Two applications of the spectrum of numbers

Christiane Frougny* and Edita Pelantová†

Abstract

Let the base $\beta$ be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet of digits. The $A$-spectrum of $\beta$ is the set

$$S_A(\beta) = \{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, a_k \in A \}.$$  

We show that the spectrum $S_A(\beta)$ has an accumulation point if and only if 0 has a particular $(\beta, A)$-representation, said to be rigid.

The first application is restricted to the case that $\beta > 1$ and the alphabet is $A = \{-M, \ldots, M\}$, $M \geq 1$ integer. We show that the set $Z_{\beta, M}$ of infinite $(\beta, A)$-representations of 0 is recognizable by a finite Büchi automaton if and only if the spectrum $S_A(\beta)$ has no accumulation point. Using a result of Akiyama-Komornik and Feng, this implies that $Z_{\beta, M}$ is recognizable by a finite Büchi automaton for any positive integer $M \geq \lceil |\beta| \rceil - 1$ if and only if $\beta$ is a Pisot number. This improves the previous bound $M \geq \lceil |\beta| \rceil$.

For the second application the base and the digits are complex. We consider the on-line algorithm for division of Trivedi and Ercegovac generalized to a complex numeration system. In on-line arithmetic the operands and results are processed in a digit serial manner, starting with the most significant digit. The divisor must be far from 0, which means that no prefix of the $(\beta, A)$-representation of the divisor can be small. The numeration system $(\beta, A)$ is said to allow preprocessing if there exists a finite list of transformations on the divisor which achieve this task. We show that $(\beta, A)$ allows preprocessing if and only if the spectrum $S_A(\beta)$ has no accumulation point.

Key words: spectrum, Pisot number, Büchi automaton

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1 Introduction

The so-called beta-numeration has been introduced by Rényi in [21] and studied by Parry in [20] in the case that $\beta$ is a real number, $\beta > 1$, and since then there have been many works in this domain, in connection with number theory, dynamical systems, and automata theory, see the survey [12] or more recent [22] for instance.

For $\beta > 1$ and $M \geq 1$ an integer, the following spectrum

$$X_M(\beta) = \{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, a_k \in \{0, 1, \ldots, M\} \}$$

has been introduced by Erdös, Joó and Komornik [8].

Since $X_M(\beta)$ is discrete its elements can be arranged into an increasing sequence

$$0 = x_0 < x_1 < \cdots$$

Denote $\ell_M(\beta) = \lim \inf_{k \to \infty} (x_{k+1} - x_k)$. Numerous works have been devoted to the study of this value, see in particular the introduction and the results of [1].

*IRIF, UMR 8243 CNRS and Université Paris-Diderot
†Doppler Institute for Mathematical Physics and Applied Mathematics, and Department of Mathematics, FNSPE, Czech Technical University in Prague
More generally, let $\beta$ be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet of digits. The $A$-spectrum of $\beta$ is the set

$$S_A(\beta) = \{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, a_k \in A \}.$$ 

Recently Feng answered an open question raised in [8], see also [1], on the density of the spectrum of $\beta$ when $\beta$ is real and the digits are consecutive integers:

**Theorem 1.1** ([9]). Let $\beta > 1$ and let $A = \{-M, \ldots, M\}$, $M$ an integer $\geq 1$. Then the spectrum $S_A(\beta)$ is dense in $\mathbb{R}$ if and only if $\beta < M + 1$ and $\beta$ is not a Pisot number.

Feng has obtained the following corollary: $\ell_M(\beta) = 0$ if and only if $\beta < M + 1$ and $\beta$ is not a Pisot number.

In this paper we use the concept of spectrum of a number to solve two problems arising in beta-numeration.

Let $\beta$ and the digits of $A$ be complex. The topological properties of the spectrum are linked with a particular representation of 0. Let $z_1z_2\cdots$ be a $(\beta, A)$-representation of 0, that is to say, $\sum_{i \geq 1} z_i \beta^{-i} = 0$. It is said to be **rigid** if $0.\overline{z_1z_2\cdots z_j}$ for all $j \geq 2$ and for all $z_2\cdots z_j$ in $A^*$. The term “rigid” comes from the preprocessing motivation, see Section 5.

We first prove that the spectrum $S_A(\beta)$ has an accumulation point if and only if 0 has a rigid $(\beta, A)$-representation, Theorem 3.5.

Then we obtain some results when the base is a complex Pisot number, which extend the real case covered by Garsia [13]. Let $\beta$ be a complex number, and let $A \subset \mathbb{Q}(\beta)$ containing 0. If $\beta$ is real and if $\beta$ or $-\beta$ is a Pisot number, or if $\beta \in \mathbb{C} \setminus \mathbb{R}$ is a complex Pisot number then $S_A(\beta)$ has no accumulation point, Theorem 3.6.

The first question we address in this work is the one of the recognizability by a finite B"uchi automaton of the set of infinite $\beta$-representations of 0 when $\beta$ is a real number and the digits are integer.

The set of infinite $\beta$-representations of 0 on the alphabet $\{-M, \ldots, M\}$, $M \geq 1$ integer, is denoted

$$Z_{\beta,M} = \{ z_1z_2\cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, z_i \in \{-M, \ldots, M\} \}.$$ 

The following result has been formulated in [12]:

**Theorem 1.2.** Let $\beta > 1$. The following conditions are equivalent:

1. the set $Z_{\beta,M}$ is recognizable by a finite B"uchi automaton for every integer $M$,

2. the set $Z_{\beta,M}$ is recognizable by a finite B"uchi automaton for one integer $M \geq \lceil \beta \rceil$,

3. $\beta$ is a Pisot number.

(3) implies (1) is proved in [10], (1) implies (3) is proved in [2] and (2) implies (1) is proved in [11].

