Arithmetics on number systems with irrational bases

P. Ambrož, C. Frougny*, Z. Masáková, E. Pelantová

Department of Mathematics, Faculty of Nuclear Sciences
and Physical Engineering, Czech Technical University
Trojanova 13, 120 00 Praha 2, Czech Republic
E-mails: ampy@linux.fjfi.cvut.cz, masakova@km1.fjfi.cvut.cz, pelantova@km1.fjfi.cvut.cz

* LIAFA, CNRS UMR 7089
2 place Jussieu, 75251 Paris Cedex 05, France
and Université Paris 8
E-mail: christiane.frougny@liafa.jussieu.fr

Abstract

For irrational $\beta > 1$ we consider the set $\mathrm{Fin}(\beta)$ of real numbers for which $|x|$ has a finite number of non-zero digits in its expansion in base β. In particular, we consider the set of β-integers, i.e. numbers whose β-expansion is of the form $\sum_{i=0}^{n} x_i\beta^i$, $n \geq 0$. We discuss some necessary and some sufficient conditions for $\mathrm{Fin}(\beta)$ to be a ring. We also describe methods to estimate the number of fractional digits that appear by addition or multiplication of β-integers. We apply these methods among others to the real solution β of $x^3 = x^2 + x + 1$, the so-called Tribonacci number. In this case we show that multiplication of arbitrary β-integers has a fractional part of length at most 5. We show an example of a β-integer x such that $x \cdot x$ has the fractional part of length 4. By that we improve the bound provided by Messaoudi [12] from value 9 to 5; in the same time we refute the conjecture of Arnoux that 3 is the maximal number of fractional digits appearing in Tribonacci multiplication.

1 Introduction

Let β be a fixed real number greater than 1 and let x be a positive real number. A convergent series $\sum_{k=-\infty}^{n} x_k\beta^k$ is called a *β-representation* of x if

$$x = \sum_{k=-\infty}^{n} x_k\beta^k$$

and for all k the coefficient x_k is a non-negative integer. If moreover for every $-\infty < N < n$ we have

$$\sum_{k=-\infty}^{N} x_k\beta^k < \beta^{N+1}$$

the series $\sum_{k=-\infty}^{n} x_k\beta^k$ is called the *β-expansion* of x. The β-expansion is an analogue of the decimal or binary expansion of reals and we sometimes use the natural notation

$$(x)_\beta = x_n x_{n-1} \cdots x_0 \bullet x_{-1} \cdots$$

Every $x \geq 0$ has a unique β-expansion which is found by the greedy algorithm [14].

We can introduce lexicographic ordering on β-representations in the following way. The β-representation $x_n\beta^n + x_{n-1}\beta^{n-1} + \cdots$ is lexicographically greater than $x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$, if $k \leq n$ and for the corresponding infinite words we have $x_n x_{n-1} \cdots \succ \underbrace{00\cdots0}_{(n-k)\text{times}} x_k x_{k-1} \cdots$, where the symbol $\prec$ means the common lexicographic ordering on words in an ordered alphabet.

Since the β-expansion is constructed by the greedy algorithm, it is the lexicographically greatest among all β-representations of x.

The set of those $x \in \mathbb{R}$, for which the β-expansion of $|x|$ has only finitely many non-zero coefficients - digits, is denoted by $\mathrm{Fin}(\beta)$. Real numbers x with β-expansion of $|x|$ of the form $\sum_{k=0}^{n} x_k\beta^k$ are called *β-integers* and the set of β-integers is denoted by $\mathbb{Z}_\beta$.

Let $x \in \mathrm{Fin}(\beta)$, $x > 0$ and let $\sum_{k=-N}^{n} x_k\beta^k$ be its β-expansion with $x_{-N} \neq 0$. If $N > 0$ then $r = \sum_{k=-N}^{-1} x_k\beta^k$ is called the *β-fractional part* of x. If $N \leq 0$ we set $\mathrm{fp}(x) = 0$, for $N > 0$ we define $\mathrm{fp}(x) = N$, i.e. $\mathrm{fp}(x)$ is the number of fractional digits in the β-expansion of x. Note that x is in $\mathbb{Z}_\beta$ if and only if $\mathrm{fp}(x) = 0$.

If $\beta \in \mathbb{Z}$, $\beta > 1$, then $\mathrm{Fin}(\beta)$ is closed under the operations of addition, subtraction and multiplication, i.e. $\mathrm{Fin}(\beta)$ is a ring. It is also easy to determine the β-expansion of $x + y$, $x - y$, and $x \times y$ with the knowledge of the β-expansions of x and y.

In case that $\beta > 1$ is not a rational integer, the situation is more complicated and generally we don't know any criterion which would decide whether $\mathrm{Fin}(\beta)$ is a ring or not. Known results for β satisfying specific conditions will be mentioned below. $\mathrm{Fin}(\beta)$ being closed under addition implies it is closed under multiplication as well. Besides the question whether results of these arithmetic operations are always finite or not, we are also interested to describe the length of the resulting β-fractional part. It is possible to convert the sum $x + y$ and the product $x \cdot y$ of two numbers $x, y \in \mathrm{Fin}(\beta)$ by multiplication by a suitable factor β^k into a sum or product of two β-integers. Therefore we define:

$$\begin{aligned}
L_\oplus = L_\oplus(\beta) \ &:= \ \max\left\{\mathrm{fp}(x+y) \mid x, y \in \mathbb{Z}_\beta,\ x+y \in \mathrm{Fin}(\beta)\right\}, \\
L_\otimes = L_\otimes(\beta) \ &:= \ \max\left\{\mathrm{fp}(x\cdot y) \mid x, y \in \mathbb{Z}_\beta,\ x\cdot y \in \mathrm{Fin}(\beta)\right\}.
\end{aligned}$$

The article is organized as follows. In Preliminaries we state some number theoretical facts and properties of $\mathbb{Z}_\beta$ used below. The paper contains results of two types. Section 3 provides some necessary and some sufficient conditions on β in order that $\mathrm{Fin}(\beta)$ has a ring structure. The proofs of the sufficient conditions are constructive and thus provide an algorithm for addition of finite β-expansions. The second type of results gives estimates on the constants $L_\oplus(\beta)$, $L_\otimes(\beta)$. In Section 4 we explain a simple and effective method for determining these bounds. In Section 5 we apply this method to the case of the Tribonacci number, β the real solution of $x^3 = x^2 + x + 1$. We show

$$5 \leq L_\oplus(\beta) \leq 6 \qquad \text{and} \qquad 4 \leq L_\otimes(\beta) \leq 5$$

which improves the bound on $L_\otimes(\beta) = 9$ given by Messaoudi [12] and refute the conjecture of Arnoux. The next Section 6 shows that the above method cannot be used for every β and therefore in Section 7 we introduce a different method that provides estimates for all Pisot numbers β. We illustrate its application on an example in Section 8. In Section 9 we mention some open problems related to the arithmetics on finite and infinite β-expansions.

2 Preliminaries

The definitions recalled below and related results can be found in the survey [11, Chap. 7].

One can decide whether a particular β-representation of a number is also its β-expansion according to Parry's condition [13]. For fixed $\beta > 1$ we define a mapping $T_\beta : [0,1] \to [0,1)$ by the prescription

$$T_\beta(x) = \beta x - [\beta x],$$

where $[z]$ is the greatest integer smaller or equal to z. The sequence

$$d_\beta(1) = t_1 t_2 t_3 \cdots$$

such that $t_i = [\beta T_\beta^{i-1}(1)]$ is called the Rényi development of 1. Obviously, the numbers t_i are non-negative integers smaller than β, $t_1 = [\beta]$ and

$$1 = \sum_{i=1}^{\infty} t_i \beta^{-i}. \tag{1}$$

In order to state the Parry condition, we introduce the sequence $d^*_\beta(1)$ in the following way: If $d_\beta(1)$ has infinitely many non-zero digits t_i, we set $d^*_\beta(1) = d_\beta(1)$. If m is the greatest index of a non-zero digit in $d_\beta(1)$, i.e.,

$$1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \cdots + \frac{t_m}{\beta^m}, \qquad t_m \neq 0,$$

we set $d^*_\beta(1) = \big(t_1 t_2 \cdots t_{m-1}(t_m - 1)\big)\big(t_1 t_2 \cdots t_{m-1}(t_m - 1)\big) \cdots$, i.e., $d^*_\beta(1)$ is an infinite periodic sequence of period of length m.

