BETA-INTEGERS AS A GROUP

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Discrete sets of numbers related to some irrational number β, called the β-integers $\mathbb{Z}_\beta$, are a natural tool to describe quasicrystals. We prove that, when β is a quadratic Pisot unit, the set $\mathbb{Z}_\beta$ is a group isomorphic to the set of integers $\mathbb{Z}$.

1 Introduction

In a previous work of us [2], we have studied mathematical properties of certain sets of numbers, called β-integers, where β is an irrational number, and describing quasicrystals. The reader wanting to know some of our physical motivations may consult this paper. The β-integers are defined via a numeration system with base β, see below. In [2], we proved that, when β is a Pisot number, the set $\mathbb{Z}_\beta$ of β-integers is a Meyer set.

Here we want to add the following comments. In many numerical simulations, in physics or elsewhere, the set $\mathbb{R}^m$ is discretized by lattices of the form

$$\Lambda = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_m$$

where $e_1, \ldots, e_m$ are linearly independent. The process is refined with smaller scales such as Λ/β^k, where β is an integer, and the stability of results is examined according to k. Our purpose is the following. We define β-grids of the form

$$\Lambda_\beta = \mathbb{Z}_\beta\, e_1 + \cdots + \mathbb{Z}_\beta\, e_m \subset \mathbb{R}^m$$

where the directions e_i are related in some algebraic sense to the irrational number β, and similarly the refinement takes the form Λ_β/β^k.

For instance, gas modeling (*e.g.* computation of ground states) can use such a β-grid, by putting or not atoms on nodes, and the problem becomes then to minimize the free energy. Therefore, by using a β-grid with a suitable Pisot number β, we obtain equilibrium structures arising from the competition between energy and entropy. In the case where β is the golden ratio, we calculated by low-temperature Monte-Carlo simulation such a Gibbs-type statistical mechanics model (see Figure 1.a). Monte-Carlo here simulates slow cooling process and leads to metastable binary structures with typical local configurations. Its diffraction pattern (see Figure 1.b) shows that such generalized lattice-gas approach defines a quasicrystalline structure.

The usual approach to the problem of finding ground state configurations is based on lattices which are Abelian groups. It is thus natural to investigate the possibilities of group structure for $\mathbb{Z}_\beta$ and thus of Λ_β. Note that Λ_β is self-similar of ratio β.

In this work we extend our previous results, by showing that, when β is a quadratic Pisot unit, the set $\mathbb{Z}_\beta$ is a group isomorphic to the set of integers $\mathbb{Z}$.

Note that the quasicrystalline structures observed in reality present self-similarity ratios which happen to be quadratic Pisot units, namely

- $\beta = (1+\sqrt{5})/2 = 2\cos\frac{\pi}{5}$ of minimal polynomial $X^2 - X - 1$: penta or decagonal symmetry
- $\beta = 1+\sqrt{2} = 1 + 2\cos\frac{\pi}{4}$ of minimal polynomial $X^2 - 2X - 1$: octogonal symmetry
- $\beta = 2+\sqrt{3} = 2 + 2\cos\frac{\pi}{6}$ of minimal polynomial $X^2 - 4X + 1$: dodecagonal symmetry.

2 Definitions

2.1 Representation of real numbers

Let $\beta > 1$ be a real number. A *representation in base* β (or a β-representation) of a real number $x \geq 0$ is an infinite sequence $(x_i)_{i\leq k}$, such that

$$x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots + x_1\beta + x_0 + x_{-1}\beta^{-1} + x_{-2}\beta^{-2} + \cdots$$

for a certain integer $k \geq 0$.