Note that in [7] Bugeaud has shown, using (1) implies (3) of Theorem 1.2, that if $\beta$ is not a Pisot number then there exists an integer $M$ such that $\ell_M(\beta) = 0$.

In this paper we first prove that the set $Z_{\beta,M}$ is recognizable by a finite B"uchi automaton if and only if the spectrum $S_A(\beta)$ has no accumulation point, Theorem 4.2.
By [1] or [2] it is known that, for $A = \{-M, \ldots, M\}$, the spectrum $S_A(\beta)$ has an accumulation point if and only if $\beta < M + 1$ and $\beta$ is not Pisot.

This result together with Theorem 4.2 proves the conjecture stated in [12]: If the set $Z_{\beta, [\beta] - 1}$ is recognizable by a finite Büchi automaton then $\beta$ must be a Pisot number.

Moreover we obtain a simpler proof of the implication $2 \Rightarrow 3$ of Theorem 1.2. Note that the value $M = [\beta] - 1$ is the best possible as $Z_{\beta, M}$ is reduced to $\{0^\omega\}$ if $M < [\beta] - 1$.

Normalization in base $\beta$ is the function which maps any $\beta$-representation on the canonical alphabet $A_\beta = \{0, \ldots, [\beta] - 1\}$ of a number $x \in [0, 1]$ onto the greedy $\beta$-expansion of $x$. Since the set of greedy $\beta$-expansions of the elements of $[0, 1]$ is computable by a finite Büchi automaton when $\beta$ is a Pisot number, see [4], the following result holds true:

Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if $\beta$ is a Pisot number.

The second utilisation of the notion of spectrum occurs in the on-line algorithm for division in a complex base.

On-line arithmetic, introduced in [25] for an integer base, is a mode of computation where operands and results are processed in a digit serial manner, starting with the most significant digit. To generate the first digit of the result, the first $\delta$ digits of the operands are required. The integer $\delta$ is called the delay of the algorithm. One of the interests of the functions that are on-line computable is that they are continuous for the usual topology on the set of infinite words on a finite alphabet.

In [5, 6] we have extended the original on-line algorithm of Trivedi-Ercegovac to a complex base. The algorithm for on-line division in a complex numeration system $(\beta, A)$ has two parameters: the delay $\delta \in \mathbb{N}$ and $D > 0$, the minimal value (in modulus) of the divisor.

When making division, we need that the divisor stays away from 0. By definition of the on-line algorithm, this means that the value of all the prefixes of the divisor $d_1 d_2 \cdots$ must be greater in absolute value than $D > 0$, so the divisor must be preprocessed before making the division.

We say that a complex numeration system $(\beta, A)$ allows preprocessing if there exists a finite list of transformations on the $(\beta, A)$-representation of the divisor which achieve this task, see Definition 5.1.

We show that a complex numeration system $(\beta, A)$ allows preprocessing if and only if the spectrum $S_A(\beta)$ has no accumulation point, Theorem 5.4.

2 Preliminaries

2.1 Words and automata

Let $A$ be a finite alphabet. A finite word $w$ on $A$ is a finite concatenation of letters from $A$, $w = w_1 \cdots w_n$, with $w_i$ in $A$. The set of all finite words over $A$ is denoted by $A^*$. An infinite word $w$ on $A$ is an infinite concatenation of letters from $A$, $w = w_1 w_2 \cdots$ with $w_i$ in $A$. The set of all infinite words over $A$ is denoted by $A^\infty$. The infinite concatenation $uuu \cdots$ is noted $u^\omega$. If $w = uv$, $u$ is a prefix of $w$.

An automaton $A = (A, Q, I, T)$ over the alphabet $A$ is a directed graph labeled by letters of $A$, with a denumerable set $Q$ of vertices called states. $I \subseteq Q$ is the set of initial states, and $T \subseteq Q$ is the set of terminal states. The automaton is said to be finite if the set of states $Q$ is finite.

An infinite path of $A$ is said to be successful if it starts in $I$ and goes infinitely often through $T$. The set of infinite words recognized by $A$ is the set of labels of
its successful infinite paths. An automaton used to recognize infinite words in this sense is called a Büchi automaton.

### 2.2 Numeration

Let $\beta$ be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite set, the alphabet of digits. We say that $(\beta, A)$ is a numeration system. A $(\beta, A)$-representation of a number $z$ is an infinite word $z_1z_2\cdots$ such that $z = \sum_{i=1}^{+\infty} z_i \beta^{-i}$ with $z_i \in A$. It should be noted that here we do not make any hypothesis on the fact that every complex number has, or does not have, a $(\beta, A)$-representation. This is a difficult problem, studied by many authors, see the pioneering works of Knuth [17], Kátai and Kovács [16], Gilbert [14], Thurston [24] for instance.

We now recall some definitions and results on the so-called beta-numeration, see [12] or [22] for a survey. Let $\beta > 1$ be a real number. Any real number $x \in [0, 1]$ can be represented by a greedy algorithm as $x = \sum_{i=1}^{+\infty} x_i \beta^{-i}$ with $x_i$ in the canonical alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor - 1\}$ for all $i \geq 1$. The greedy sequence $(x_i)_{i \geq 1}$ corresponding to a given real number $x$ is the greatest in the lexicographical order, and is said to be the $\beta$-expansion of $x$, see [21]. It is denoted by $d_\beta(x) = (x_i)_{i \geq 1}$. When the expansion ends in infinitely many 0's, it is said to be finite, and the 0's are omitted.

The greedy $\beta$-expansion of 1 is denoted $d_\beta(1) = (t_i)_{i \geq 1}$. When it is finite, of the form $d_\beta(1) = t_1 \cdots t_m$, the quasi-greedy $\beta$-expansion of 1 is defined as $d'_\beta(1) = (t_1 \cdots t_{m-1} (t_m - 1))$. If it is infinite, set $d'_\beta(1) = d_\beta(1)$. The sequence $d'_\beta(1)$ is the lexicographically greatest infinite representation of 1 in the base $\beta$ and the alphabet $\mathbb{N}$. It is known from [20] that a sequence of integers $x_1x_2\cdots$ is the greedy $\beta$-expansion of some $x$ from $[0, 1]$ if and only if, for all $j \geq 1$, $x_jx_{j+1}\cdots$ is less than or equal to $d'_\beta(1)$ in the lexicographic order.