Theorem 2.1 (Parry). *Let $\sum_{k=-\infty}^{n} x_k \beta^k$ be a β-representation of a number $x > 0$. Then $\sum_{k=-\infty}^{n} x_k \beta^k$ is the β-expansion of x if and only if for all $i \leq n$ the sequence $x_i x_{i-1} x_{i-2} \cdots$ is strictly lexicographically smaller than the sequence $d^*_\beta(1)$, symbolically*

$$x_i x_{i-1} x_{i-2} \cdots \prec d^*_\beta(1).$$

Notice that a finite string $c_n c_{n-1} \cdots c_0$ of non-negative integers satisfies the Parry condition above, if

$$c_i c_{i-1} \cdots c_0 \prec d_\beta(1) \qquad \text{for all } i = 0, 1, \ldots, n.$$

Such strings are called *admissible*. A finite string of non-negative integers is called *forbidden*, if it is not admissible.

Numbers $\beta > 1$ with eventually periodic Rényi development of 1 are called *beta-numbers* [13]. It is easy to see that every beta-number is a solution of an equation

$$x^n - a_{n-1}x^{n-1} - \cdots - a_1 x - a_0 = 0\,, \qquad a_{n-1}, \ldots, a_1, a_0 \in \mathbb{Z}\,. \tag{2}$$

Therefore β is an algebraic integer. If d is the minimal degree of a polynomial of the form (2), with root β, we say that d is the degree of β. The other roots $\beta^{(2)}, \ldots, \beta^{(d)}$ of the polynomial are called the algebraic conjugates of β. As usually, we denote by $\mathbb{Q}[\gamma]$ the minimal subfield of complex numbers $\mathbb{C}$ which contains γ and the rationals $\mathbb{Q}$. For an algebraic number β it is known that

$$\mathbb{Q}[\beta] = \{g(\beta) \mid g \text{ is a polynomial with coefficients in } \mathbb{Q}\}\,.$$

For $2 \le i \le d$ the mapping $\mathbb{Q}[\beta] \mapsto \mathbb{Q}[\beta^{(i)}]$ given by prescription

$$z = g(\beta) \quad \mapsto \quad z^{(i)} = g(\beta^{(i)})$$

is an isomorphism of fields $\mathbb{Q}[\beta]$ and $\mathbb{Q}[\beta^{(i)}]$. In case that we consider only one resp. two conjugates of β we use instead of $\beta^{(2)}$, resp. $\beta^{(3)}$ the notation β', resp. β'', and similarly we use z', resp. z'' for $z^{(2)}$, resp. $z^{(3)}$.

Besides the eventually periodic Rényi development of 1, beta-numbers have another nice property: the set of distances between neighbors in $\mathbb{Z}_\beta$ is finite and can be determined with the knowledge of $d_\beta(1)$. The distances have the values

$$\Delta_i = \sum_{k=0}^{\infty} \frac{t_{k+i}}{\beta^{k+1}}\,, \qquad \text{for some } i = 1, 2, \ldots\,,$$

see [17]. From the definition of $d_\beta(1)$ it is obvious that the largest distance between neighboring β-integers is

$$\Delta_1 = \sum_{k=0}^{\infty} \frac{t_{k+1}}{\beta^{k+1}} = 1\,.$$

It is not simple to give an algebraic description of the set of beta-numbers [16], however, it is known that every Pisot number is a beta-number [5]. Recall that Pisot numbers are those algebraic integers $\beta > 1$ whose all conjugates are in modulus less than 1. Other results about beta-numbers can be found in [6, 7].

3 Algorithm for addition of positive β-integers

In this section we shall investigate some necessary and some sufficient conditions on β in order that $\mathrm{Fin}(\beta)$ is a ring. In [9] it is shown that if $\mathrm{Fin}(\beta)$ is a ring, then necessarily β is a Pisot number.

According to our definition from the previous section, $\mathrm{Fin}(\beta)$ contains both positive and negative numbers. Therefore we first justify why, in order to decide about $\mathrm{Fin}(\beta)$ being a ring, we shall study only the question of addition of positive numbers, as indicated in the title of the section.

Proposition 3.1. *Let $\beta > 1$ and $d_\beta(1)$ be its development of unit.*

(i) If $d_\beta(1)$ is infinite, then $\mathrm{Fin}(\beta)$ *is not a ring.*

(ii) If $d_\beta(1)$ is finite, then $\mathrm{Fin}(\beta)$ *is a ring if and only if* $\mathrm{Fin}(\beta)$ *is closed under addition of positive elements.*

Proof. (i) Let $d_\beta(1) = t_1 t_2 t_3 \cdots$ be infinite. Then (1) implies

$$1 - \frac{1}{\beta} = \frac{t_1 - 1}{\beta} + \frac{t_2}{\beta^2} + \frac{t_3}{\beta^3} + \cdots \tag{3}$$

Since $(t_1 - 1)t_2 t_3 \cdots \prec d_\beta(1)$, the expression on the right hand side of (3) is the β-expansion of $1 - \beta^{-1}$ which therefore does not belong to $\mathrm{Fin}(\beta)$.

(ii) Let

$$1 = \frac{t_1}{\beta} + \frac{t_2}{\beta^2} + \cdots + \frac{t_m}{\beta^m}\,, \tag{4}$$

and let $\mathrm{Fin}(\beta)$ be closed under addition of positive numbers. Consider arbitrary $x \in \mathrm{Fin}(\beta)$ and arbitrary $\ell \in \mathbb{Z}$ such that $x > \beta^\ell$. Then the β-expansion of x has the form $x = \sum_{i=-N}^{n} x_i \beta^i$, where $n \geq \ell$. Repeated applications of (4) allows us to create a representation of x, say $x = \sum_{i=-M}^{\ell} \tilde{x}_i \beta^i$ such that $\tilde{x}_\ell \geq 1$. Then

$$(\tilde{x}_\ell - 1)\beta^\ell + \sum_{i=-M}^{\ell-1} \tilde{x}_i \beta^i$$

is a finite β-representation of $x - \beta^\ell$. Such representation can be interpreted as a sum of a finite number of positive elements of $\mathrm{Fin}(\beta)$, which is, according to the assumption, again in $\mathrm{Fin}(\beta)$.

It suffices to realize that subtraction $x - y$ of arbitrary $x, y \in \mathrm{Fin}(\beta)$, $x > y > 0$ is a finite number of subtractions of some powers of β. Therefore $\mathrm{Fin}(\beta)$ being closed under addition of positive elements implies being closed under addition of arbitrary $x, y \in \mathrm{Fin}(\beta)$.

Since multiplication of numbers $x, y \in \mathrm{Fin}(\beta)$ is by the distributive law addition of a finite number of summands from $\mathrm{Fin}(\beta)$, the proposition is proved. □

Let us mention that $d_\beta(1)$ infinite does not exclude $\mathrm{Fin}(\beta)$ to be closed under addition of positive elements, see Remark 3.9.