A particular β-representation — called the *β-expansion* — can be computed by the "greedy algorithm" (see [6] and [5]):

Denote by $\lfloor y \rfloor$ and $\{y\}$ the integer part and the fractional part of a number y. There exists $k \in \mathbb{Z}$ such that $\beta^k \leq x < \beta^{k+1}$. Let $x_k = \lfloor x/\beta^k \rfloor$ and

$r_k = \{x/\beta^k\}$. Then for $i < k$, put $x_i = \lfloor \beta r_{i+1} \rfloor$, and $r_i = \{\beta r_{i+1}\}$. We get an expansion $x = x_k\beta^k + x_{k-1}\beta^{k-1} + \cdots$. If $x < 1$ then $k < 0$, and we put $x_0 = x_{-1} = \ldots = x_{k+1} = 0$. The β-expansion of x is denoted by

$$\langle x \rangle_\beta = x_k x_{k-1} \ldots x_1 x_0 \cdot x_{-1} x_{-2} \ldots$$

The digits x_i obtained by this algorithm are integers from the set $A = \{0, \ldots, \beta - 1\}$ if β is an integer, or the set $A = \{0, \ldots, \lfloor \beta \rfloor\}$ if β is not an integer. The set A is called the *canonical alphabet*. We will sometimes omit the splitting point between the integer part and the fractional part of the β-expansion; then the infinite sequence is just an element of $A^{\mathbb{N}}$. If an expansion ends in infinitely many zeros, it is said to be *finite*, and the ending zeros are omitted.

There is a representation which plays an important role in the theory. The β-expansion of 1, denoted by $d_\beta(1)$, is computed by the following process [6]. Let the β-transform be defined on $[0, 1]$ by $T_\beta(x) = \beta x \bmod 1$. Then $d_\beta(1) = (t_i)_{i \geq 1}$, where $t_i = \lfloor \beta T_\beta^{i-1}(1) \rfloor$. Note that $d_\beta(1)$ belongs to $A^{\mathbb{N}}$.

Beta-integers

The set of *β-integers* is the set of real numbers having a zero fractional part in their β-expansion

$$\begin{aligned} \mathbb{Z}_\beta &= \{x \in \mathbb{R} \mid \langle |x| \rangle_\beta = x_k \cdots x_0\} \\ &= \mathbb{Z}_\beta^+ \cup (-\mathbb{Z}_\beta^+) \end{aligned}$$

where $\mathbb{Z}_\beta^+$ is the set of non-negative β-integers.

$\mathbb{Z}_\beta$ is self-similar and symmetrical with respect to the origin

$$\beta \mathbb{Z}_\beta \subset \mathbb{Z}_\beta \,, \; \mathbb{Z}_\beta = -\mathbb{Z}_\beta$$

2.2 Representation of integers

Let $U = (u_n)_{n \geq 0}$ be a strictly increasing sequence of integers with $u_0 = 1$. A *representation in the system U* — or a *U-representation* — of a nonnegative integer N is a finite sequence of integers $(d_i)_{0 \leq i \leq k}$ such that

$$N = \sum_{i=0}^{k} d_i u_i.$$

Such a representation will be written $d_k \cdots d_0$, most significant digit first.

Among all possible U-representations $d_k \cdots d_0$ of a given integer N one is distinguished and called the *greedy U-representation* of N: the greatest in

the lexicographical ordering. It is obtained by the following greedy algorithm (see [4]):
Given integers m and p let us denote by $q(m,p)$ and $r(m,p)$ the quotient and the remainder of the Euclidean division of m by p.
Let $k \geq 0$ such that $u_k \leq N < u_{k+1}$ and let $d_k = q(N, u_k)$ and $r_k = r(N, u_k)$, $d_i = q(r_{i+1}, u_{n+i})$ and $r_i = r(r_{i+1}, u_{n+i})$ for $i = k-1, \cdots, 0$. Then $N = d_k u_k + \cdots + d_0 u_0$.

The greedy representation is often called the *normal* representation. The greedy representation of N in the system U is denoted by $\langle N \rangle_U$.

By convention the greedy representation of 0 is the empty word ε. Under the hypothesis that the ratio u_{n+1}/u_n is bounded by a constant as n tends to infinity, the integers of the greedy U-representation of any integer N are bounded and contained in a *canonical* finite alphabet A_U associated to U.