Notation: The numerical value $y_{m-1}\beta^{m-1} + \cdots + y_0 + y_{-1}\beta^{-1} + y_{-2}\beta^{-2} + \cdots$ is denoted by $y_{m-1}\cdots y_0y_{-1}y_{-2}\cdots$.

### 2.3 Numbers

A number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic is a Parry number. It is a simple Parry number if $d_\beta(1)$ is finite.

A Pisot number is an algebraic integer greater than 1 such that all its Galois conjugates have modulus less than 1. Every Pisot number is a Parry number, see [3] and [22].

A complex Pisot number is an algebraic integer $\beta$ such that $|\beta| > 1$ and such that all its Galois conjugates different from its complex conjugate $\overline{\beta}$ have modulus less than 1.

### 3 Spectrum and rigid representation of 0

Let $\beta$ be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet. We introduce the $A$-spectrum of $\beta$ as

$$S_A(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, a_k \in A \right\}.$$ 

The topological properties of $S_A(\beta)$ are linked with a particular representation of 0.
Definition 3.1. Let $z_1z_2\cdots$ be a $\beta$-representation of 0 on $A$, that is to say, 
$\sum_{i>0} z_i\beta^{-i} = 0$. It is said to be rigid if $0.z_1z_2\cdots z_j \neq 0.0z'_1\cdots z'_j$ for all $j \geq 2$ and for all $z'_1\cdots z'_j$ in $A^*$. 

Example 3.2. The signed digit $(-1)$ is denoted $\overline{1}$. In base 2 with alphabet $\{\overline{1}, 0, 1\}$, 0 has two representations, namely $0 = 0.\overline{1}T\overline{1}T\overline{1}T\cdots = 0.\overline{1}T1111\cdots$. They are not rigid, since $0.1\overline{1}T = 0.01$ and $0.\overline{1}T = 0.0\overline{1}T$. 

Definition 3.3. Let $z_1z_2\cdots$ be a $(\beta, A)$-representation of 0. For $n \in \mathbb{N}$, its $n$-th tail is $r_n = 0.z_{n+1}z_{n+2}z_{n+3}\cdots$.

Lemma 3.4. Let $z_1z_2\cdots$ be a $(\beta, A)$-representation of 0.

1. If the sequence $(r_n)_{n \in \mathbb{N}}$ is injective, then the spectrum $S_A(\beta)$ has an accumulation point.

2. If the representation of 0 is rigid, then the sequence $(r_n)_{n \in \mathbb{N}}$ is injective.

Proof. Since $0 = 0.z_1z_2z_3\cdots$, the $n$th tail $r_n = \sum_{k=1}^{+\infty} z_{n+k}\beta^{-k} = -\sum_{k=0}^{n-1} z_{n-k}\beta^k$. It means that $-r_n$ belongs to the spectrum $S_A(\beta)$ and moreover

$$|r_n| \leq \frac{\alpha}{|\beta|-1}, \quad \text{where } \alpha = \max\{|a| : a \in A\}.$$ 

1) If the sequence $(r_n)_{n \in \mathbb{N}}$ is injective, then the ball centered at 0 with radius $\frac{\alpha}{|\beta|-1}$ contains infinitely many elements $(-r_n)$ of the spectrum, and thus the spectrum has an accumulation point.

2) Suppose that the representation of 0 is rigid. We show by contradiction the injectivity of $(r_n)_{n \in \mathbb{N}}$. Let us assume that $r_i = r_j$ for some indices $i < j$. Then $\sum_{k=0}^{j-1} z_j \beta^k = \sum_{k=0}^{i-1} z_i \beta^k$ and thus $0.z_1z_2\cdots z_j = 0.0\cdots0 z_1\cdots z_i$ — a contradiction with the rigidity of the representation of zero. $\Box$

Theorem 3.5. Let $\beta$ be a complex number, $|\beta| > 1$, and let $A \subset \mathbb{C}$ be a finite alphabet. The spectrum $S_A(\beta)$ has an accumulation point if and only if 0 has a rigid $(\beta, A)$-representation.

Proof. $(\Rightarrow)$ Let $s$ be an accumulation point of $S_A(\beta)$. There exists an injective sequence $(x^{(n)})_{n \in \mathbb{N}}$ of points from $S_A(\beta)$ such that $\lim(x^{(n)})_{n \in \mathbb{N}} = s$. For any $x \in S_A(\beta)$ denote

$$\rho(x) = \min\{n \in \mathbb{N} : x = \sum_{k=0}^{n} a_k \beta^k, \text{ with } a_k \in A\}.$$ 

Set $\rho_n = \rho(x^{(n)})$, then $x^{(n)} = \sum_{k=0}^{\rho_n} x_k^{(n)} \beta^k$. The sequence $(\rho_n)_{n \in \mathbb{N}}$ is unbounded, as there exists only a finite number of strings of a given length over a finite alphabet. Without loss of generality assume that $(\rho_n)_{n \in \mathbb{N}}$ is strictly increasing. Clearly,

$$\frac{x^{(n)}}{\beta^{1+\rho_n}} = 0.x_0^{(n)} x_1^{(n)} x_2^{(n)} x_1^{(n)} x_0^{(n)} x_0^{(n)} \cdots \to 0 \quad (3.1)$$

since the nominators tend to $s$. The fact that $A^\mathbb{N}$ endowed with the product topology is a compact space implies the existence of a string $x_1x_2x_3\cdots$ which is the limit of a subsequence of $(x^{(n)})_{n \in \mathbb{N}}$. It means that for any $N \in \mathbb{N}$ one can find $n \in \mathbb{N}$ such that $\rho_n > N$ and $x^{(n)}_{\rho_n} x^{(n)}_{\rho_n} \cdots x^{(n)}_{\rho_n}$ is a prefix of $x_1x_2x_3\cdots$. The
definition of \( \rho_n \) and the fact (3.1) forces \( 0.x_1x_2x_3 \cdots \) to be a rigid representation of 0.