From now on, we focus on addition $x + y$ for $x, y \in \mathrm{Fin}(\beta)$, $x, y \geq 0$. As we have already explained above, it suffices to consider $x, y \in \mathbb{Z}_\beta$. Let $x, y \in \mathbb{Z}_\beta$, $x, y \geq 0$ with β-expansions $x = \sum_{k=0}^{n} a_k \beta^k$, $y = \sum_{k=0}^{n} b_k \beta^k$. Then $\sum_{k=0}^{n} (a_k + b_k)\beta^k$ is a β-representation of the sum $x + y$. If the sequence of coefficients $(a_n + b_n)(a_{n-1} + b_{n-1}) \cdots (a_0 + b_0)$ verifies the Parry condition (Theorem 2.1), we have directly the β-expansion of $x + y$. In the opposite case, the sequence must contain a forbidden string.

The so-called minimal forbidden strings defined below play a special role in our considerations.

Definition 3.2. Let $\beta > 1$. A forbidden string $u_k u_{k-1} \cdots u_0$ of non-negative integers is called *minimal*, if

(i) $u_{k-1}\cdots u_0$ and $u_k\cdots u_1$ are admissible, and

(ii) $u_i \geq 1$ implies $u_k\cdots u_{i+1}(u_i-1)u_{i-1}\cdots u_0$ is admissible, for all $i = 0, 1, \ldots, k$.

Obviously, a minimal forbidden string $u_k u_{k-1}\cdots u_0$ contains at least one non-zero digit, say $u_i \geq 1$. The forbidden string $u_k u_{k-1}\cdots u_0$ with $u_i \geq 1$ is a β-representation of the addition of the two β-integers $w = u_k\beta^k + \cdots + u_{i+1}\beta^{i+1} + (u_i - 1)\beta^i + u_{i-1}\beta^{i-1} + \cdots + u_0$ and $\tilde{w} = \beta^i$. The β-expansion of a number is lexicographically the greatest among all its β-representations, and thus if the sum $w + \tilde{w}$ belongs to $\mathrm{Fin}(\beta)$, then there exists a finite β-representation of $w+\tilde{w}$ lexicographically strictly greater than $u_k u_{k-1}\cdots u_0$ (the β-expansion of $w + \tilde{w}$).

We have thus shown a necessary condition on β in order that $\mathrm{Fin}(\beta)$ is closed under addition of two positive elements. The condition will be formulated as Proposition 3.4. For that we need the following definition.

Definition 3.3. Let $k, p \in \mathbb{Z}$, $k \geq p$ and $z = z_k\beta^k + z_{k-1}\beta^{k-1} + \cdots + z_p\beta^p$, where $z_i \in \mathbb{N}_0$ for $p \leq i \leq k$. A finite β-representation of z of the form $v_n\beta^n + v_{n-1}\beta^{n-1} + \cdots + v_\ell\beta^\ell$, $\ell \leq n$, is called a *transcription* of $z_k\beta^k + z_{k-1}\beta^{k-1} + \cdots + z_p\beta^p$ if

$$k \leq n \qquad \text{and} \qquad v_n v_{n-1}\cdots v_\ell \succ \underbrace{00\cdots 0}_{(n-k)\ \text{times}}\, z_k\cdots z_p\,.$$

Proposition 3.4 (Property T). *Let $\beta > 1$. If $\mathrm{Fin}(\beta)$ is closed under addition of positive elements, then β satisfies Property T:*

For every minimal forbidden string $u_k u_{k-1}\cdots u_0$ there exists a transcription of $u_k\beta^k + u_{k-1}\beta^{k-1} + \cdots + u_1\beta + u_0$.

If Property T is satisfied, the transcription of a β-representation of a number z can be obtained in the following way. Every β-representation of z which contains a forbidden string can be written as a sum of a minimal forbidden string $\beta^j(u_k\beta^k + \cdots + u_1\beta + u_0)$ and a β-representation of some number $\tilde{z}$. The new transcribed β-representation of z is obtained by digit-wise addition of the transcription $\beta^j(v_n\beta^n + \cdots + v_\ell\beta^\ell)$ of the minimal forbidden string and the β-representation of $\tilde{z}$.

Example 3.5. Let us illustrate the notions we have introduced sofar on the simple example of the numeration system based on the golden ratio $\tau = \frac{1}{2}(1+\sqrt{5})$. Since τ satisfies $x^2 = x+1$, its Rényi development of 1 is $d_\tau(1) = 11$. The Parry condition implies that every finite word lexicographically greater or equal to 11 is forbidden. Thus admissible are only those finite words $x_k\cdots x_\ell$ for which $x_i \in \{0,1\}$ and $x_i x_{i-1} = 0$, for $\ell < i \leq k$. According to the definition, minimal forbidden words are 2 and 11. The golden ratio τ satisfies Property T, since $2 = \tau + \tau^{-2}$ and $\tau + 1 = \tau^2$, where $02 \prec 1001$ and $011 \prec 100$. The rules we have derived can be used for obtaining the τ-expansion starting from any τ-representation of a given number x. For example

$$3 = 1 + 2 = 1 + \tau + \frac{1}{\tau^2} = \tau^2 + \frac{1}{\tau^2}\,.$$

The τ-representations $\tau + 1 + \tau^{-2}$ and $\tau^2 + \tau^{-2}$ are transcriptions of 3. In the same time $100 \bullet 01$ is the τ-expansion of the number $x = 3$, since it does not contain any forbidden string.

Let $z_k\beta^k+z_{k-1}\beta^{k-1}+\cdots+z_p\beta^p$ be a β-representation of z such that the string $z_kz_{k-1}\cdots z_p$ is not admissible. Repeating the above described process we obtain a lexicographically increasing sequence of transcriptions. In general, it can happen that the procedure may be repeated infinitely many times without obtaining the lexicographically greatest β-representation of z, i.e. the β-expansion of z. The following theorems provide sufficient conditions in order that such situation is avoided.

Theorem 3.6. *Let $\beta > 1$. Suppose that for every minimal forbidden string $u_ku_{k-1}\cdots u_0$ there exists a transcription $v_n\beta^n+v_{n-1}\beta^{n-1}+\cdots+v_\ell\beta^\ell$ of $u_k\beta^k+u_{k-1}\beta^{k-1}+\cdots+u_1\beta+u_0$ such that*

$$v_n+v_{n-1}+\cdots+v_\ell \le u_k+u_{k-1}+\cdots+u_0\,.$$

Then $\mathrm{Fin}(\beta)$ is closed under addition of positive elements. Moreover, for every positive $x,y\in\mathrm{Fin}(\beta)$, the β-expansion of $x+y$ can be obtained from any β-representation of $x+y$ using finitely many transcriptions.

Proof. Without loss of generality, it suffices to decide about finiteness of the sum $x+y$ where $x=\sum_{k=0}^n a_k\beta^k$ and $y=\sum_{k=0}^n b_k\beta^k$ are the β-expansions of x and y, respectively. We prove the theorem by contradiction, i.e., suppose that we can apply a transcription to the β-representation $\sum_{k=0}^n(a_k+b_k)\beta^k$ infinitely many times.

We find $M\in\mathbb{N}$ such that $x+y<\beta^{M+1}$. Then the β-representation of $x+y$ obtained after the k-th transcription is of the form

$$x+y=\sum_{i=\ell_k}^{M} c_i^{(k)}\beta^i\,,$$

where ℓ_k is the smallest index of non-zero coefficient in the β-representation in the k-th step.

Since for every exponent $i\in\mathbb{Z}$ there exists a non-negative integer f_i such that $x+y\le f_i\beta^i$, we have that $c_i^{(k)}\le f_i$ for every step k.

