Beta-representations of $0$ and Pisot numbers

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Résumé. Soient $\beta$ un nombre réel supérieur à $1$, $d$ un entier positif, et soit
$$Z_{\beta,d} = \{z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, \; z_i \in \{-d, \ldots, d\}\}$$
l’ensemble des mots infinis sur l’alphabet $\{-d, \ldots, d\} \subset \mathbb{Z}$ ayant
la valeur $0$ en base $\beta$. En utilisant un résultat de Feng sur le spectre de $\beta$ nous prouvons que si $Z_{\beta,\lceil \beta \rceil-1}$ est reconnaissable par
un automate de Büchi fini, alors $\beta$ doit être un nombre de Pisot.
En conséquence de résultats antérieurs, $Z_{\beta,d}$ est reconnaissable
par un automate de Büchi fini pour tout entier positif $d$ si et seulennement si $Z_{\beta,d}$ est reconnaissable par un automate de Büchi
fini pour un $d \geq \lceil \beta \rceil - 1$. Ces conditions sont équivalentes à ce que $\beta$ soit un nombre de Pisot. La borne $\lceil \beta \rceil - 1$ ne peut pas être
réduite.

Abstract. Let $\beta > 1$, $d$ a positive integer, and
$$Z_{\beta,d} = \{z_1 z_2 \cdots \mid \sum_{i \geq 1} z_i \beta^{-i} = 0, \; z_i \in \{-d, \ldots, d\}\}$$
be the set of infinite words having value $0$ in base $\beta$ on the alphabet
$\{-d, \ldots, d\} \subset \mathbb{Z}$. Based on a result of Feng on spectra of numbers,
we prove that if the set $Z_{\beta,\lceil \beta \rceil-1}$ is recognizable by a finite Büchi
automaton then $\beta$ must be a Pisot number. As a consequence
of previous results, the set $Z_{\beta,d}$ is recognizable by a finite Büchi
automaton for every positive integer $d$ if and only if $Z_{\beta,d}$ is rec-
ognizable by a finite Büchi automaton for one $d \geq \lceil \beta \rceil - 1$. These
conditions are equivalent to the fact that $\beta$ is a Pisot number.
The bound $\lceil \beta \rceil - 1$ cannot be further reduced.

1. Introduction

Let $\beta$ be a real number $> 1$. The so-called beta-numeration has been
introduced by Rényi in [13], and since then there are been many works in
this domain, in connection with number theory, dynamical systems, and
automata theory, see the survey [10] or more recent [14] for instance.
By a greedy algorithm each number of the interval \([0, 1]\) is given a \(\beta\)-expansion, which is an infinite word on a canonical alphabet of non-negative integers. When \(\beta\) is an integer, we obtain the classical numeration systems. When \(\beta\) is not an integer, a number \(x\) may have different \(\beta\)-representations. The \(\beta\)-expansion obtained by the greedy algorithm is the greatest in the lexicographic ordering of all the \(\beta\)-representations of \(x\). The question of converting a \(\beta\)-representation into another one is equivalent to the study of the \(\beta\)-representations of 0. We focus on the question of the recognizability by a finite automaton of the set of \(\beta\)-representations of 0.

Let \(d\) be a positive integer, and let
\[
Z_{\beta,d} = \{z_1z_2\cdots | \sum_{i \geq 1} z_i \beta^{-i} = 0, \ z_i \in \{-d, \ldots, d\}\}
\]
be the set of infinite words having value 0 in base \(\beta\) on the alphabet \(\{-d, \ldots, d\} \subset \mathbb{Z}\).

The following result has been formulated in [10]:

**Theorem 1.1.** Let \(\beta > 1\). The following conditions are equivalent:
1. the set \(Z_{\beta,d}\) is recognizable by a finite Büchi automaton for every integer \(d\),
2. the set \(Z_{\beta,d}\) is recognizable by a finite Büchi automaton for one integer \(d \geq \lceil \beta \rceil\),
3. \(\beta\) is a Pisot number.

(3) implies (1) is proved in [8], (1) implies (3) is proved in [2] and (2) implies (1) is proved in [9].

Recently Feng answered an open question raised by Erdős, Joó and Komornik [6], see also [1], on accumulation points of the set
\[
Y_d(\beta) = \{\sum_{k=0}^{n} a_k \beta^k | n \in \mathbb{N}, \ a_k \in \{-d, \ldots, d\}\}.
\]

**Theorem 1.2 ([7]).** Let \(\beta > 1\). Then \(Y_d(\beta)\) is dense in \(\mathbb{R}\) if and only if \(\beta < d + 1\) and \(\beta\) is not a Pisot number.