3 Numeration systems associated with β

A *Pisot number* is an algebraic integer > 1 such that all its Galois conjugates have modulus less than one.

There is a general theory valid for any Pisot number β but here we consider only the case where β is a quadratic Pisot unit. There are two cases to be considered.

Case 1: β is the root > 1 of the polynomial $X^2 - aX - 1$, with $a \geq 1$

The canonical alphabet is equal to $A = \{0, \dots, a\}$, the β-expansion of 1 is finite, $d_\beta(1) = a1$, and every positive number of $\mathbb{Z}[\beta]$ has a finite β-expansion [3].

Let U_β be the sequence of integers defined by

$$u_{n+2} = au_{n+1} + u_n$$

$$u_0 = 1,\ u_1 = [\beta] + 1 = a + 1.$$

It will be useful for our purpose to extend the recurrence to elements with negative indices such as u_{-n}.

We recall some facts (see [1]).

Lemma 1 . *An infinite sequence is the β-expansion of a positive real number, and a finite sequence is the greedy U_β-representation of a positive integer if and only if it does not contain a factor of the form $a1$, $a2$, ..., aa.*

Case 2: β is the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$

The canonical alphabet is equal to $A = \{0, \ldots, a-1\}$, the β-expansion of 1 is eventually periodic, $d_\beta(1) = (a-1)(a-2)^\omega$, and every positive number of $\mathbb{Z}[\beta]$ has an eventually periodic β-expansion, which is finite for numbers from $\mathbb{N}[\beta]$ ([3]).

Let U_β be the sequence of integers defined by

$$u_{n+2} = a u_{n+1} - u_n$$

$$u_0 = 1,\ u_1 = [\beta] + 1 = a.$$

We also extend the recurrence to elements of the form u_{-n}.

The following properties are well-known, see [1].

LEMMA **2** . *An infinite sequence is the β-expansion of a positive real number, and a finite sequence is the greedy U_β-representation of a positive integer if and only if it does not contain a factor of the form $(a-1)(a-2)^n(a-1)$.*

Lemmas 1 and 2 mean that the dynamical systems defined by β-expansions of real numbers and by greedy U_β-representations of natural numbers are the same.

Let us consider the set $\mathbb{Z}[X, X^{-1}]$ of formal polynomials in X and X^{-1}. We define two applications. The first one is

$$\begin{array}{c} \pi_\beta : \mathbb{Z}[X, X^{-1}] \to \mathbb{Z}[\beta, \beta^{-1}] \subset \mathbb{R} \\ F = f_k X^k + \cdots + f_{-j} X^{-j} \mapsto f_k \beta^k + \cdots + f_{-j} \beta^{-j} \end{array}$$

The second application is

$$\begin{array}{c} \pi_{U_\beta} : \mathbb{Z}[X, X^{-1}] \to \mathbb{Z} \\ F = f_k X^k + \cdots + f_{-j} X^{-j} \mapsto f_k u_k + \cdots + f_{-j} u_{-j} \end{array}$$

Now let

$$\begin{array}{l} \Phi\ :\ \mathbb{Z}[\beta, \beta^{-1}] \to \mathbb{Z} \\ \Phi = \pi_{U_\beta} \circ \pi_\beta^{-1} \end{array}$$

Remark that Φ is a linear map. We then have the following result [1]

PROPOSITION **1** . *Let β be a quadratic Pisot unit. Let $x \in \mathbb{Z}_\beta^+$ such that its β-expansion is $\langle x \rangle_\beta = x_k \cdots x_0$, and let $N = \Phi(x)$. Then the greedy U_β-representation of N is exactly $\langle N \rangle_{U_\beta} = x_k \cdots x_0$.*

EXAMPLE **1** . The numeration system associated with $\tau = (1 + \sqrt{5})/2$ is the Fibonacci numeration system with $u_0 = 1$ and $u_1 = 2$.