Realize that for every index $p\in\mathbb{Z}$, $p\le M$, there are only finitely many sequences $c_Mc_{M-1}\cdots c_p$ satisfying $0\le c_i\le f_i$ for all $i=M,M-1,\ldots,p$. Since in every step k the sequence $c_M^{(k)}c_{M-1}^{(k)}\cdots$ lexicographically increases, we can find for every index p the step κ, so that the digits $c_M^{(k)}$, $c_{M-1}^{(k)}$, …, $c_p^{(k)}$ are constant for $k\ge\kappa$. Formally, we have

$$(\forall p\in\mathbb{Z},\, p\le M)(\exists\kappa\in\mathbb{N})(\forall k\in\mathbb{N},\, k\ge\kappa)(\forall i\in\mathbb{Z},\, M\ge i\ge p)(c_i^{(k)}=c_i^{(\kappa)}) \tag{5}$$

Since by assumption of the proof, the transcription can be performed infinitely many times, it is not possible that the digits $c_i^{(\kappa)}$ for $i<p$ are all equal to 0. Let us denote by t the maximal index $t<p$ with non-zero digit, i.e., $c_t^\kappa\ge 1$.

In order to obtain the contradiction, we use the above idea (5) repeatedly. For $p=0$ we find κ_1 and t_1 satisfying

$$x+y=\sum_{i=0}^{M} c_i^{(\kappa_1)}\beta^i+c_{t_1}^{(\kappa_1)}\beta^{t_1}+\sum_{i=\ell_{\kappa_1}}^{t_1-1} c_i^{(\kappa_1)}\beta^i\,.$$

In further steps $k \geq \kappa_1$ the digit sum $\sum_{i=0}^{M} c_i^{(k)}$ remains constant, since the digits $c_i^{(k)}$ remain constant. The digit sum $\sum_{i=t_1}^{(-1)} c_i^{(k)} \geq 1$, because the sequence of digits lexicographically increases. For every $k \geq \kappa_1$ we therefore have

$$\sum_{i=t_1}^{M} c_i^{(k)} \quad \geq \quad 1 + \sum_{i=0}^{M} c_i^{(k)} .$$

We repeat the same considerations for $p = t_1$. Again, we find the step $\kappa_2 > \kappa_1$ and the position $t_2 < t_1$, so that for every $k \geq \kappa_2$

$$\sum_{i=t_2}^{M} c_i^{(k)} \quad \geq \quad 1 + \sum_{i=t_1}^{M} c_i^{(k)} .$$

In the same way we apply (5) and find steps $\kappa_3 < \kappa_4 < \kappa_5 < \dots$ and positions $t_3 > t_4 > t_5 \cdots$ such that the digit sum $\sum_{i=t_s}^{M} c_i^{(\kappa_s)}$ increases with s at least by 1. Since there are infinitely many steps, the digit sum increases with s to infinity, which contradicts the fact that we started with the digit sum $\sum_{k=0}^{n}(a_k + b_k)$ and the transcription we use do not increase the digit sum. $\square$

Let us comment on the consequences of the proof for β satisfying the assumptions of the theorem. The β-expansion of the sum of two β-integers can be obtained by finitely many transcriptions where the order in which we transcribe the forbidden strings in the β-representation of $x + y$ is not important. However, the proof does not provide an estimate on the number of steps needed. Recall that for rational integers the number of steps depends only on the number of digits of the summated numbers. It is an interesting open problem to determine the complexity of the summation algorithm for β-integers.

In order to check whether β satisfies Property T we have to know all the minimal forbidden strings. Let the Rényi development of 1 be finite, i.e., $d_\beta(1) = t_1 t_2 \cdots t_m$. Then a minimal forbidden string has one of the forms

$$(t_1 + 1), \quad t_1(t_2 + 1), \quad t_1 t_2(t_3 + 1), \quad \dots, \quad t_1 t_2 \cdots t_{m-2}(t_{m-1} + 1), \quad t_1 t_2 \cdots t_{m-1} t_m.$$

Note that not all the above forbidden strings must be minimal. For example if β has the Rényi development $d_\beta(1) = 111$, the above list of strings is equal to 2, 12, 111. However, 12 is not minimal.

In [9] it is shown that if β has a finite development of 1 with decreasing digits, then $\mathrm{Fin}(\beta)$ is closed under addition. The proof includes an algorithm for addition. Let us show that the result of [9] is a consequence of our Theorem 3.6.

Corollary 3.7. *Let* $d_\beta(1) = t_1 \cdots t_m$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 1$. *Then* $\mathrm{Fin}(\beta)$ *is closed under addition of positive elements.*

Proof. We shall verify the assumptions of Theorem 3.6. Consider the forbidden string $t_1 t_2 \cdots t_{i-1}(t_i + 1)$, for $1 \leq i \leq m - 1$. Clearly, the following equality is verified

$$t_1\beta^{i-1} + \cdots + t_{i-2}\beta^2 + t_{i-1}\beta + (t_i + 1) =$$
$$= \beta^i + (t_1 - t_{i+1})\beta^{-1} + \cdots + (t_{m-i} - t_m)\beta^{-m+i} + t_{m-i+1}\beta^{-m+i-1} + \cdots + t_m\beta^{-m}.$$

The assumption of the corollary assure that the coefficients on the right hand side are non-negative. The digit sum on the left and on the right is the same. Thus

$$1\underbrace{0\,0\cdots 0}_{i\ \text{times}}\,(t_1-t_{i+1})\,(t_2-t_{i+2})\cdots(t_{m-i}-t_m)$$

is the desired finite string lexicographically strictly greater than $0\,t_1t_2\cdots t_{i-1}(t_i+1)$.

It remains to transcribe the string $t_1t_2\cdots t_{m-1}t_m$ into the lexicographically greater string $1\underbrace{0\,0\cdots 0}_{m\ \text{times}}$. □

The conditions of Theorem 3.6 are however satisfied also for other irrationals that do not fulfil assumptions of Corollary 3.7. As an example we may consider the minimal Pisot number. It is known that the smallest among all Pisot numbers is the real solution β of the equation $x^3 = x + 1$. The Rényi development of 1 is $d_\beta(1) = 10001$. The number β thus satisfies relations

$$\beta^3 = \beta + 1 \qquad \text{and} \qquad \beta^5 = \beta^4 + 1\,.$$

The minimal forbidden strings are 2, 11, 101, 1001, and 10001. Their transcription according to Property T is the following:

$$\begin{aligned} 2 &= \beta^2 + \beta^{-5} \\ \beta + 1 &= \beta^3 \\ \beta^2 + 1 &= \beta^3 + \beta^{-3} \\ \beta^3 + 1 &= \beta^4 + \beta^{-5} \\ \beta^4 + 1 &= \beta^5 \end{aligned}$$

The digit sum in every transcription is smaller than or equal to the digit sum of the corresponding minimal forbidden string. Therefore $\mathrm{Fin}(\beta)$ is according to Theorem 3.6 closed under addition of positive numbers. Since $d_\beta(1)$ is finite, by Proposition 3.1 $\mathrm{Fin}(\beta)$ is a ring. This was shown already in [2].

In the assumptions of Theorem 3.6 the condition of non-increasing digit sum can be replaced by another requirement. We state it in the following theorem. Its proof uses the idea and notation of the proof of Theorem 3.6.