(\( \Leftarrow \)) Let \( 0 = 0.z_1z_2z_3 \cdots \) be a rigid representation of zero. Then by Point 2 of Lemma 3.3, the sequence of its tails is injective and by Point 1 of the same lemma, the spectrum has an accumulation point.

We now turn to the Pisot case. The real case is due to Garsia \([13]\), and we follow his idea.

**Theorem 3.6.** Let \( \beta \) be a complex number, \( |\beta| > 1 \), and let \( A \subset \mathbb{Q}(\beta) \) containing 0.

1. If \( \beta \) is real and if \( \beta \) or \( -\beta \) is Pisot
2. or if \( \beta \in \mathbb{C} \setminus \mathbb{R} \) is complex Pisot

then \( S_A(\beta) \) has no accumulation point.

**Proof.** Let \( \beta = \beta_1 \) be a complex Pisot number of degree \( r \) with conjugates \( \beta_2 = \overline{\beta_1}, \beta_3, \ldots, \beta_r \), i.e. \( |\beta_k| < 1 \) for \( k = 3, 4, \ldots, r \). We denote \( \sigma_k : \mathbb{Q}(\beta_1) \to \mathbb{Q}(\beta_k) \) the isomorphism induced by \( \beta_1 \mapsto \beta_k \). As \( A \) is finite there exists \( q \in \mathbb{N} \) such that \( qA \) belongs to the ring of integers of the field \( \mathbb{Q}(\beta) \). In particular, the norm \( N(qa) = q^r \prod_{k=1}^{r} |\sigma_k(a)| \) is an integer for any letter \( a \) in \( A \).

Consider \( x, y \in S_A(\beta), x \neq y \). Then the difference between \( x \) and \( y \) can be expressed as \( x - y = v = \sum_{j=0}^{n} b_j \beta^j \), for some \( n \in \mathbb{N} \) and \( b_j \) in \( A - A \).

Let us denote \( A_k = \max\{|\sigma_k(a)| : a \in A\} \). For \( k = 3, 4, \ldots, r \), the modulus of the \( k \)-th conjugate of \( v \) satisfies

\[
|\sigma_k(v)| = \sum_{j=0}^{n} |b_j| |\beta_k|^j \leq 2A_k \sum_{j=0}^{\infty} |\beta_k|^j = 2A_k \frac{|\beta_k|}{1 - |\beta_k|}.
\]

Since \( \beta \) and \( q\beta_k \) are algebraic integers, \( qv \) is an algebraic integer as well and its norm is a rational non-zero integer. Compute the norm of \( qv \)

\[
1 \leq |N(qv)| = q^r \prod_{k=1}^{r} |\sigma_k(v)| = q^r v^r \prod_{k=3}^{r} |\sigma_k(v)| \leq (2q)^r q^r \prod_{k=3}^{r} \frac{A_k |\beta_k|}{1 - |\beta_k|}.
\]

It means that the squared distance \( v^r \) of two different points from the spectrum \( S_A(\beta) \) is bounded from below by the constant \( (2q)^{-r} \prod_{k=3}^{r} \frac{1 - |\beta_k|}{A_k |\beta_k|} \). Consequently, the spectrum has no accumulation point.

The case \( \beta \) real is analogous.

If the base \( \beta \) is real and the alphabet is a symmetric set of consecutive integers, Theorem 3.5 together with the following theorem answers completely the question of the existence of a rigid representation of zero.

**Theorem 3.7** (Akiyama and Komornik \([1]\), Feng \([9]\)). Let \( \beta > 1 \) and let \( A = \{-M, \ldots, M\} \). Then \( S_A(\beta) \) has an accumulation point if and only if \( \beta < M + 1 \) and \( \beta \) is not Pisot.

If the base \( \beta \) is real but the alphabet is not symmetric we have only the following partial observation.

**Proposition 3.8.** Let \( \beta > 1 \) and \( \{-1, 0, 1\} \subset A = \{m, \ldots, 0, \ldots, M\} \subset \mathbb{Z} \).

1. Zero has a non-trivial \((\beta, A)\)-representation if and only if \( \beta \leq \max\{M + 1, -m + 1\} \).
2. If $\beta \leq \max\{M + 1, -m + 1\}$, and $\beta$ is not a Parry number, then zero has a rigid $(\beta, A)$-representation.

**Proof.** Let $d_\beta(1) = t_1t_2t_3\cdots$ be the greedy expansion of 1. Then $\beta - 1 \leq t_1 < \beta$, $t_i \leq t_1$ and

$$0 = 0.t_1t_2t_3\cdots = 0.1t_1t_2t_3\cdots$$

We have two non-trivial representations of 0 over the alphabets $\{-[\beta]+1, \ldots, \bar{\beta}, 0, 1\}$ and $\{\bar{0}, 0, 1, \ldots, [\beta] - 1\}$ respectively.

Therefore, if $\{-1, 0, 1, \ldots, t_1\} \subset A$ or $\{-t_1, \ldots, -1, 0, 1\} \subset A$, zero has a non-trivial $(\beta, A)$-representation. Let us note that $t_1 \in A$ means $M \geq t_1 \geq \beta - 1$. Similarly $-t_1 \in A$ implies $m \leq -t_1 \leq -\beta + 1$.