$\mathbb{Z}$	Fibonacci	$\mathbb{Z}_\tau$
1	1	1
2	10	τ
3	100	τ^2
4	101	τ^2+1
5	1000	τ^3
6	1001	τ^3+1

The set $\mathbb{Z}_\beta$ is ordered: let b_n be the n-th β-integer. As a consequence of Proposition 1, we have that

$$\Phi(b_n) = n.$$

4 $\mathbb{Z}_\beta$ is a group

In this section we prove that, when β is a quadratic Pisot unit, the set $\mathbb{Z}_\beta$, equipped with a suitable addition law, is a group isomorphic to the set of integers $\mathbb{Z}$. The proof uses an extension of the proofs of Proposition 4.1 and Proposition 4.2 of [2]. For the sake of completeness, we give a self-contained and unified proof.

4.1 β is the root >1 of the polynomial X^2-aX-1, with $a \geq 1$

Let $'$ be the Galois automorphism $' : \beta \mapsto -1/\beta$.
LEMMA **3** . (i) *Let z be in $\mathbb{Z}[\beta]$, $z>0$, and let $z_k z_{k-1}\cdots z_{-m}$ be its β-expansion, with $z_{-m}\neq 0$. If $m>0$ then $z' \notin]-1,\beta[$; moreover, if $z<\beta^{-1}$, then $z' \notin]-\beta^2,\beta[$.*
(ii) *The set $\mathbb{Z}_\beta^+$ is equal to*

$$\mathbb{Z}_\beta^+ = \{z \in \mathbb{Z}[\beta], z\geq 0 \mid z' \in]-1,\beta[\}.$$

Proof. (i) Since $z=\sum_{i=-m}^k z_i\beta^i$, $z'=\sum_{i=-m}^k (-1)^i z_i \beta^{-i}$. Since $z_{-m}\neq 0$, by admissibility $z_{-m+1}\leq a-1$ and the other z_i's are $\leq a$. If m is even, then

$$z' > -a(\cdots+\beta^{m-5}+\beta^{m-3})-(a-1)\beta^{m-1}+\beta^m = -a\frac{\beta^{m-1}}{\beta^2-1}+\beta^{m-1}+\beta^{m-2} = \beta^{m-1}$$

hence $z' \notin]-1,\beta[$. If m is odd,

$$z' < a(\cdots+\beta^{m-5}+\beta^{m-3})+(a-1)\beta^{m-1}-\beta^m = -\beta^{m-1}$$

hence $z' \notin]-1,\beta[$. Note that if $0<z<\frac{1}{\beta}$ then $m\geq 2$.

(ii) Let z be in $\mathbb{Z}_\beta^+$ and let $z_k \cdots z_0$ be its β-expansion. Thus $z' = \sum_{i=0}^k (-1)^i z_i \beta^{-i}$. Since

$$\cdots - z_3\beta^{-3} - z_1\beta^{-1} \leq z' \leq z_0 + z_2\beta^{-2} + \cdots$$

$-1 < z' < a\beta^2/(\beta^2-1) = \beta$. Hence $\mathbb{Z}_\beta^+ \subseteq \{z \in \mathbb{Z}[\beta] \mid z' \in]-1, \beta[\}$. Conversely, let $z \geq 0$, $z \in \mathbb{Z}[\beta]$ such that $-1 < z' < \beta$. It follows from (i) that $\langle z \rangle_\beta = z_k \cdots z_{-m}$, where m has to be ≤ 0, thus $z \in \mathbb{Z}_\beta^+$. ∎

LEMMA 4 . *Let x and y be in $\mathbb{Z}_\beta^+$. Then $x+y$ can be uniquely written as*

$$x + y = g + \frac{\eta}{\beta}$$

with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$, and $g + \eta$ is still in $\mathbb{Z}_\beta^+$.

Proof. Let $z = x + y$. Then $z \in \mathbb{Z}[\beta]$, $z \geq 0$, and by Lemma 3 (ii), $-2 < z' < 2\beta$.