Theorem 3.8. *Let $\beta > 1$ be an algebraic integer satisfying Property T, and let at least one of its conjugates, say β', belong to $(0, 1)$. Then $\mathrm{Fin}(\beta)$ is closed under addition of positive elements. Moreover, for every positive $x, y \in \mathrm{Fin}(\beta)$, the β-expansion of $x+y$ can be obtained from any β-representation of $x + y$ using finitely many transcriptions.*

Proof. If it was possible to apply a transcription on the β-representation of $x + y$ infinitely many times, then we obtain the sequence of β-representations

$$x + y = \sum_{i=\ell_k}^{M} c_i^{(k)} \beta^i\,,$$

where the smallest indices of the non-zero digits ℓ_k satisfy $\lim_{k\to\infty} \ell_k = -\infty$. Here we have used the notation of the proof of Theorem 3.6. Now we use the isomorphism between algebraic fields $\mathbb{Q}[\beta]$ and $\mathbb{Q}[\beta']$ to obtain

$$(x+y)' = x' + y' = \sum_{i=\ell_k}^{M} c_i^{(k)} (\beta')^i \geq (\beta')^{\ell_k} .$$

The last inequality follows from the fact that $\beta' > 0$ and $c_i^{(k)} \geq 0$ for all k and i. Since $\beta' < 1$ we have $\lim_{k\to\infty} (\beta')^{\ell_k} = +\infty$, which is a contradiction. □

Remark 3.9. Let us point out that an algebraic integer β with at least one conjugate in the interval $(0,1)$ must have an infinite Rényi development of 1. Such β has necessarily infinitely many minimal forbidden strings. The only examples known to the authors of β satisfying Property T and having a conjugate $\beta' \in (0,1)$ have been treated in [9], namely those which have eventually periodic $d_\beta(1)$ with period of length 1,

$$d_\beta(1) = t_1 t_2 \cdots t_{m-1} t_m t_m t_m \cdots , \qquad \text{with } t_1 \geq t_2 \geq \cdots \geq t_m \geq 1 . \tag{6}$$

In such a case every minimal forbidden string has a transcription with digit sum strictly smaller than its own digit sum. Thus closure of $\mathrm{Fin}(\beta)$ under addition of positive elements follows already by Theorem 3.6. This means that we don't know any β for which Theorem 3.8 would be necessary.

From the above remark one could expect that the fact that $\mathrm{Fin}(\beta)$ is closed under addition forces the digit sum of the transcriptions of minimal forbidden strings to be smaller than or equal to the digit sum of the corresponding forbidden string. It is not so. For example let β be the solution of $x^3 = 2x^2 + 1$. Then $d_\beta(1) = 201$ and the minimal forbidden string 3 has the β-expansion

$$3 = \beta + \frac{1}{\beta} + \frac{1}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4} .$$

The digit sum of this transcription of 3 is equal to 5. If there exists another transcription of 3 with digit sum ≤ 4, it must be lexicographically strictly larger than 03 and strictly smaller than 101111, because the β-expansion of a number is lexicographically the greatest among all its β-representations. It can be shown easily that a string with the above properties does not exist.

In the same time $\mathrm{Fin}(\beta)$ is closed under addition. This follows from the results of Akiyama who shows that for a cubic Pisot unit β the set $\mathrm{Fin}(\beta)$ is a ring if and only if $d_\beta(1)$ is finite, see [2]. We can also use the result of Hollander [10] who shows that $\mathrm{Fin}(\beta)$ is a ring for $\beta > 1$ root of the equation

$$x^m = a_{m-1}x^{m-1} + \cdots + a_1 x + a_0 , \qquad \text{such that } a_{m-1} > a_{m-2} + \cdots + a_1 + a_0 \quad a_i \in \mathbb{N} .$$

On the other hand, Property T is not sufficient for $\mathrm{Fin}(\beta)$ to be closed under addition of positive elements. As an example we can mention β with the Rényi development of 1 being $d_\beta(1) = 100001$. Such β satisfies $\beta^6 = \beta^5 + 1$. Among the conjugates of β there is a pair of complex conjugates, say β', $\beta'' = \overline{\beta'}$, with absolute value $|\beta'| = |\beta''| \simeq 1.0328$. Thus β is not a Pisot number and according to the result of [9], $\mathrm{Fin}(\beta)$ cannot be closed under addition of positive elements. However, Property T is satisfied for β. All minimal forbidden string can be transcribed as follows:

$$\begin{aligned}
2 &= \beta+\beta^{-6}+\beta^{-7}+\beta^{-8}+\beta^{-9}+\beta^{-10}\\
\beta+1 &= \beta^{2}+\beta^{-6}+\beta^{-7}+\beta^{-8}+\beta^{-9}\\
\beta^{2}+1 &= \beta^{3}+\beta^{-6}+\beta^{-7}+\beta^{-8}\\
\beta^{3}+1 &= \beta^{4}+\beta^{-6}+\beta^{-7}\\
\beta^{4}+1 &= \beta^{5}+\beta^{-6}\\
\beta^{5}+1 &= \beta^{6}
\end{aligned}$$

The expressions on the right hand side are the desired transcriptions, since they are finite and lexicographically strictly greater than the corresponding minimal forbidden string.

4 Upper bounds on $L_\oplus$ and $L_\otimes$

As we have mentioned in the beginning of the previous section, $\mathrm{Fin}(\beta)$ is a ring only for β a Pisot number. However, it is meaningful to study upper bounds on the number of fractional digits that appear as a result of addition and multiplication of β-integers also in the case that $\mathrm{Fin}(\beta)$ is not a ring. We shall explain two methods for determining upper estimates on $L_\oplus(\beta)$ and $L_\otimes(\beta)$. The first method is applicable even in the case that β is not a Pisot number.

The first method stems from the theorem that we cite from [8]. The idea of the theorem is quite simple and was in some way used already by other authors [12, 18]. It uses the isomorphism between the fields $\mathbb{Q}[\beta]$ and $\mathbb{Q}[\beta']$ which to a β-integer $z=\sum_{i=0}^{n} c_i\beta^i$ assigns its algebraic conjugate $z'=\sum_{i=0}^{n} c_i\beta'^i$.

Theorem 4.1. *Let β be an algebraic number, $\beta>1$, with at least one conjugate β' satisfying*

$$\begin{aligned}
H &:= \sup\{|z'| \mid z\in\mathbb{Z}_\beta\} < +\infty\,,\\
K &:= \inf\{|z'| \mid z\in\mathbb{Z}_\beta\setminus\beta\mathbb{Z}_\beta\} > 0\,.
\end{aligned}$$

Then

$$\left(\frac{1}{|\beta'|}\right)^{L_\oplus} < \frac{2H}{K} \qquad \textit{and} \qquad \left(\frac{1}{|\beta'|}\right)^{L_\otimes} < \frac{H^2}{K}\,. \tag{7}$$

Remark 4.2.

- Since $H\geq\sup\{|\beta'^k| \mid k\in\mathbb{N}\}$, the condition $H<+\infty$ implies that $|\beta'|<1$. In this case
$$H\leq\sum_{i=0}^{\infty}[\beta]|\beta'|^i=\frac{[\beta]}{1-|\beta'|}\,.$$
- If $\beta'\in(0,1)$, we have for $z\in\mathbb{Z}_\beta\setminus\beta\mathbb{Z}_\beta$ that $|z'|=\sum_{i=0}^{n}c_i\beta'^i\geq c_0\geq 1$. The value 1 is achieved for $z=1$. Therefore $K=1$.

If the considered algebraic conjugate β' of β is negative or complex, it is complicated to determine the value of K and H. However, for obtaining bounds on $L_\oplus$, $L_\otimes$ it suffices to have "reasonable" estimates on K and H. In order to determine good approximation of K and H we introduce some notation. For $n \in \mathbb{N}$ we shall consider the set

$$E_n := \{z \in \mathbb{Z}_\beta \mid 0 \leq z < \beta^n\}.$$

In fact this is the set of all $a_0 + a_1\beta + \cdots + a_{n-1}\beta^{n-1}$ where $a_{n-1}\cdots a_1 a_0$ is an admissible β-expansion. We denote

$$\begin{aligned} \min\nolimits_n &:= \min\{|z'| \mid z \in E_n, z \notin \beta\mathbb{Z}_\beta\}, \\ \max\nolimits_n &:= \max\{|z'| \mid z \in E_n\}. \end{aligned}$$

Lemma 4.3. *Let $\beta > 1$ be an algebraic number with at least one conjugate $|\beta'| < 1$. Then*

(i) For all $n \in \mathbb{N}$ we have $K \geq K_n := \min_n - |\beta'|^n H$.

(ii) $K > 0$ if and only if there exists $n \in \mathbb{N}$ such that $K_n > 0$.