On the other hand, let $M < \beta - 1$ and $m > -\beta + 1$. Then for $z = \sum_{i \geq 1} z_i \beta^{-i}$ with $z_i \in A$ and $z_1 \geq 1$, we have $z \geq \frac{1}{\beta} + \sum_{i \geq 2} \frac{m}{\beta} = \frac{\beta - 1 + m}{\beta(\beta - 1)} > 0$. Analogously, if $z_1 \leq -1$, then $z < 0$. Consequently, 0 has only the trivial representation.

Now assume that $\beta$ is not a Parry number. Then the sequence of the $n^{th}$ tails of the $\beta$-expansion of 1, $r_n = 0.t_{n+1}t_{n+2}\cdots$, is injective. By Lemma 3.3 and Theorem 3.5, zero has a rigid $(\beta, A)$-representation.

**Remark 3.9.** A numeration system with negative base $-\beta < -1$ and an alphabet $A_{-\beta} = \{0, \ldots, [\beta]\}$ was introduced by Ito and Sadahiro in [15]. Liao and Steiner in [19] defined an Yrrap number as an analogy of a Parry number for numeration systems with negative base. This definition implies that if $\beta$ is not Yrrap, then there exists a rigid $(-\beta, A)$-representation of 0 over the alphabet $A = \{1, \ldots, [\beta] + 1\}$.

### 4 A problem in automata theory

#### 4.1 Representations of 0

Let $\beta$ be a real number $> 1$. We consider infinite $\beta$-representations of 0 on an alphabet of the form $\{-M, \ldots, M\}$, $M \geq 1$ integer. Let

$$Z_{\beta,M} = \{z_1z_2 \cdots | \sum_{i \geq 1} z_i \beta^{-i} = 0, z_i \in \{-M, \ldots, M\}\}$$

be the set of infinite words having value 0 in base $\beta$ on the alphabet $\{-M, \ldots, M\}$.

Proposition 3.8 says that 0 has a non-trivial representation only if $M \geq [\beta] - 1$. Therefore, we consider only $M$ satisfying this inequality.

Note that, if $Z_{\beta,M}$ is recognizable by a finite Büchi automaton, then, for every $c < M$, $Z_{\beta,c} = Z_{\beta,M} \cap \{c, \ldots, c\}^\mathbb{N}$ is recognizable by a finite Büchi automaton as well.

We briefly recall the construction of the (not necessarily finite) Büchi automaton recognizing $Z_{\beta,M}$, see [10] and [12]:

- the set of states is $Q_M \subset \{\sum_{k=0}^n a_k \beta^k | n \in \mathbb{N}, a_k \in \{-M, \ldots, M\}\} \cap \left[\frac{M}{\beta - 1}, \frac{-M}{\beta - 1}\right]$

- for $s, t \in Q_M$, $a \in \{-M, \ldots, M\}$ there is an edge

$$s \xrightarrow{a} t \iff t = \beta s + a$$

- the initial state is 0

- all states are terminal.
Theorem 4.2. The initial state is 0, and all the states are terminal.

Example 4.1. Take $\beta = \varphi = \frac{1 + \sqrt{5}}{2}$ the Golden Ratio. It is a Pisot number, with $d_\varphi(1) = 11$. A finite Büchi automaton recognizing $Z_{\varphi,1}$ is designed in Figure 1. The initial state is 0, and all the states are terminal.

Theorem 4.2. Let $\beta > 1$ and $A = \{-M, \ldots, M\}$ with $M$ a fixed integer $\ge 1$. The set $Z_{\beta, M}$ is recognizable by a finite Büchi automaton if and only if the spectrum $S_A(\beta)$ has no accumulation point.

Proof. To any string $z = z_1z_2\cdots \in Z_{\beta, M}$ we assign the sequence of polynomials $P_n(z)(X) = z_1X^{n-1} + z_2X^{n-2} + \cdots + z_{n-1}X + z_n$. Denote $R_n(z)$ the remainder of the Euclidean division of the polynomial $P_n(z)(X)$ by the polynomial $(X - \beta)$. It means that there exists a polynomial $Q_n(z)(X)$ such that $P_n(z)(X) = (X - \beta)Q_n(z)(X) + R_n(z)$. Clearly $P_n(z)(\beta) = R_n(z)$. Denote $R = \{R_n(z) : z \in Z_{\beta, M} \text{ and } n \in \mathbb{N}\}$.

As $z = z_1z_2\cdots$ is a $(\beta, A)$-representation of 0, the value $P_n(z)(\beta) = -0.z_{n+1}z_{n+2}\cdots$ belongs to the spectrum $S_A(\beta)$ and $-P_n(z)(\beta)$ is the $n$th tail of the $(\beta, A)$-representation of 0. Consequently,

$$R \subset S_A(\beta) \quad \text{and} \quad R \quad \text{is bounded.} \quad (4.1)$$

To prove the theorem, we apply Proposition 3.1 from [10]. It says that $Z_{\beta, M}$ is recognizable by a finite Büchi automaton if and only if the set $R$ is finite.

$(\Leftarrow)$ If $Z_{\beta, M}$ is not recognizable by finite automaton, then $R$ is infinite and by (4.1) the spectrum has an accumulation point.

$(\Rightarrow)$ If $S_A(\beta)$ has an accumulation point, then by Theorem 3.5 zero has a rigid representation $z_1z_2\cdots \in Z_{\beta, M}$. By Point 2 of Lemma 3.4 the sequence of its tails $(r_n)$ is injective. Since $-r_n = P_n(z)(\beta) = R_n(z) \in R$, the set $R$ is not finite and therefore $Z_{\beta, M}$ is not recognizable by finite automaton.

Combining Theorems 4.1 and 4.2, we answer a conjecture raised in [12] and obtain the following result.

Theorem 4.3. Let $\beta > 1$. The following conditions are equivalent:

1. the set $Z_{\beta, M}$ is recognizable by a finite Büchi automaton for every positive integer $M$;

2. the set $Z_{\beta, M}$ is recognizable by a finite Büchi automaton for one $M \ge \lceil \beta \rceil - 1$;

3. $\beta$ is a Pisot number.