• If $-1 < z' < \beta$ then $z \in \mathbb{Z}_\beta^+$ by Lemma 3 (ii).

• If $\beta \leq z' < 2\beta$, set $g = z + \beta^{-1}$. Then $g' = z' - \beta \in]0, \beta[$. Note that $g' \neq 0$ because $z' \neq \beta$, since z cannot be equal to $-\beta^{-1}$. By Lemma 3 (ii), $g \in \mathbb{Z}_\beta^+$, and $z = g - \beta^{-1}$. Now, $(g-1)' = g' - 1 \in]-1, \beta - 1[$, thus $g - 1 \in \mathbb{Z}_\beta^+$.

• If $-2 < z' \leq -1$ then $z' \in]-\beta^2, \beta[$, and $z > \beta - 1$ by Lemma 3 (i). Then $z' \neq -1$. Let $g = z - \beta^{-1}$, then $g' = z' + \beta \in]\beta - 2, \beta - 1[$. Hence $z = g + \beta^{-1}$, with $g \in \mathbb{Z}_\beta^+$. Since $(g+1)' = g' + 1 \in]\beta - 1, \beta[$, $g + 1$ is also in $\mathbb{Z}_\beta^+$. ∎

LEMMA 5 . *Let x and y be in $\mathbb{Z}_\beta^+$, $x \geq y$. Then $x - y$ can be uniquely written as*

$$x - y = g + \frac{\eta}{\beta}$$

with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$, and $g + \eta$ is still in $\mathbb{Z}_\beta^+$.

Proof. Let $z = x - y \geq 0$, then $-1 - \beta < z' < 1 + \beta$ by Lemma 3.

• If $-1 < z' < \beta$ then $z \in \mathbb{Z}_\beta^+$.

• If $\beta \leq z' < 1 + \beta$, let $g = z + \beta^{-1}$. Note that $z' \neq \beta$ since $z \geq 0$. Then $g' = z' - \beta \in]0, 1[$, and $z = g - \beta^{-1}$, where $g \in \mathbb{Z}_\beta^+$. Now, $(g-1)' = g' - 1 \in]-1, 0[$, and $g - 1 \in \mathbb{Z}_\beta^+$.

• If $-1 - \beta < z' \leq -1$, by Lemma 3 (i), we get that $z \geq \beta^{-1}$. Thus, let $g = z - \beta^{-1}$, $g' = z' + \beta \in]-1, \beta - 1[$ ($z' \neq -1$). Therefore $z = g + \beta^{-1}$, with $g \in \mathbb{Z}_\beta^+$. We have that $(g+1)' \in]0, \beta[$, hence $g + 1 \in \mathbb{Z}_\beta^+$ as well. ∎

Now we can define two internal operations on $\mathbb{Z}_\beta$: Let x and y be in $\mathbb{Z}_\beta^+$, and let $x + y = g + \frac{\eta}{\beta}$ with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$ as in Lemma 4. Define

$$x \oplus y = g + \eta \in \mathbb{Z}_\beta^+.$$

If $x \geq y$, let $x - y = g + \frac{\eta}{\beta}$ with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$ as in Lemma 5. Define

$$x \ominus y = g + \eta \in \mathbb{Z}_\beta^+.$$

Remark that in the case where β is the golden ratio τ, $x \oplus y$ (resp. $x \ominus y$) is actually the τ-integer closest to $x + y$ (resp. $x - y$).

THEOREM **1** . *Let β be the root > 1 of $X^2 - aX - 1$, with $a \geq 1$. The set $\mathbb{Z}_\beta$ equipped with the laws $\oplus$ and $\ominus$ is a group isomorphic to $\mathbb{Z}$.*

It is a corollary of the following proposition.