Remark that the problem studied by Feng is closely related to the following one. For \(\beta > 1\) let
\[
X_d(\beta) = \{\sum_{k=0}^{n} a_k \beta^k | n \in \mathbb{N}, \ a_k \in \{0, \ldots, d\}\}.
\]
Since \(X_d(\beta)\) is discrete its elements can be arranged into an increasing sequence
\[
0 = x_0(\beta, d) < x_1(\beta, d) < \cdots
\]
Denote $\ell_d(\beta) = \liminf_{n \to \infty} (x_{k+1}(\beta, d) - x_k(\beta, d))$. Feng has obtained the following corollary: $\ell_d(\beta) = 0$ if and only if $\beta < d + 1$ and $\beta$ is not a Pisot number.

Previously in [5] Bugeaud had shown, using (1) implies (3) of Theorem 1.1, that if $\beta$ is not a Pisot number then there exists an integer $d$ such that $\ell_d(\beta) = 0$.

In the present note we use Feng's theorem to prove the conjecture stated in [10]:

If the set $Z_{\beta, \lfloor \beta \rfloor - 1}$ is recognizable by a finite Büchi automaton then $\beta$ is a Pisot number.

Moreover we obtain a simpler proof of the implication (2) $\Rightarrow$ (3) of Theorem 1.1. Note that the value $d = \lfloor \beta \rfloor - 1$ is the best possible as $Z_{\beta, d}$ is reduced to $\{0^\omega\}$ if $d < \lfloor \beta \rfloor - 1$.

To summarize, this gives the following result:

The set $Z_{\beta, d}$ is recognizable by a finite Büchi automaton for every positive integer $d$ if and only if $Z_{\beta, d}$ is recognizable by a finite Büchi automaton for one $d \geq \lfloor \beta \rfloor - 1$ if and only if $\beta$ is a Pisot number.

Normalization in base $\beta$ is the function which maps a $\beta$-representation on the canonical alphabet $A_\beta = \{0, \ldots, \lfloor \beta \rfloor - 1\}$ of a number $x \in [0, 1]$ onto the greedy $\beta$-expansion of $x$. Since the set of greedy $\beta$-expansions of the elements of $[0, 1]$ is recognizable by a finite Büchi automaton when $\beta$ is a Pisot number, see [3], the following result holds true:

Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if $\beta$ is a Pisot number.

We end this paper by considering finite $\beta$-representations of 0. We recall an old result of [8]:

Let $\beta$ be a complex number. The set $W_{\beta, d}$ of finite $\beta$-representations of 0 is recognizable by a finite automaton for every $d$ if and only if $\beta$ is an algebraic number with no conjugate of modulus 1.

In view of previous results and examples we conclude with a conjecture:

Let $\beta > 1$ be an algebraic number such that $W_{\beta, \lfloor \beta \rfloor - 1} \neq \{0^\omega\}$. If $\beta$ has a conjugate of modulus 1 then $W_{\beta, \lfloor \beta \rfloor - 1}$ is not recognizable by a finite automaton.

2. Preliminaries

2.1. Words and automata. Let $A$ be a finite alphabet. A finite word $w$ on $A$ is a finite concatenation of letters from $A$, $w = w_1 \cdots w_n$ with $w_i$ in $A$. The set of all finite words over $A$ is denoted by $A^*$. An infinite word $w$ on $A$ is an infinite concatenation of letters from $A$, $w = w_1w_2\cdots$ with $w_i$
in $A$. The set of all infinite words over $A$ is denoted by $A^\omega$. The infinite concatenation $uuu \cdots$ is noted $w^\omega$. If $w = uv$, $u$ is a prefix of $w$.

An automaton $A = (A, Q, I, T)$ over $A$ is a directed graph labeled by letters of the alphabet $A$, with a denumerable set $Q$ of vertices called states. $I \subseteq Q$ is the set of initial states, and $T \subseteq Q$ is the set of terminal states. The automaton is said to be finite if the set of states $Q$ is finite.

A finite path of $A$ is successful if it starts in $I$ and terminates in $T$. The set of finite words recognized by $A$ is the set of labels of its successful finite paths.