Remark 4.4. The fact that, if $\beta$ is not a Pisot number, then the set $Z_{\beta, M}$ is not recognizable by a finite Büchi automaton for any $M \ge \lceil \beta \rceil$ was already settled in Theorem 1.2, but the proof given above is more direct than the original one.
4.2 Normalization

Normalization in base $\beta$ is the function which maps a $\beta$-representation on the canonical alphabet $A_\beta = \{0, \ldots, [\beta] - 1\}$ of a number $x \in [0, 1]$ onto the greedy $\beta$-expansion of $x$. From the Büchi automaton $Z$ recognizing the set of representations of 0 on the alphabet $\{-[\beta] + 1, \ldots, [\beta] - 1\}$, one constructs a Büchi automaton (a converter) $C$ on the alphabet $A_\beta \times A_\beta$ that recognizes the set of couples on $A_\beta$ that have the same value in base $\beta$, as follows:

$$s \xrightarrow{(a,b)} t \text{ in } C \iff s \xrightarrow{a-b} t \text{ in } Z,$$

see [12] for details. Obviously $C$ is finite if and only if $Z$ is finite.

Then we take the intersection of the set of second components with the set of greedy $\beta$-expansions of the elements of $[0, 1]$, which is recognizable by a finite Büchi automaton when $\beta$ is a Pisot number, see [4]. Thus the following result holds true.

**Corollary 4.5.** Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if $\beta$ is a Pisot number.

5 On-line division in complex base

5.1 Trivedi-Ercegovac algorithm

On-line arithmetic, introduced in [25], is a mode of computation where operands and results are processed in a digit serial manner, starting with the most significant digit. To generate the first digit of the result, the first $d$ digits of the operands are required. The integer $d$ is called the delay of the algorithm.

In [5, 6] we have extended the original on-line algorithm of Trivedi-Ercegovac to the complex case.

The algorithm for on-line division in a complex numeration system $(\beta, A)$ has two parameters: the delay $d \in \mathbb{N}$ and $D > 0$, the minimal value (in modulus) of the divisor.

The $(\beta, A)$-representation of the nominator is $n = \sum_{i=1}^{\infty} n_i \beta^{-i}$, of the divisor is $d = \sum_{i=1}^{\infty} d_i \beta^{-i}$, and of their quotient $q = \sum_{i=1}^{\infty} q_i \beta^{-i}$. Partial sums are denoted by $N_k = \sum_{i=1}^{k} n_i \beta^{-i}$, $D_k = \sum_{i=1}^{k} d_i \beta^{-i}$, and $Q_k = \sum_{i=1}^{k} q_i \beta^{-i}$.

The inputs of the algorithm are two infinite strings $0.n_1n_2\cdots n_{\delta+1}n_{\delta+2}\cdots$ with $n_i \in A$ and $n_1 = n_2 = \cdots = n_{\delta} = 0$ and $0.d_1d_2\cdots$ with $d_i \in A$ satisfying $|D_j| \geq D$ for all $j \in \mathbb{N}$, $j \geq 1$.

The output is a string $q_1q_2q_3\cdots$ corresponding to a $(\beta, A)$-representation of the quotient $q = n/d = 0.q_1q_2q_3\cdots$. The settings of the algorithm ensure that the representation of $q$ starts behind the fractional point.

Set $W_0 = q_0 = Q_0 = 0$. Then, for $k \geq 1$ compute

$$W_k = \beta(W_{k-1} - q_{k-1}D_{k-1+\delta}) + (n_{k+\delta} - Q_{k-1}D_{k+\delta})\beta^{-\delta}.$$

The $k$-th digit $q_k$ of the representation of the quotient is evaluated by a function Select, function of the values of the auxiliary variable $W_k$ and the interim representation $D_{k+\delta}$, so that

$$q_k = \text{Select}(W_k, D_{k+\delta}) \in A.$$

It can be shown that for any $k \geq 1$:

$$W_k = \beta^k(N_{k+\delta} - Q_{k-1}D_{k+\delta}).$$

Moreover, if the sequence $(W_k)$ is bounded, then $q = \lim_{k \to \infty} Q_k = \frac{q}{\beta}$.

Conditions on the system $(\beta, A)$ so that the definition of the function Select ensures the correctness of the on-line division algorithm are given in [5, 6].
5.2 Preprocessing of divisors

When making division, we need that the divisor stays away from 0. By definition of the on-line algorithm, this means that the value of all the prefixes of the divisor $d_1d_2\cdots$ must be greater in absolute value than a parameter $D > 0$.

**Definition 5.1.** We say that a complex numeration system $(\beta, A)$ allows preprocessing if there exists $D > 0$ and a finite list $\mathcal{L}$ of identities of the type $0.u_k \cdots u_0 = 0.0u_{k-1} \cdots u_0$ with digits in $A$ such that any string $d_1d_2\cdots$ on $A$ without prefix $w_k \cdots u_0$ from $\mathcal{L}$ satisfies $|0.d_1d_2\cdots d_j| > D$ for all $j \in \mathbb{N}$.

We must have at least $d_1 \neq 0$ after preprocessing, so the preprocessing consists first of all in shifting the fractional point to the most significant non-zero digit of the $(\beta, A)$-representation of the divisor. Of course, after preprocessing the value of the original divisor $w$ has been changed into a new one $d$ which is just a shift of the original one, that is to say $d = w\beta^k$ for some $k \in \mathbb{Z}$. This will have to be taken into account to give the result of the division.