PROPOSITION **2** . *Let b_n and b_m be the n-th and the m-th β-integers. Then*

$$b_n \oplus b_m = b_{n+m}.$$

If $n \geq m$, then

$$b_n \ominus b_m = b_{n-m}.$$

Proof. Let $b_n + b_m = g + \eta\beta^{-1}$ with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$. Then $b_n \oplus b_m = g + \eta$. We have that $\Phi(b_n + b_m) = n + m$. Let $g_k \cdots g_0$ be the β-expansion of g. Then $\Phi(g) = g_k u_k + \cdots + g_0 u_0$, thus $n + m = g_k u_k + \cdots + g_0 u_0 + \eta u_{-1} = g_k u_k + \cdots + g_1 u_1 + (g_0 + \eta) u_0$ because $u_{-1} = 1 = u_0$, thus $\Phi(b_n \oplus b_m) = n + m$.

The proof for the subtraction is similar. ■

4.2 β is the root > 1 of the polynomial $X^2 - aX + 1$, with $a \geq 3$

Let $'$ be the Galois automorphism $' : \beta \mapsto 1/\beta$.

LEMMA **6** . (i) *The β-expansion of $z \in \mathbb{Z}[\beta]$, $z \geq 0$, is finite if and only if $0 < z' < \infty$.*

(ii) *The set $\mathbb{Z}_\beta^+$ is equal to*

$$\mathbb{Z}_\beta^+ = \{z \in \mathbb{Z}[\beta], z \geq 0 \mid z' \in [0, \beta[\}.$$

(iii) *The set $\{z \in \mathbb{Z}[\beta] \mid z' \in]-\beta, \beta[\}$ is included into the set $\mathbb{Z}_\beta + \{0, \pm\beta^{-1}\}$.*

Proof. (i) Suppose that the β-expansion of z is finite, equal to $z_k \cdots z_{-m}$. Then $z' = \sum_{-m}^{k} z_i \beta^{-i} > 0$.

Conversely, let $z = c\beta + d > 0$, $c, d \in \mathbb{Z}$, with $z' = c/\beta + d > 0$. So $c + d\beta > 0$.

• If $c,\ d \geq 0$, then z is in $\mathbb{N}[\beta]$, and we know that z has a finite β-expansion by [3].

• Let $c < 0$, $d > 0$. Since there exists $l \in \mathbb{N}$ such that $\beta^{l-1} \leq -c < \beta^l$, there exists $k \in \mathbb{N}$, $k \leq l$ such that $\beta^k z \in \mathbb{N}[\beta]$. To find such a k, we go through the following procedure. From $z = c\beta + d > 0$ and $\beta = a - 1/\beta$, it follows that $ca + d > c/\beta$. If $ca + d \geq 0$, then $\beta z = (ca + d)\beta - c \in \mathbb{N}[\beta]$. Otherwise $\beta z = c_1\beta + d_1 > 0$, $c_1 < 0$, $d_1 > 0$ and $c_1 a + d_1 > (ca + d)/\beta > c/\beta^2$. Since

$c/\beta^l > -1$, it follows that there exists a $k \leq l$ such that $\beta^k z \in \mathbb{N}[\beta]$ and this implies that z has a finite β-expansion.

• Let $c > 0, d < 0$. The previous case implies that $\beta z' = d\beta + c > 0$ has a finite β-expansion and so z has a finite β-expansion.

(ii) Let $z \in \mathbb{Z}_\beta^+$ and let $z_k \cdots z_0$ be its β-expansion. Then $z_k \cdots z_0 \leq_{lex} (a-1)(a-2)\cdots(a-2)$, where $\leq_{lex}$ denotes the lexicographic order. Therefore $z_0 \cdots z_k$ is a β-expansion as well. From this follows that $0 \leq z' = z_0 + z_1\beta^{-1} + \cdots + z_k\beta^{-k} < \beta$. Conversely, let $z \geq 0$, $z \in \mathbb{Z}[\beta]$ such that $z' \in [0, \beta[$. From (i) we know that the β-expansion of z is finite, $\langle z \rangle_\beta = z_k \cdots z_{-m}$, with $z_{-m} \neq 0$, $m, k \in \mathbb{N}$. If $m = 0$, then $z \in \mathbb{Z}_\beta^+$. If $m \neq 0$, then $z = \sum_{i=0}^k z_i\beta^i + \sum_{i=-m}^{-1} z_i\beta^i$, and $z' = \sum_{i=0}^k z_i\beta^{-i} + \sum_{i=-m}^{-1} z_i\beta^{-i}$, where $0 \leq \sum_{i=0}^k z_i\beta^{-i} < \beta$. Since $z_{-m} \neq 0$ we have $\sum_{i=-m}^{-1} z_i\beta^{-1} \geq \beta^m$ and hence $z' \notin [0, \beta[$. Therefore $m = 0$. Hence $\mathbb{Z}_\beta^+ = \{z \in \mathbb{Z}[\beta], z \geq 0 \mid z' \in [0, \beta[\}$.