An infinite path of $A$ is successful if it starts in $I$ and goes infinitely often through $T$. The set of infinite words recognized by $A$ is the set of labels of its successful infinite paths. An automaton used to recognize infinite words in this sense is called a Büchi automaton.

### 2.2. Beta-numeration

We now recall some definitions and results on the so-called beta-numeration, see [10] or [14] for a survey. Let $\beta > 1$ be a real number. Any real number $x \in [0, 1]$ can be represented by a greedy algorithm as $x = \sum_{i=1}^{+\infty} x_i \beta^{-i}$ with $x_i$ in the canonical alphabet $A_\beta = \{0, \ldots, \lceil \beta \rceil - 1\}$ for all $i \geq 1$. The greedy sequence $(x_i)_{i \geq 1}$ corresponding to a given real number $x$ is the greatest in the lexicographical order, and is said to be the $\beta$-expansion of $x$, see [13]. It is denoted by $d_\beta(x) = (x_i)_{i \geq 1}$.

When the expansion ends in infinitely many 0's, it is said finite, and the 0's are omitted. The $\beta$-expansion of 1 is denoted $d_\beta(1) = (t_i)_{i \geq 1}$.

An infinite (resp. finite) $\beta$-representation of 0 on an alphabet $\{-d, \ldots, d\}$ is an infinite (resp. finite) sequence $z_1z_2\cdots$ of letters from this alphabet such that $\sum_{i \geq 1} z_i \beta^{-i} = 0$. If $d < \lceil \beta \rceil - 1$, then 0 has only the trivial $\beta$-representation on the alphabet $\{-d, \ldots, d\}$. On the other hand, $(-1)t_1t_2\cdots$ is a nontrivial $\beta$-representation of 0 on the alphabet $\{-\lceil \beta \rceil + 1, \ldots, \lceil \beta \rceil - 1\}$, and thus we consider only alphabets $\{-d, \ldots, d\}$ with $d \geq \lceil \beta \rceil - 1$.

Note that, if $Z_{\beta,d}$ is recognizable by a finite Büchi automaton, then, for every $c < d$, $Z_{\beta,c} = Z_{\beta,d} \cap \{-c, \ldots, c\}^\omega$ is recognizable by a finite Büchi automaton as well.

We briefly recall the construction of the (not necessarily finite) Büchi automaton recognizing $Z_{\beta,d}$, see [8] and [10]:

- the set of states is $Q_d \subset \{\sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, a_k \in \{-d, \ldots, d\}\} \cap [-\frac{d}{\beta^n}, \frac{d}{\beta^n}]$
- for $s, t \in Q_d$, $a \in \{-d, \ldots, d\}$ there is an edge $s \xrightarrow{a} t \iff t = \beta s + a$
- the initial state is 0
- all states are terminal.
Example 2.1. Take $\beta = \varphi = \frac{1 + \sqrt{5}}{2}$ the Golden Ratio. It is a Pisot number, with $d_\varphi(1) = 11$. A finite Büchi automaton recognizing $\mathbb{Z}_{\varphi,1}$ is designed in Figure 1. The initial state is 0, and all the states are terminal. The signed digit $(-1)$ is denoted $\bar{1}$.

![Figure 1. Finite Büchi automaton recognizing $\mathbb{Z}_{\varphi,1}$ for $\varphi = \frac{1 + \sqrt{5}}{2}$.

Notation: In the sequel $y_{m-1} \cdots y_0 y_{-1} y_{-2} \cdots$ denotes the numerical value $y_{m-1} \beta^{m-1} + \cdots + y_0 + y_{-1} \beta^{-1} + y_{-2} \beta^{-2} + \cdots$.

2.3. Numbers. A number $\beta > 1$ such that $d_\beta(1)$ is eventually periodic is a Parry number. It is a simple Parry number if $d_\beta(1)$ is finite.

A Pisot number is an algebraic integer greater than 1 such that all its Galois conjugates have modulus less than 1. Every Pisot number is a Parry number, see [3] and [15]. The converse is not true, see for instance Example 4.4 below.

A Salem number is an algebraic integer greater than 1 such that all its Galois conjugates have modulus $\leq 1$ with at least one conjugate with modulus 1.

3. Infinite representations

3.1. Main result. We answer a conjecture raised in [10] and obtain the following result.

Theorem 3.1. Let $\beta > 1$. The following conditions are equivalent:

1. the set $\mathbb{Z}_{\beta,d}$ is recognizable by a finite Büchi automaton for every positive integer $d$,
2. the set $\mathbb{Z}_{\beta,d}$ is recognizable by a finite Büchi automaton for one $d \geq \lceil \beta \rceil - 1$,
3. $\beta$ is a Pisot number.