Proof. (i) Let $z \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$. Then the β-expansion of z is $z = \sum_{i=0}^{N} b_i\beta^i$, $b_0 \neq 0$. The triangle inequality gives

$$|z'| \geq \left|\sum_{i=0}^{n-1} b_i {\beta'}^i\right| - \left|\sum_{i=n}^{N} b_i {\beta'}^i\right| \geq \min\nolimits_n - |\beta'|^n \left|\sum_{i=n}^{N} b_i {\beta'}^{i-n}\right| > \min\nolimits_n - |\beta'|^n H = K_n\,.$$

Hence taking the infimum on both sides we obtain $K \geq K_n$. (ii) From the definition of $\min_n$ it follows that $\min_n$ is a decreasing sequence with $\lim_{n\to\infty} \min_n = K$. If there exists $n \in \mathbb{N}$ such that $\min_n - |\beta'|^n H > 0$ we have $K > 0$ from (i). The opposite implication follows easily from the fact that $\lim_{n\to\infty} K_n = K$. $\square$

For a fixed β, the determination of $\min_n$ for small n is relatively easy. It suffices to find the minimum of a finite set with small number of elements. If for such n we have $K_n = \min_n - |\beta'|^n H > 0$, we obtain using (7) bounds on $L_\oplus$ and $L_\otimes$. We illustrate this procedure on the real solution β of $x^3 = x^2 + x + 1$, the so-called Tribonacci number.

5 $L_\oplus$, $L_\otimes$ for the Tribonacci number

Let β be the real root of $x^3 = x^2 + x + 1$. The arithmetics on β-expansions was already studied in [12]. Messaoudi [12] finds the upper bound on the number of β-fractional digits for the Tribonacci multiplication as 9. Arnoux, see [12], conjectures that $L_\otimes = 3$. We refute the conjecture of Arnoux and find a better bound on $L_\otimes$ than 9. Moreover, we find the bound for $L_\oplus$, as well.

It turns out that the best estimates on $L_\oplus$, $L_\otimes$ are obtained by Theorem 4.1 with approximation of K by K_n for $n = 9$. By inspection of the set E_9 we obtain

$$\min\nolimits_9 = |1 + {\beta'}^2 + {\beta'}^4 + {\beta'}^7| \simeq 0.5465$$

and

$$\max_9 = |1 + {\beta'}^3 + {\beta'}^6| \simeq 1.5444$$

Consider $y \in \mathbb{Z}_\beta$, $y = \sum_{k=0}^{N} a_k \beta^k$. Then from the triangle inequality

$$\begin{aligned} |y'| &\leq \left|\sum_{k=0}^{8} a_k {\beta'}^k\right| + |\beta'|^9 \left|\sum_{k=9}^{17} a_k {\beta'}^{k-9}\right| + |\beta'|^{18} \left|\sum_{k=18}^{26} a_k {\beta'}^{k-18}\right| + \cdots \\ &< \max_9 \left(1 + |\beta'|^9 + |\beta'|^{18} + \cdots\right) = \frac{\max_9}{1 - |\beta'|^9}. \end{aligned}$$

In this way we have obtained an upper estimate on H, i.e., $H \leq \frac{\max_9}{1-|\beta'|^9}$. This implies

$$K_9 = \min_9 - |\beta'|^9 H \geq \min_9 - |\beta'|^9 \frac{\max_9}{1 - |\beta'|^9}.$$

Hence

$$\begin{aligned} \left(\frac{1}{|\beta'|}\right)^{L_\oplus} &< \frac{2H}{K} \leq \; 2\frac{\max_9}{1-|\beta'|^9}\left(\min_9 - |\beta'|^9 \frac{\max_9}{1-|\beta'|^9}\right)^{-1} \simeq 7.5003 \\ \left(\frac{1}{|\beta'|}\right)^{L_\otimes} &< \frac{H^2}{K} \leq \; \left(\frac{\max_9}{1-|\beta'|^9}\right)^2 \left(\min_9 - |\beta'|^9 \frac{\max_9}{1-|\beta'|^9}\right)^{-1} \simeq 6.1908 \end{aligned}$$

Since

$$\left(\frac{1}{|\beta'|}\right)^5 \simeq 4.5880, \quad \left(\frac{1}{|\beta'|}\right)^6 \simeq 6.2222, \quad \left(\frac{1}{|\beta'|}\right)^7 \simeq 8.4386,$$

we conclude that $L_\oplus \leq 6$, $L_\otimes \leq 5$.

In order to determine the lower bounds on $L_\oplus$, $L_\otimes$ we have used a computer program [3] to perform additions and multiplications on a large set of β-expansions. As a result we have obtained examples of a sum with 5 and product with 4 fractional digits, namely:

$$\begin{aligned} 1001011010 + 1001011011 &= 10100100100 \bullet 10101 \\ 110100100101101 \times 110100100101101 &= 11001000100010000100100101101 1 \bullet 0011 \end{aligned}$$

We can thus sum up our results as

$$5 \leq L_\oplus \leq 6 \qquad \text{and} \qquad 4 \leq L_\otimes \leq 5.$$

Let us mention that Bernat [4] recently improved the result for addition to $L_\oplus = 5$.

6 Case $K = 0$

The above mentioned method cannot be used in case that $K = 0$. It is however difficult to prove that $K = 0$ for a given algebraic β and its conjugate β'. Particular situation is solved by the following proposition.

Proposition 6.1. *Let $\beta > 1$ be an algebraic number and $\beta' \in (-1, 0)$ its conjugate such that $\frac{1}{\beta'^2} < [\beta]$. Then $K = 0$.*

Proof. Set $\gamma := \beta'^{-2}$. Digits in the γ-expansion take values in the set $\{0, 1, \ldots, [\gamma]\}$. Since $[\gamma] \leq [\beta] - 1$ and the Rényi development of unit $d_\beta(1)$ is of the form $d_\beta(1) = [\beta]t_2t_3\cdots$, every sequence of digits in $\{0, 1, \ldots, [\gamma]\}$ is lexicographically smaller than $d_\beta(1)$ and thus is an admissible β-expansion. Since $1 < -\beta'^{-1} < \gamma$, the γ-expansion of $-\beta'^{-1}$ has the form

$$-\beta'^{-1} = c_0 + c_1\gamma^{-1} + c_2\gamma^{-2} + c_3\gamma^{-3} + \cdots \tag{8}$$

where all coefficients $c_i \leq [\beta] - 1$.

Let us define the sequence $z_n := 1 + c_0\beta + c_1\beta^3 + c_2\beta^5 + \cdots + c_n\beta^{2n+1}$. Clearly, $z_n \in \mathbb{Z}_\beta \setminus \beta\mathbb{Z}_\beta$ and $z_n' := 1 + \beta'(c_0 + c_1\beta'^2 + c_2\beta'^4 + \cdots + c_n\beta'^{2n})$. According to (8) we have $\lim_{n\to\infty} z_n' = 0 = \lim_{n\to\infty} |z_n'|$. Finally, this implies $K = 0$. □

An example of an algebraic number satisfying assumptions of Propositions 6.1 is $\beta > 1$ solution of the equation $x^3 = 25x^2 + 15x + 2$. The algebraic conjugates of $\beta \simeq 25.5892$ are $\beta' \simeq -0.38758$ and $\beta'' \simeq -0.20165$, and so $K = 0$ for both of them. Hence Theorem 4.1 cannot be used for determining the bounds on $L_\oplus$, $L_\otimes$. We thus present another method for finding these bounds and illustrate it further on the mentioned example.

Note that similar situation happens infinitely many times, for example for a class of totally real cubic numbers, solutions of $x^3 = p^6x^2 + p^4x + p$, for $p \geq 3$. Theorem 4.1 cannot be applied to any of them which justifies utility of a new method.