(iii) Let $z \geq 0$ such that $0 \leq z' < \beta$, then $z \in \mathbb{Z}_\beta^+$ by (i). If $z \geq 0$ and $-\beta < z' < 0$, then $z' + \beta = (z + \beta^{-1})' \in]0, \beta[$, and $z + \beta^{-1} \geq 0$. By (i), $z + \beta^{-1} \in \mathbb{Z}_\beta^+$ and $z \in \mathbb{Z}_\beta^+ + \{-\beta^{-1}\}$. The case $z < 0$ is symmetrical. ■

Lemma **7** . *Let x and y be in $\mathbb{Z}_\beta^+$. Then the β-expansion of $x + y$ is of the form*

$$\langle x + y \rangle_\beta = g_k \cdots g_0 \cdot g_{-1}.$$

Proof. Let $z = x + y$. By Lemma 6 (ii), $0 \leq z' < 2\beta < \beta^2$ since $\beta > 2$. Then $z \in \frac{\mathbb{Z}_\beta^+}{\beta}$ by Lemma 6 (ii) again. ■

In general the difference of two positive β-integers may have an infinite eventually periodic β-expansion. Nevertheless we have the following result.

Lemma **8** . *Let x and y be in $\mathbb{Z}_\beta^+$ such that $x \geq y$. Then $x - y$ can be uniquely written as*

$$x - y = g + \frac{\eta}{\beta}$$

with $g \in \mathbb{Z}_\beta^+$ and $\eta \in \{-1, 0, 1\}$.

Proof. Let $z = x - y$. Then $z' \in]-\beta, \beta[$ by Lemma 6 (ii). By Lemma 6 (iii), we get that $z = g + \eta/\beta$ with $g \in \mathbb{Z}_\beta^+$ and $\eta \in \{-1, 0, 1\}$. ■

We define two internal operations on $\mathbb{Z}_\beta$: Let x and y be in $\mathbb{Z}_\beta^+$, such that $\langle x + y \rangle_\beta = g_k \cdots g_0 \cdot g_{-1}$ as in Lemma 7. Define

$$x \oplus y = g = g_k\beta^k + \cdots + g_0 \in \mathbb{Z}_\beta^+$$

If $x \geq y$, let $x - y = g + \frac{\eta}{\beta}$ with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$ as in Lemma 8. Define

$$x \ominus y = g \in \mathbb{Z}_\beta^+$$

One can see $x \oplus y$ (resp. $x \ominus y$) as the "integer part" of $x + y$ (resp. $x - y$).

THEOREM **2** . *Let β be the root > 1 of $X^2 - aX + 1$, with $a \geq 3$. The set $\mathbb{Z}_\beta$ equipped with the laws $\oplus$ and $\ominus$ is a group isomorphic to $\mathbb{Z}$.*

It is a corollary of the following proposition.

PROPOSITION **3** . *Let b_n and b_m be the n-th and the m-th β-integers. Then*

$$b_n \oplus b_m = b_{n+m}.$$

If $n \geq m$, then

$$b_n \ominus b_m = b_{n-m}.$$

Proof. Suppose that $\langle b_n + b_m \rangle_\beta = g_k \cdots g_0 \cdot g_{-1}$. Then $b_n \oplus b_m = g_k \beta^k + \cdots + g_0$. Thus $n + m = \Phi(b_n + b_m) = g_k u_k + \cdots + g_0 u_0 + g_{-1} u_{-1} = \Phi(b_n \oplus b_m)$, because $u_{-1} = 0$.