It will be a consequence of the result which follows.

Proposition 3.2. Let $d \geq \lceil \beta \rceil - 1$. If $\beta$ is not a Pisot number then the set $\mathbb{Z}_{\beta,d}$ is not recognizable by a finite Büchi automaton.
Proof. Since \( \beta \) is not a Pisot number and \( d \geq \lceil \beta \rceil - 1 \), by Feng [7], the set
\[
Y_d(\beta) = \left\{ \sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, \ a_k \in \{-d, \ldots, d\} \right\}
\]
is dense in \( \mathbb{R} \). In particular there exists a sequence \( (r_n)_{n \in \mathbb{N}} \) of elements of \( Y_d(\beta) \) such that \( r_n \neq r_m \) for all \( n \neq m \), and \( \lim_{n \to \infty} r_n = 0 \). As \( r_n \) belongs to \( Y_d(\beta) \), it can be written as \( r_n = \sum_{k=0}^{\ell_n} a_k^{(n)} \beta^k \) where \( a_0^{(n)}, a_1^{(n)}, \ldots, a_{\ell_n}^{(n)} \) are in \( \{-d, \ldots, d\} \), \( a_{\ell_n}^{(n)} \neq 0 \) and \( \ell_n \) is minimal.

Clearly the number
\[
0.a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} = \frac{r_n}{\beta^{\ell_n}} \to 0
\]
as \( n \) tends to \( \infty \).

Let us fix \( k \) in \( \mathbb{N} \). Since there exist finitely many words of length \( k \) on \( \{-d, \ldots, d\} \), there exists a word \( S_k \) of length \( k \) which is a prefix of infinitely many elements of the sequence \( (a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)})_n \). Set \( S_k = x_1 \cdots x_k \).

As \( a_{\ell_n}^{(n)} \neq 0 \), \( x_1 \neq 0 \) as well.

Since infinitely many \( a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} \omega \) have \( S_k \) as a prefix, there exists a letter \( x_{k+1} \) such that \( S_{k+1} = x_1 \cdots x_k x_{k+1} \) is a prefix of infinitely many \( a_{\ell_n}^{(n)} a_{\ell_n-1}^{(n)} \cdots a_0^{(n)} \omega \). We continue in this manner and we get an infinite string \( x_1 x_2 \cdots \). To formulate an important property of it we need the following notion.

Definition 3.3. Let \( z_1 z_2 \cdots \) be a \( \beta \)-representation of 0 on \( \{-d, \ldots, d\} \). It is said to be rigid if \( 0.z_1 z_2 \cdots z_j \neq 0.0 z_2 \cdots z_j' \) for all \( j \geq 2 \) and for all \( z_2 \cdots z_j \) in \( \{ -d, \ldots, d \}^* \).

Claim 1 \( x_1 x_2 \cdots \) is a rigid \( \beta \)-representation of 0.

First we prove that it is indeed a \( \beta \)-representation of 0: by the construction, for every \( n \in \mathbb{N} \) there exist infinitely many \( N \)'s such that the common prefix of \( x_1 x_2 \cdots \) and \( a_{\ell_N}^{(N)} a_{\ell_{N-1}}^{(N)} \cdots a_0^{(N)} \omega \) is longer than \( n \). Thus
\[
|0.x_1 x_2 \cdots - 0.a_{\ell_N}^{(N)} a_{\ell_{N-1}}^{(N)} \cdots a_0^{(N)}| \leq \frac{1}{\beta^n} \frac{2d}{\beta - 1}.
\]
Therefore, by (3.1),
\[
|0.x_1 x_2 \cdots| < |0.x_1 x_2 \cdots - 0.a_{\ell_N}^{(N)} a_{\ell_{N-1}}^{(N)} \cdots a_0^{(N)}| + |0.a_{\ell_N}^{(N)} a_{\ell_{N-1}} \cdots a_0^{(N)}| \leq \frac{1}{\beta^n} \frac{2d}{\beta - 1} + \frac{r_N}{\beta^{\ell_N}} \to 0
\]
and thus \( x_1 x_2 \cdots \) is a \( \beta \)-representation of 0.