7 Upper bounds on $L_\oplus$, $L_\otimes$ for β Pisot numbers

The second method of determining upper bounds on $L_\oplus$, $L_\otimes$ studied in this paper is applicable to β being a Pisot number, i.e., an algebraic integer $\beta > 1$ with conjugates in modulus less than 1. This method is based on the so-called cut-and-project scheme.

Let $\beta > 1$ be an algebraic integer of degree d, let $\beta^{(2)}, \ldots, \beta^{(s)}$ be its real conjugates and let $\beta^{(s+1)}$, $\beta^{(s+2)} = \overline{\beta^{(s+1)}}$, $\ldots$, $\beta^{(d-1)}$, $\beta^{(d)} = \overline{\beta^{(d-1)}}$ be its non-real conjugates. Then there exists a basis $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d$ of the space $\mathbb{R}^d$ such that every $\vec{x} = (a_0, a_1, \ldots, a_{d-1}) \in \mathbb{Z}^d$ has in this basis the form

$$\vec{x} = \alpha_1\vec{x}_1 + \alpha_2\vec{x}_2 + \cdots + \alpha_d\vec{x}_d\,,$$

where

$$\alpha_1 = a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} =: z \in \mathbb{Q}[\beta]$$

and

$$\begin{array}{ll} \alpha_i = z^{(i)} & \text{for } i = 2, 3, \ldots, s, \\ \alpha_j = \Re(z^{(j)}) & \text{for } s < j \leq d,\ j \text{ odd}, \\ \alpha_j = \Im(z^{(j)}) & \text{for } s < j \leq d,\ j \text{ even}. \end{array}$$

Technical details of the construction of the basis $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_d$ can be found in [1, 8, 18]. Let us denote

$$\mathbb{Z}[\beta] := \{a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1} \mid a_i \in \mathbb{Z}\}\,.$$

For β an algebraic integer, the set $\mathbb{Z}[\beta]$ is a ring and moreover it can be geometrically interpreted as a projection of the lattice $\mathbb{Z}^d$ on a suitable chosen straight line in $\mathbb{R}^d$. The correspondence $(a_0, a_1, \ldots, a_{d-1}) \mapsto a_0 + a_1\beta + a_2\beta^2 + \cdots + a_{d-1}\beta^{d-1}$ is a bijection of the lattice $\mathbb{Z}^d$ on the ring $\mathbb{Z}[\beta]$.

In the following, we shall consider β an irrational Pisot number. Important property that will be used is the inclusion

$$\mathbb{Z}_\beta \subset \mathbb{Z}[\beta]\,. \tag{9}$$

Let us recall that $\mathbb{Z}_\beta$ is a proper subset of $\mathbb{Z}[\beta]$, since $\mathbb{Z}[\beta]$ is dense in $\mathbb{R}$ as a projection of the lattice $\mathbb{Z}^d$, whereas $\mathbb{Z}_\beta$ has no accumulation points. Since $\mathbb{Z}[\beta]$ is a ring,

$$\mathbb{Z}_\beta + \mathbb{Z}_\beta \subset \mathbb{Z}[\beta] \qquad \text{and} \qquad \mathbb{Z}_\beta \cdot \mathbb{Z}_\beta \subset \mathbb{Z}[\beta]\,.$$

Consider an $x \in \mathbb{Z}_\beta$ with the β-expansion $x = \sum_{k=0}^{n} a_k \beta^k$. Then

$$|x^{(i)}| = \left|\sum_{k=0}^{n} a_k (\beta^{(i)})^k\right| < \sum_{k=0}^{\infty} [\beta]\, \left|\beta^{(i)}\right|^k = \frac{[\beta]}{1-|\beta^{(i)}|}\,,$$

for every $i = 2, 3, \ldots, d$. Therefore we can define

$$H_i := \sup\{|x^{(i)}| \mid x \in \mathbb{Z}_\beta\} \tag{10}$$

The inclusion (9) thus can be precised to

$$\mathbb{Z}_\beta \subset \{x \in \mathbb{Z}[\beta] \mid |x^{(i)}| < H_i,\ i = 2, 3, \ldots, d\}\,.$$

Another important property needed for determining upper bounds of $L_\oplus$, $L_\otimes$ is the finiteness of the set

$$C(l_1, l_2, \ldots, l_d) := \{x \in \mathbb{Z}[\beta] \mid |x| < l_1,\ |x^{(i)}| < l_i,\ i = 2, 3, \ldots, d\}\,,$$

for every choice of positive $l_1, l_2, \ldots, l_d$. A point $a_0 + a_1\beta + \cdots + a_{d-1}\beta^{d-1}$ belongs to $C(l_1, l_2, \ldots, l_d)$ only if the point $(a_0, a_1, \ldots, a_{d-1})$ of the lattice $\mathbb{Z}^d$ has all coordinates in the basis $\vec{x}_1, \ldots, \vec{x}_d$ in a bounded interval $(-l_i, l_i)$, i.e., $(a_0, a_1, \ldots, a_{d-1})$ belongs to a centrally symmetric parallelepiped. Every parallelepiped contains only finitely many lattice points. Let us mention that notation $C(l_1, l_2, \ldots, l_d)$ is kept in accordance with [1], where Akiyama finds some conditions for $\mathrm{Fin}(\beta)$ to be a ring according to the properties of $C(l_1, l_2, \ldots, l_d)$. Our aim here is to use the set for determining the bounds on the length of the β-fractional part of the results of additions and multiplications in $\mathbb{Z}_\beta$.

Theorem 7.1. *Let β be a Pisot number of degree d, and let $H_2, H_3, \ldots, H_d$ be defined by* (10). *Then*

$$\begin{aligned} L_\oplus &\le \max\{\mathrm{fp}(r) \mid r \in \mathrm{Fin}(\beta) \cap C(1, 3H_2, 3H_3, \ldots, 3H_d)\}\,, \\ L_\otimes &\le \max\{\mathrm{fp}(r) \mid r \in \mathrm{Fin}(\beta) \cap C(1, H_2^2 + H_2, \ldots, H_d^2 + H_d)\}\,. \end{aligned}$$

Proof. Consider $x, y \in \mathbb{Z}_\beta$ such that $x + y > 0$, $x + y \in \mathrm{Fin}(\beta)$. Set $z := \max\{w \in \mathbb{Z}_\beta \mid w \le x + y\}$. Then $r := x + y - z$ is the β-fractional part of $x + y$ and thus $r \in \mathrm{Fin}(\beta)$ and $\mathrm{fp}(r) = \mathrm{fp}(x + y)$, and $0 \le r < 1$. Numbers x, y, z belong to the ring $\mathbb{Z}[\beta]$ and hence also $r \in \mathbb{Z}[\beta]$. From the triangle inequality

$$|r^{(i)}| = |x^{(i)} + y^{(i)} - z^{(i)}| \le 3H_i$$

for all $i = 2, 3, \ldots, d$. Therefore r belongs to the finite set $C(1, 3H_2, 3H_3, \ldots, 3H_d)$, which together with the definition of $L_\oplus$ gives the statement of the theorem for addition. The upper bound on $L_\otimes$ is obtained analogically. $\square$

8 Application to β solution of $x^3 = 25x^2 + 15x + 2$.

We apply the above Theorem 7.1 on $\beta > 1$ solution of the equation $x^3 = 25x^2 + 15x + 2$. Recall that such β satisfies the conditions of Proposition 6.1 for both conjugates β', β'' and thus Theorem 4.1 cannot be used for determining the bounds on $L_\oplus$, $L_\otimes$.

Since $[\beta] = 25$, β-expansions are words in the 26-letter alphabet, say $\{(0), (1), \dots, (25)\}$. The Rényi development of 1 is $d_\beta(1) = (25)(15)(2)$. Since the digits of $d_\beta(1)$ are decreasing, Corollary 3.7 implies that the set $\mathrm{Fin}(\beta)$ is a ring.

