On the context-freeness of the θ-expansions of the integers

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Abstract

Let $\theta > 1$ be a non-integral real number such that the θ-expansion of every positive integer is finite. If the set of θ-expansions of all the positive integers is a context-free language, then θ must be a quadratic Pisot unit.

Résumé

Soit $\theta > 1$ un nombre réel non entier tel que le θ-développement de tout entier positif soit fini. Si l'on suppose que l'ensemble des θ-développements des entiers positifs forme un langage algébrique, alors θ doit être un nombre de Pisot quadratique unitaire.

In [1], to which this note is an addendum, it was proved (Theorem 2) that, if θ is a quadratic Pisot unit, then there exists a letter-to-letter finite two-tape automaton that maps the representation of any integer in a linear numeration system canonically associated with θ onto the "folded" θ-expansion of that integer (we refer to [1] for definitions and references).

As an immediate consequence ([1, Corollary 4]), it follows that, if θ is a quadratic Pisot unit, then the set of θ-expansions of all the positive integers is a *linear context-free language*. The purpose of this note is to establish the converse of that statement.

Let θ be a real number greater than 1. By a greedy algorithm every nonnegative real number x can be expanded as an infinite sum of powers of θ, *i.e.* x may be written as $x = \sum_{n=-\infty}^{k} x_n\theta^n$, where the x_n are digits of a canonical alphabet A_θ given by the

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algorithm. Such a *θ-expansion* of x is denoted by $x = x_k \cdots x_0 . x_{-1} x_{-2} \cdots$, with a radix point.

A *Perron number* is an algebraic integer $\theta > 1$ such that any other root of its minimal polynomial is strictly smaller in modulus than θ. A *Pisot number* is an algebraic integer $\theta > 1$ such that any other root of its minimal polynomial has modulus < 1. A *Salem number* is an algebraic integer $\theta > 1$ such that any other root of its minimal polynomial has modulus ≤ 1, with at least one root of modulus 1. A *unit* is an algebraic integer such that the constant term of its minimal polynomial is equal to ± 1. When θ is a quadratic Pisot unit, then the θ-expansion of every integer is finite ([2]).

We can now state:

THEOREM 1 *Latex 2e* *Let $\theta > 1$ be a non-integral real number such that the θ-expansion of every positive integer is finite, and let $A_\theta = \{0, \ldots, \lfloor\theta\rfloor\}$ be the canonical alphabet. Let $R_\theta \subset A_\theta^* . A_\theta^*$ be the set of θ-expansions of all the positive integers. If R_θ is a context-free language, then θ must be a quadratic Pisot unit.*

Proof. From [2] it is known that if the θ-expansion of every positive integer is finite, then θ must be a Pisot or a Salem number. Now, if R_θ is a context-free language, by the pumping lemma (Bar Hillel, Perles, Shamir theorem, see [3]), there exists a constant $N(R_\theta)$ such that if $w \in R_\theta$ and $|w| \geq N(R_\theta)$, then we may write $w = \alpha f \beta g \gamma$ such that
i) $|fg| \geq 1$
ii) $|f\beta g| \leq N(R_\theta)$
iii) $\forall n \geq 0$, $\alpha f^n \beta g^n \gamma \in R_\theta$.

Let us introduce the following notation. If $h = h_1 \cdots h_k$, then $\pi(h) = h_1 \theta^{k-1} + \cdots + h_k$ and $\bar{\pi}(h) = h_1 \theta^{-1} + \cdots + h_k \theta^{-k}$.

When N has a θ-expansion of the form $w = w'.w''$, the word w' to the left of the radix point is said to be the "integral part" of N, and the word w'' to the right of the radix point is said to be the "fractional part" of N.

Claim 1 *The radix point in w cannot belong to f or g.*

For otherwise one would get, by the pumping lemma, a word not in R_θ.

Claim 2 *The radix point in w cannot belong to α.*

For, if $\alpha = \alpha' . \alpha''$, then

$$\forall n \geq 0 \,, \qquad \pi(\alpha') + \bar{\pi}(\alpha'' f^n \beta g^n \gamma) = K_n \in \mathbb{N} \ .$$

Thus $K_{n+1} - K_n$ is an integer between -1 and 1, and hence 0, which is impossible.