Second, we prove that this representation is rigid. Suppose the opposite, that is to say that there exist some index \( j \) and a word \( x'_2 \cdots x'_j \)
in \( \{-d, \ldots, d\}^* \) such that \( 0.x_1x_2 \cdots x_j = 0.0x'_2 \cdots x'_j \). There exists some \( N \) such that \( a_{\ell N}^{(N)} a_{\ell N-1}^{(N)} \cdots a_0^{(N)} 0^\omega \) has \( x_1x_2 \cdots x_j \) as a prefix and thus \( r_N = a_{\ell N}^{(N)} a_{\ell N-1}^{(N)} \cdots a_0^{(N)} \cdot = 0x'_2 \cdots x'_j a_{\ell N-j}^{(N)} \cdots a_0^{(N)} \cdot \), a contradiction with the choice of \( \ell_N \).

**Claim 2** The set \( \{0.x_{n+1}x_{n+2} \cdots | n \in \mathbb{N}\} \) is infinite.

We show that the elements of the sequence \( (0.x_{n+1}x_{n+2} \cdots)_n \) with distinct indices do not coincide. Let us suppose that there exist \( k < n \) such that \( 0.x_{k+1}x_{k+2} \cdots = 0.x_{n+1}x_{n+2} \cdots \). Then

\[-x_1x_2 \cdots x_k, = -x_1x_2 \cdots x_n.\]

This implies that there is some \( r_N \) such that

\[
\begin{align*}
r_N &= a_{\ell N}^{(N)} a_{\ell N-1}^{(N)} \cdots a_0^{(N)} \\
&= x_1 x_2 \cdots x_n a_{\ell N}^{(N)} a_{\ell N-n}^{(N)} \cdots a_0^{(N)}.
\end{align*}
\]

a contradiction with the choice of \( \ell_N \).

Define polynomials \( P_n(X) = \sum_{k=0}^{n-1} x_{n-k} X^k \). Then the division by \( (X - \beta) \) gives \( P_n(X) = Q_n(X)(X - \beta) + s_n \), and

\[
P_n(\beta) = s_n = x_1\beta^{n-1} + \cdots + x_{n-1}\beta + x_n = -0.x_{n+1}x_{n+2} \cdots
\]

Consequently to Claim 2, the set of remainders of the division by \( (X - \beta) \) is infinite. By Proposition 3.1 in [8] the set \( Z_{\beta,d} \) is not recognizable by a finite Büchi automaton.

**Remark 3.4.** The fact that, if \( \beta \) is not a Pisot number, then the set \( Z_{\beta,d} \) is not recognizable by a finite Büchi automaton for any \( d \geq \lceil \beta \rceil \) was already settled in Theorem 1.1, but the proof given in Proposition 3.2 is more direct than the original one.

### 3.2. Normalization

Normalization in base \( \beta \) is the function which maps a \( \beta \)-representation on the canonical alphabet \( A_{\beta} = \{0, \ldots, \lceil \beta \rceil - 1\} \) of a number \( x \in [0,1) \) onto the greedy \( \beta \)-expansion of \( x \). From the Büchi automaton \( Z \) recognizing the set of representations of 0 on the alphabet \( \{-\lceil \beta \rceil + 1, \ldots, \lceil \beta \rceil - 1\} \), one constructs a Büchi automaton (a converter) \( C \) on the alphabet \( A_{\beta} \times A_{\beta} \) that recognizes the set of couples on \( A_{\beta} \) that have the same value in base \( \beta \), as follows:

\[
s \xrightarrow{(a,b)} t \text{ in } C \iff s \xrightarrow{a-b} t \text{ in } Z,
\]

see [10] for details. Obviously \( C \) is finite if and only if \( Z \) is finite.

Then we take the intersection of the set of second components with the set of greedy \( \beta \)-expansions of the elements of \([0,1]\), which is recognizable.
by a finite Büchi automaton when $\beta$ is a Pisot number, see [4]. Thus the following result holds true.

**Corollary 3.5.** Normalization in base $\beta > 1$ is computable by a finite Büchi automaton on the alphabet $A_\beta \times A_\beta$ if and only if $\beta$ is a Pisot number.

### 3.3. F-Numbers

In [11] Lau defined for $1 < \beta < 2$ the following notion, that we extend to any $\beta > 1$.

**Definition 3.6.** $\beta > 1$ is said to be a **F-number** if

$$L_{\lfloor \beta \rfloor - 1}(\beta) = Y_{\lfloor \beta \rfloor - 1}(\beta) \cap \left[ -\frac{\lfloor \beta \rfloor - 1}{\beta - 1}, \frac{\lfloor \beta \rfloor - 1}{\beta - 1} \right]$$

is finite.