If zero has only the trivial $(\beta, A)$-representation the situation is simple. This fact can be equivalently rewritten as

$$\inf \mathcal{R} > 0, \quad \text{where} \quad \mathcal{R} = \left\{ |\sum_{i \geq 1} z_i\beta^{-i}| : z_1 \neq 0, z_i \in A \right\}.$$

In this case the numeration system $(\beta, A)$ allows preprocessing, since we can take $D = \inf \mathcal{R}$ and the list of rewriting rules is empty.

**Example 5.2.** If $\beta = 4$ and $A = \{\overline{1}, \overline{1}, 0, 1, 2\}$, then zero has only the trivial representation and for $D$ one can take $\frac{1}{12} = \min \mathcal{R}$.

**Example 5.3.** If $\beta = 2$ and $A = \{\overline{1}, 0, 1\}$, zero has two non-trivial representations $0 = 0.1\overline{1} 1 1 \cdots = 0.1 \overline{1} 1 1 1 \cdots$. Therefore, preprocessing is a little bit more sophisticated. Consider the list

$$0.\overline{1} = 0.0\overline{1} \quad \text{and} \quad 0.1\overline{1} = 0.01$$

If a string $d_1d_2\cdots$ has no prefix $\overline{1}$ neither $1\overline{1}$, then

$$|0.d_1d_2\cdots d_j| \geq 0.10\overline{1} 1 1 \cdots = \frac{1}{4}$$

and thus one can take $D = \frac{1}{4}$.

**Theorem 5.4.** A complex numeration system $(\beta, A)$ allows preprocessing if and only if the spectrum $S_A(\beta)$ has no accumulation point.

The result is proved by the following three lemmas, in which we use the notation

$$H = \max \{|\sum_{i \geq 1} d_i\beta^{-i}| : d_i \in A \quad \text{for all} \ i \in \mathbb{N}\}.$$

**Lemma 5.5.** If 0 has a rigid $(\beta, A)$-representation then the numeration system $(\beta, A)$ does not allow preprocessing.

**Proof.** Let $0 = 0.z_1z_2z_3\cdots$ be a rigid representation of 0. Assume that preprocessing is possible with $D > 0$. Find $j$ such that $\frac{H}{|z_j|} < D$. Consider the number $0.z_1z_2z_3\cdots z_j000\cdots$. Since the representation of zero is rigid, no prefix of the string $z_1z_2z_3\cdots z_j$ is contained in the list of the rewriting rules. But $|0.z_1z_2z_3\cdots z_j| = |0.00\cdots 0. z_j+1z_j+2\cdots | < \frac{H}{|z_j|} < D$ — a contradiction.\[\Box\]
Lemma 5.6. Let us assume that $S_A(\beta)$ has no accumulation point and fix $K > 0$. Then there exists $m \in \mathbb{N}$ such that any string $x_{m-1}x_{m-2}\cdots x_1x_0$ of length $m$ over $A$ satisfies either

$$|x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0| \geq K$$

or there exists a string $y_{k-1}x_{k-2}\cdots y_1y_0$ of length $k < m$ over $A$ such that

$$x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0 = y_{k-1}\beta^{k-1} + y_{k-2}\beta^{k-2} + \cdots + y_1\beta + y_0.$$  

Proof. Since $S_A(\beta)$ has no accumulation point, the set $P = \{z \in S_A(\beta) : |z| < K\}$ is finite. Denote $m = 1 + \max\{\rho(z) : z \in P\}$. Let $x = x_{m-1}\beta^{m-1} + x_{m-2}\beta^{m-2} + \cdots + x_1\beta + x_0$. Obviously, $x \in S_A(\beta)$. Then either $|x| \geq K$ or $x \in P$ and thus $x = y_{k-1}\beta^{k-1} + y_{k-2}\beta^{k-2} + \cdots + y_1\beta + y_0$, where $k \leq \max\{\rho(z) : z \in P\} \leq m - 1$. □

Lemma 5.7. If $S_A(\beta)$ has no accumulation point, then there exists $D > 0$ and $m \in \mathbb{N}$ such that for all infinite strings $d_1d_2\cdots$ over $A$ one has

1. either $|0.d_1d_2\cdots d_j| \geq D$ for all $j \in \mathbb{N}$,
2. or $0.d_1d_2\cdots d_m = 0.0d_2d_3\cdots d_m'$ for some string $d_2d_3\cdots d_m' \in A^*.$

Proof. Let us take $\mu > 0$ and apply Lemma 5.6 with $K = H + \mu$ to get $m \in \mathbb{N}$. Denote $D = \{|0.d_1d_2\cdots d_j| : j < m$ and $0.d_1d_2\cdots d_j \neq 0.0d_2d_3\cdots d_j\}$. The set $D$ is finite and does not contain zero. Therefore, $D' = \min D > 0$.

To prove the lemma, consider an infinite string $d_1d_2\cdots$ and assume that $0.d_1d_2\cdots d_m \neq 0.0d_2d_3\cdots d_m'$ for all strings $d_2d_3\cdots d_m' \in A^*$. We distinguish two cases

- $j < m, j \in \mathbb{N}$. Then $0.d_1d_2\cdots d_j \neq 0.0d_2d_3\cdots d_j'$, otherwise $0.d_1d_2\cdots d_m = 0.0d_2d_3\cdots d_j' \cdots d_m$ — a contradiction. Therefore, $|0.d_1d_2\cdots d_j| \geq D'$.

- $j \geq m, j \in \mathbb{N}$. Then

$$|0.d_1d_2\cdots d_j| \geq |0.d_1d_2\cdots d_m| - \frac{1}{|\beta|^m} |0.d_{m+1}d_{m+2}\cdots d_j| \geq \frac{1}{|\beta|^m}K - \frac{1}{|\beta|^m}H = \frac{\mu}{|\beta|^m}$$

Thus we can set $D = \min\{D', \frac{\mu}{|\beta|^m}\}$. □

The previous lemma gives a hint for creating the list of rewriting rules. We take the index $m$ found by the lemma and inspect all strings $d_1d_2\cdots d_m$ over $A$. If $0.d_1d_2\cdots d_m = 0.0d_2d_3\cdots d_m'$ for some string $d_2d_3\cdots d_m'$ we put it into the list.