In case that some of the algebraic conjugates of β is a real number, the bounds from Theorem 7.1 can be refined. In our case β is totally real. Let $x \in \mathbb{Z}_\beta$, $x = \sum_{i=0}^{n} a_i\beta^i$. Since $\beta' < 0$, we have

$$x' = \sum_{i=0}^{n} a_i(\beta')^i \le \sum_{i=0, i \text{ even}}^{n} a_i(\beta')^i < \sum_{i=0}^{\infty} (25)(\beta')^{2i} = \frac{25}{1-\beta'^2} = H_1\,.$$

The lower bound on x' is

$$x' = \sum_{i=0}^{n} a_i(\beta')^i \ge \sum_{i=0, i \text{ odd}}^{n} a_i(\beta')^i > \beta' H_1\,.$$

Similarly for x'' we obtain

$$\beta'' H_2 < x'' < \frac{25}{1-\beta''^2} = H_2\,.$$

Consider $x, y \in \mathbb{Z}_\beta$ such that $x + y > 0$. Again, the β-fractional part of $x + y$ has the form $r = x + y - z$ for some $z \in \mathbb{Z}_\beta$. Thus

$$\begin{array}{rcccl} (2\beta'-1)H_1 = \beta' H_1 + \beta' H_1 - H_1 & < & r' = x' + y' - z' & < & H_1 + H_1 - \beta' H_1 = (2-\beta')H_1 \\ (2\beta''-1)H_2 & < & r'' = x'' + y'' - z'' & < & (2-\beta'')H_2 \end{array}$$

We have used a computer to calculate explicitly the set of remainders $r = A + B\beta + C\beta^2$, $A, B, C \in \mathbb{Z}$, satisfying

$$\begin{array}{rcl} 0 < & A + B\beta + C\beta^2 & < 1 \\ (2\beta'-1)H_1 < & A + B\beta' + C\beta'^2 & < (2-\beta')H_1 \\ (2\beta''-1)H_2 < & A + B\beta'' + C\beta''^2 & < (2-\beta'')H_2 \end{array}$$

where for β', β'' we use numerical values, (see Section 6). The set has 93 elements, which we shall not list here. For every element of the set we have found the corresponding β-expansion. The maximal length of the β-fractional part is 5. Thus $L_\oplus \le 5$.

On the other hand, using the algorithm described in Section 3 we have found a concrete example of addition of numbers $(x)_\beta = (25)(0)(25)$ and $(y)_\beta = (25)(0)(25)$, so that $x + y$ has the β-expansion $(x + y)_\beta = (1)(24)(12)(11) \bullet (23)(0)(14)(13)(2)$. Thus we have found the value

$$L_\oplus = 5\,.$$

In order to obtain bounds on $L_\otimes$, we have computed the list of all $r = A + B\beta + C\beta^2$, $A, B, C \in \mathbb{Z}$, satisfying the inequalities

$$\begin{array}{rcl} 0 < & A + B\beta + C\beta^2 & < 1 \\ \beta' H_1^2 - H_1 \; < & A + B\beta' + C\beta'^2 & < \; H_1^2 - \beta' H_1 \\ \beta'' H_2^2 - H_2 \; < & A + B\beta'' + C\beta''^2 & < \; H_2^2 - \beta'' H_2 \end{array}$$

In this case we have obtained 8451 candidates on the β-fractional part of multiplication. The longest of them has 7 digits. From the other hand we have for $(x)_\beta = (25)$ and $(y)_\beta = (25)$, $(x \cdot y)_\beta = (24)(10) \bullet (21)(24)(16)(7)(16)(13)(2)$. Therefore

$$L_\otimes = 7\,.$$

Let us mention that the above method can be applied also to the case of the Tribonacci number, but the bounds obtained in this way are not better than those from Theorem 4.1. We get $L_\oplus \leq 6$, $L_\otimes \leq 6$.

9 Comments and open problems

1. It is clear from the second method of estimate that for β a Pisot number $L_\oplus$, $L_\otimes$ are finite numbers. Does there exist a β non Pisot such that $L_\oplus = +\infty$ or $L_\otimes = +\infty$?
2. Whenever β satisfies Property T, it is possible to apply repeatedly the transcription on the β-representation of $x + y$, $x, y \in \mathrm{Fin}(\beta)$, $x, y > 0$. If the transcription can be applied infinitely many times, what is the order of choice of forbidden strings so that the sequence of β-representations converges rapidly to the β-expansion of $x + y$?
3. It is known [5, 15] that for β a Pisot number every $x \in \mathbb{Q}[\beta]$ has a finite or eventually periodic β-expansion. It would be interesting to have algorithms working with periodic expansions.

Acknowledgements

The authors acknowledge financial support by the Grant Agency of the Czech Republic GA 201/01/0130; by the Barrande Project number 03900WJ, Z.M. and E.P. are grateful for the hospitality of LIAFA, Université Paris 7, and Ch.F. thanks the Department of Mathematics, FNSPE Czech Technical University, where part of the work was done.

References

[1] S. Akiyama, *Self affine tiling and Pisot numeration system,* in *Number Theory and its Applications,* Kyoto 1997, Eds. K. Györy and S. Kanemitsu, Kluwer Acad. Publ., Dordrecht (1999), 7–17.

[2] S. Akiyama, *Cubic Pisot Units with finite beta expansions,* in *Algebraic Number Theory and Diophantine Analysis,* Graz 1998, Eds. F.Halter-Koch and R.F. Tichy, de Gruyter, Berlin, (2000), 11–26.

[3] P. Ambrož, Computer program performing arithmetical operations on β-integers for a family of Pisot numbers:

`http://linux.fjfi.cvut.cz/~ampy/dp/BetaArith/BetaArithmetics.exe`

[4] J. Bernat, private communication (2003).

[5] A. Bertrand, *Développements en base de Pisot et répartition modulo 1,* C. R. Acad. Sc. Paris, Série A, t.285, (1977), 419–421.

[6] D. W. Boyd, *On the beta expansions for Salem numbers of degree* 6, Math. Comp. **65** (1996) 861–875.

[7] D. W. Boyd, *Salem numbers of degree four have periodic expansions,* Théorie des nombres (Québec, PQ 1987), 57–64, de Gruyter, Berlin-New York, (1989).

[8] L.S. Guimond, Z. Masáková, E. Pelantová, *Arithmetics on β-expansions,* to appear in Acta Arith. (2003).

[9] Ch. Frougny and B. Solomyak, *Finite β-expansions,* Ergodic Theory Dynamical Systems **12** (1994), 713–723.

[10] M. Hollander, *Linear numeration systems, finite beta-expansions, and discrete spectrum of substitution dynamical systems,* Ph.D. Thesis, Washington University (1996).

[11] M. Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, 2002.

[12] A. Messaoudi, *Généralisation de la multiplication de Fibonacci,* Math. Slovaca **50** (2000), 135–148.

[13] W. Parry, *On the β-expansions of real numbers,* Acta Math. Acad. Sci. Hung. **11** (1960), 401–416.

[14] A. Rényi, *Representations for real numbers and their ergodic properties,* Acta Math. Acad. Sci. Hung. **8** (1957), 477–493.

[15] K. Schmidt, *On periodic expansions of Pisot numbers and Salem numbers,* Bull. London Math. Soc. **12** (1980), 269–278.

[16] B. Solomyak, *Conjugates of beta-numbers and the zero-free domain for a class of analytic functions,* Proc. London Math. Soc. **68** (1994), 477–498.

[17] W.P. Thurston, *Groups, tilings, and finite state automata,* Geometry supercomputer project research report GCG1, University of Minnesota (1989).

[18] J.-L. Verger-Gaugry, J. P. Gazeau, *Geometric study of the set of beta-integers for a Perron number and mathematical quasicrystals,* to appear in J. Théor. Nombres Bordeaux (2003).