Feng proved in [7] that if $1 < \beta < 2$ is a F-number then $\beta$ is a Pisot number. This property extends to any $\beta > 1$. Another way of proving it consists in realizing that the set of states $Q_{\lfloor \beta \rfloor - 1}$ of the automaton for $Z_{\beta, \lfloor \beta \rfloor - 1}$ is included into $L_{\lfloor \beta \rfloor - 1}(\beta)$, and we get

**Corollary 3.7.** $\beta$ is a F-number if and only if $\beta$ is a Pisot number.

### 4. Finite representations

One can ask: what are the results in case we consider only finite representations of numbers. The situation is quite different. Let $\beta$ be any complex number and let

$$W_{\beta, d} = \{z_1z_2 \cdots z_n \mid n \geq 1, \sum_{i=1}^{n} z_i \beta^{-i} = 0, z_i \in \{-d, \ldots, d\}\}$$

be the set of finite representations of 0 on $\{-d, \ldots, d\}$.

**Theorem 4.1** ([8]). Let $\beta$ be a complex number. The set $W_{\beta, d}$ is recognizable by a finite automaton for every $d$ if and only if $\beta$ is an algebraic number with no conjugate of modulus 1.

**Remark 4.2.** The construction of the automaton recognizing $Z_{\beta, d}$ presented in Section 2.2 can be extended to the case that $\beta$ is a complex number, $|\beta| > 1$, with set of states $Q_d \subset \{\sum_{k=0}^{n} a_k \beta^k \mid n \in \mathbb{N}, a_k \in \{-d, \ldots, d\}, |\sum_{k=0}^{n} a_k \beta^k| \leq \frac{d}{|\beta| - 1}\}$.

The automaton recognizing $W_{\beta, d}$ is the subautomaton formed by the finite paths starting and ending in state 0.

**Example 4.3.** Take $\beta = \varphi = \frac{1 + \sqrt{5}}{2}$ the Golden Ratio. A finite automaton recognizing $W_{\varphi, 1}$ is designed in Figure 2. The initial state is 0, which is the only terminal state.
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\[ \varphi = \frac{1 + \sqrt{5}}{2} \]

\[ \overline{1 \varphi + 1} \quad 1 \quad 1 \quad 0 \quad 1 \quad 1 \quad \varphi - 1 \]

**Figure 2.** Finite automaton recognizing \( W_{\varphi, 1} \) for \( \varphi = \frac{1 + \sqrt{5}}{2} \).

From Remark 4.2 we know that if \( Z_{\beta, d} \) is recognizable by a finite Büchi automaton then \( W_{\beta, d} \) is recognizable by a finite automaton, but the converse is false, as shown by the following example.

**Example 4.4 ([10]).** Let \( \beta \) be the root > 1 of the polynomial \( X^4 - 2X^3 - 2X^2 - 2 \). Then \( d_\beta(1) = 2202 \) and \( \beta \) is a simple Parry number which is not a Pisot number and has no root of modulus 1. The set \( W_{\beta, 2} \) is recognizable by a finite automaton but \( Z_{\beta, 2} \) is not recognizable by a finite Büchi automaton.

The following gives an example of a Salem number which is a Parry number such that the set of representations of 0 is not recognizable by a finite automaton.

**Example 4.5 ([12]).** Let \( \beta \) be the root > 1 of \( X^4 - 2X^3 + X^2 - 2X + 1 \). \( \beta \) is a Salem number, and \( d_\beta(1) = 1(1100)^2 \). It has been proved by an adhoc construction that \( W_{\beta, 1} \) is not recognizable by a finite automaton.

In the case that \( \beta > 1 \) we have from [9] that \( W_{\beta, d} \) is recognizable by a finite automaton for every \( d \) if and only if \( W_{\beta, d} \) is recognizable by a finite automaton for one \( d \geq \lceil \beta \rceil \). So we raise the following conjecture.

**Conjecture 4.6.** Let \( \beta > 1 \) be an algebraic number such that \( W_{\beta, \lceil \beta \rceil - 1} \neq \{0^n \} \). If \( \beta \) has a conjugate of modulus 1 then \( W_{\beta, \lceil \beta \rceil - 1} \) is not recognizable by a finite automaton.

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