Let $b_n - b_m = g + \eta\beta^{-1}$ with $g \in \mathbb{Z}_\beta^+$, $\eta \in \{-1, 0, 1\}$, let $\langle g \rangle_\beta = g_k \cdots g_0$. Then $b_n \ominus b_m = g_k \beta^k + \cdots + g_0$. Thus $n - m = \Phi(b_n - b_m) = g_k u_k + \cdots + g_0 u_0 + \eta u_{-1} = \Phi(b_n \ominus b_m)$, because $u_{-1} = 0$. ■

4.3 Consequence

As an interesting consequence, the group structure we have exhibited for $\mathbb{Z}_\beta$ and so for the β-grid Λ_β allows one to build up the whole structure from a finite germ, exactly as it is done with standard lattices.

COROLLARY **1** . *Let β be a quadratic Pisot unit. Then $\mathbb{Z}_\beta = \cup_{k \geq 0} \mathbb{Z}_\beta^{(k)}$, where $\mathbb{Z}_\beta^{(0)} = \{0\}$ and $\mathbb{Z}_\beta^{(k+1)} = \mathbb{Z}_\beta^{(k)} \oplus \{\pm 1\}$.*

The interpretation of the function $z \mapsto z \oplus 1$ in the linear numeration system associated with β is the successor function which maps the integer N onto $N + 1$.

EXAMPLE **2** . For $\tau = \frac{1+\sqrt{5}}{2}$

$$1 + 1 = \tau^2 - 1/\tau$$
$$1 \oplus 1 = \tau^2 - 1 = \tau$$

and 10 is the Fibonacci representation of 2.

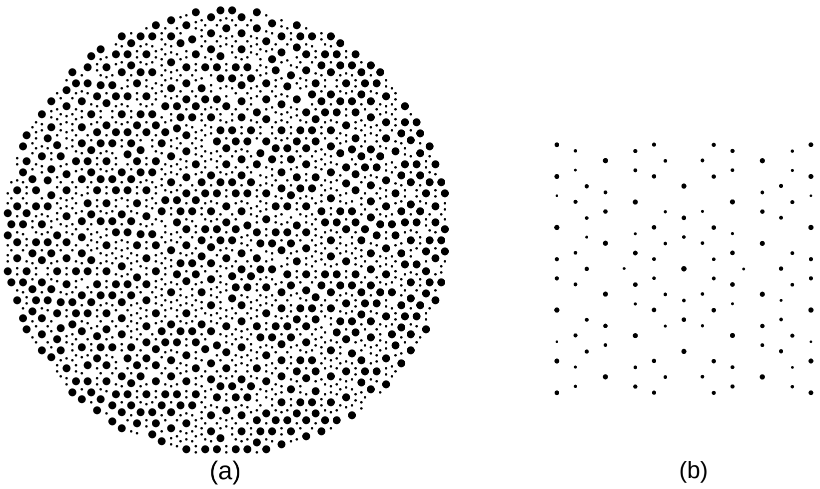
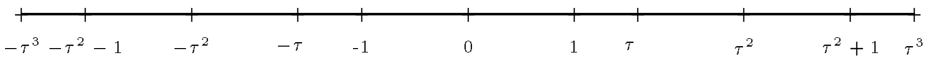


Figure 1. Gas modelling on the τ-grid. (a) Quasicrystalline structure. (b) Corresponding diffraction pattern.

We draw below the set $\mathbb{Z}_\tau^{(5)}$. Of course, the positive semiline is tiled by large and small intervals distributed according to the Fibonacci substitution

$$\begin{aligned} L &\to LS \\ S &\to L \end{aligned}$$

The negative part is just symmetrical.



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