Example 5.8. Let $\beta = \phi = \frac{1 + \sqrt{5}}{2}$ and $A = \{1, 0, 1\}$. The minimal polynomial of $\phi$ is $X^2 - X - 1$. In this numeration system, 0 has countably many finite representations and uncountably many infinite representations. As the alphabet is symmetric, the rewriting rules appear in pairs. For example, as $10\overline{T}$ can be rewritten to 010, also $101$ can be rewritten to $01\overline{T}$. To shorten our list, we put into it only one rule of each pair, namely the rule, where the first digit is 1. First we consider the list

$L_0$: $10\overline{T} \rightarrow 010, 1\overline{T} \rightarrow 001, 1\overline{T} \rightarrow 000.$

Claim: If no rule from $L_0$ can be applied to the string $d_1d_2\cdots$, then $|d| \geq D = \frac{1}{\phi^2}$,

where $d = 0.d_1d_2\cdots$.

Proof. WLOG $d_1 = 1$.

If $d_2 = 0$, then $|d| \geq 0$ and thus $|d| \geq \frac{1}{\phi} - \sum_{k \geq 2} \phi^{-k} = \frac{1}{\phi} - \frac{1}{\phi^2} = \frac{1}{\phi^2} \geq D.$

If $d_2 = 1$, then $|d| \geq \frac{1}{\phi} + \frac{1}{\phi^2} - \sum_{k \geq 3} \phi^{-k} = 1 - \frac{1}{\phi} = \frac{1}{\phi^2} \geq D.$

If $d_2 = \overline{T}$, then $d_3 = 1$. Therefore, $|d| \geq \frac{1}{\phi} - \frac{1}{\phi^2} + \frac{1}{\phi^2} - \sum_{k \geq 4} \phi^{-k} = \frac{1}{\phi^2} \geq D.$ □
We can extend the list of rewriting rules to increase the lower bound $D$. Let us consider the whole families of rules $L$:

- $(11)^k 0 \rightarrow 00(10)^{k-1}1$ for $k \geq 1$.
- $(11)^k 1 \rightarrow 00(10)^{k-1}10$ for $k \geq 1$.
- $(11)^k 10 \rightarrow 01(00)^{k-1}10$ for $k \geq 1$.
- $(11)^k 100 \rightarrow 01(00)^{k-1}11$ for $k \geq 1$.
- $10^k 0 \rightarrow 0^{k+1}11$ for $k \geq 0$.
- $10^k 1 \rightarrow 0^{k+1}101$ for $k \geq 1$.

**Claim:** If no rule from $L$ can be applied to the string $d_1d_2\cdots$, then $|d| \geq D = \frac{1}{\phi^2}$, where $d = 0.d_1d_2\cdots$.

**Proof.** WLOG $d_1 = 1$. Our string has a prefix 11 or a prefix $(11)^k 101$ for $k \geq 0$. Therefore either

$$|d| \geq 0.11(1)^\omega = \frac{1}{\phi^2} \quad \text{or} \quad |d| \geq 0.(11)^k 101(1)^\omega = \frac{1}{\phi^2} + \frac{1}{\phi^2 + 1}.$$

Some examples where the base is a complex number can be found in [6].

## 6 Comments and open questions

### 6.1 F-number

In [18] Lau defined for $1 < \beta < 2$ the following notion, that we extend to any $\beta > 1$.

**Definition 6.1.** Let $\beta > 1$ and $B_\beta = \{-\lceil \beta \rceil + 1, \ldots, \lceil \beta \rceil - 1\}$ be the symmetrized alphabet of the canonical alphabet $A_\beta$. Then $\beta$ is said to be a F-number if the set

$$L_{(\lceil \beta \rceil - 1)}(\beta) = S_{B_\beta}(\beta) \cap \left[ -\frac{\lceil \beta \rceil - 1}{\beta - 1}, \frac{\lceil \beta \rceil - 1}{\beta - 1}\right]$$

is finite.

Feng proved in [9] that $1 < \beta < 2$ is a F-number if and only if it is a Pisot number. This property extends readily to any $\beta > 1$. Another way of proving it consists in realizing that the set of states $Q_{(\lceil \beta \rceil - 1)}$ of the automaton for $Z_{\beta, \lceil \beta \rceil - 1}$ is included into $L_{(\lceil \beta \rceil - 1)}(\beta)$.

### 6.2 Open questions

- A motivation for introducing the notion of “rigid representation of zero” comes from on-line division in a numeration system $(\beta, A)$. A more elementary question is “Has zero a non-trivial $(\beta, A)$-representation”? The answer is easy for real bases and alphabets of the form $A = \{m, \ldots, 0, \ldots, M\}$, see Proposition 5.8. The same question for complex bases is an open problem.

- In the case that the base is real and the alphabet is $A = \{-M, \ldots, M\}$, Theorem 4.2 says that recognizability by a finite automaton is equivalent to the fact that the spectrum $S_A(\beta)$ has no accumulation point.

An analogous result can be proved for complex bases as well. But for complex bases the question about the existence of accumulation points in the spectrum $S_A(\beta)$ is not yet investigated. Nevertheless, it is often easy to check that a $(\beta, A)$-representation of 0 is **not** rigid.

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If $\beta > 1$ is a non-Pisot base then $A = \{-\lfloor \beta \rfloor + 1, \ldots, \lfloor \beta \rfloor - 1\}$ is the smallest symmetric alphabet of consecutive integers for which the spectrum $S_A(\beta)$ has an accumulation point. What is the minimal size of an alphabet $A = \{-M, \ldots, M\} \subset \mathbb{Z}$ for which the spectrum of a non-Pisot complex number $\beta$ has an accumulation point?

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