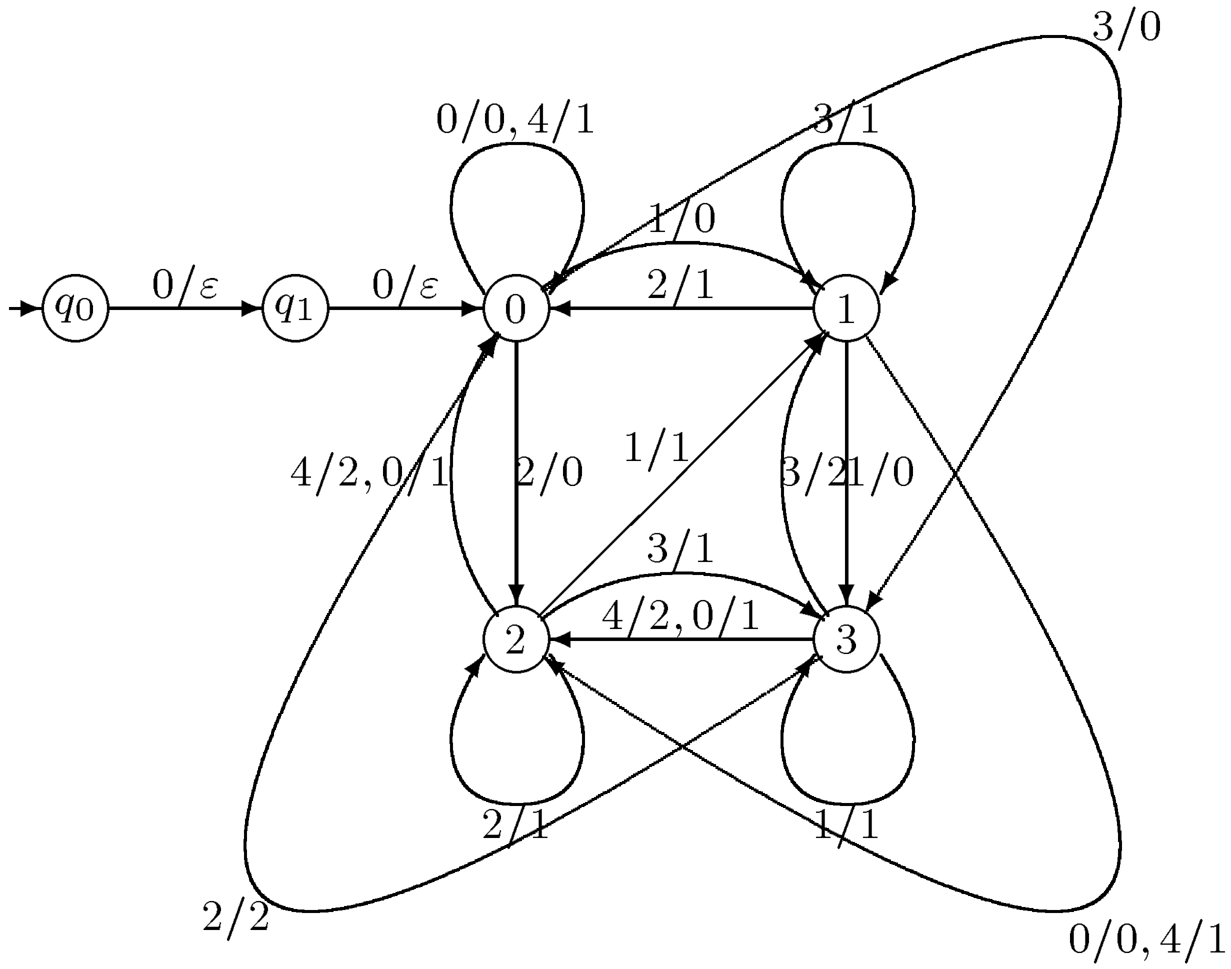
$$q_0 \stackrel{d_1/\varepsilon}{\longrightarrow} q_1 \cdots \stackrel{d_\delta/\varepsilon}{\longrightarrow} q_\delta \stackrel{d_{\delta+1}/a_1}{\longrightarrow} q_{\delta+1} \stackrel{d_{\delta+2}/a_2}{\longrightarrow} q_{\delta+2} \cdots$$

then $\sum_{j\geq 1} d_j \beta^{-j} = \sum_{j \geq 1} a_j \beta^{-j}$, with the digits a_j in A, and $\chi((d_j)_{j\geq 1}) = (a_j)_{j \geq 1}$.

EXAMPLE **5** . β integer ≥ 2. On $A = \{0, \cdots, \beta\}$, the representation of numbers is redundant. Addition on A is computable by an on-line finite state automaton, with delay 2.

For $\beta = 2$, this representation is the well-known carry-save representation, used for multiplication.

On-line finite automaton for addition in base $\beta = 2$: $\mathcal{A} = (Q, \{0, \ldots, 4\} \times (\{0, 1, 2\} \cup \varepsilon), E, q_0)$, with $Q = Q_t \cup Q_s$. All input words begin with 00.
Transient states $Q_t = \{q_0, q_1\}$.
Synchronous states $Q_s = \{q(X) \in \mathbb{Z}[X]/(X - 2) \mid 0 \leq q(2) < 4\} = \{0, 1, 2, 3\}$.



On-line automaton with delay 2 for addition base 2 on $\{0, 1, 2\}$

EXAMPLE **6** . $\beta = \frac{1+\sqrt{5}}{2}$, $A = \{0,1\}$.

Formula (*) gives $\delta = 4$ for addition, with $D = \{0,1,2\}$. The on-line automaton constructed with delay 4 is not minimal in the number of states, but it is equivalent to an on-line automaton with delay 3. The construction given in Theorem 1 and in Theorem 2 works with delay $\delta' = 3$, with bound on the states

$$0 \leq q(\beta) \leq \mu$$

with $\mu = \beta + 2$, and Condition (*) being replaced by

$$\beta\mu + d \leq \lfloor \beta \rfloor \beta^{\delta'} + \mu \qquad (**)$$

Let $\mathcal{A} = (Q, \{0,1,2\} \times (\{0,1\} \cup \varepsilon), E, \{q_0\})$. Input words begin with 00. Elements of Λ are denoted by words of length 2.

The transient part of $\mathcal{A}$ is of the form

$q_0 \xrightarrow{0/\varepsilon} q_1 \xrightarrow{0/\varepsilon} q_2$

$q_2 \xrightarrow{0/\varepsilon} 00$; $\quad q_2 \xrightarrow{1/\varepsilon} 01$; $\quad q_2 \xrightarrow{2/\varepsilon} 02$.

	00	01	02	11	12	10	$\bar{1}2$	03	$1\bar{1}$	20
00	0/0	1/0	2/0							
01				1/0	2/0	0/0				
02	1/1	2/0								0/0
11	0/1	1/1	2/1							
12				1/1	2/1	0/1				
10				0/0	1/0		2/1			
$\bar{1}2$				2/0		1/0			0/0	
03				2/1		1/1			0/1	
$1\bar{1}$		0/0	1/0					2/0		
20		0/1	1/1					2/1		

Synchronous part of $\mathcal{A}$, on-line automaton for addition base $\frac{1+\sqrt{5}}{2}$