Claim 3 *The radix point in w cannot belong to γ.*

Indeed, if $\gamma = \gamma' \cdot \gamma''$ then the equation

$$\forall n \geq 0\,, \qquad \alpha f^n \beta g^n \gamma' \cdot \gamma'' \in R_\theta \ ,$$

holds, which means that there exist infinitely many integers having the same "fractional part". And this is shown to be impossible by the following lemma.

Lemma 1 *Let $\theta > 1$ be a non-integral Pisot or Salem number and let $z = \sum_{i=-p}^{k} c_i \theta^i$ be a finite θ-representation of an integer z, with $c_i \in \mathbb{Z}$. Let $c = \max |c_i|$, and θ_j be an algebraic conjugate of θ of smallest modulus (< 1). Then*

$$p > \frac{\log(|z| c^{-1}(1 - |\theta_j|))}{\log(|\theta_j|^{-1})}.$$

Proof. We have $z = \sum_{i=-p}^{k} c_i \theta^i$, and also $z = \sum_{i=-p}^{k} c_i \theta_j^i$, so

$$|z| < c \sum_{i=-p}^{\infty} |\theta_j|^i = c \frac{|\theta_j|^{-p}}{1 - |\theta_j|},$$

and the result follows. ∎

Thus, the fractional part of $K_n = \pi(\alpha f^n \beta g^n \gamma') + \bar{\pi}(\gamma'')$ cannot be the same for all n, since, from the lemma, $|\gamma''|$ must increase with K_n. This proves Claim 3.

So w is of the form $w = \alpha f \beta' \cdot \beta'' g \gamma$. Note that if $f = \varepsilon$, then $\forall n \geq 0$, $\alpha \beta' \cdot \beta'' g^n \gamma \in R_\theta$, and this is impossible by the same method as in Claim 2. Similarly, if $g = \varepsilon$, then $\alpha f^n \beta' \cdot \beta'' \gamma \in R_\theta$, $\forall n \geq 0$, and the same argument as in Claim 3 shows this is impossible. So both f and g are non-empty.

Let $|f| = i$, $|g| = l$, $|\beta'| = a$, $|\beta''| = d$, and $x = \pi(\alpha f) - \pi(\alpha) \neq 0$, $y = \bar{\pi}(g\gamma) - \bar{\pi}(\gamma) \neq 0$. Let $K_n = \pi(\alpha f^n \beta') + \bar{\pi}(\beta'' g^n \gamma) \in \mathbb{N}$. Thus

$$\forall n \geq 0,\ K_{n+1} - K_n = \theta^{a+in} x + \theta^{-d-ln} y = A_n \in \mathbb{Z}.$$

Lemma 2 *Let θ be a non-integral Perron number. Suppose that for some integers i, l, a, d and for some nonzero reals x, y,*

$$\theta^{a+in} x + \theta^{-d-ln} y = A_n \in \mathbb{Z}, \quad n \geq 0. \tag{1}$$

Then $i = l$ and θ is a quadratic Pisot unit.

Proof. Since θ is an algebraic integer, θ^i is an algebraic integer as well. Let $P(X) = X^m - k_1 X^{m-1} - \ldots - k_m$ in $\mathbb{Z}[X]$ be the minimal polynomial for θ^i. Then the sequence $\{\theta^{a+in} x\}_{n \geq 0}$ satisfies the linear recurrence relation

$$u_{n+m} = k_1 u_{n+m-1} + \cdots + k_m u_n. \tag{2}$$

Claim. The sequence $\{A_n\}_{n\geq 0}$ satisfies the recurrence relation (2) for n sufficiently large ($n \geq K$). Indeed,

$$\varepsilon_n = A_{n+m} - k_1 A_{n+m-1} - \cdots - k_m A_n = y[\theta^{-d-l(n+m)} - k_1\theta^{-d-l(n+m-1)} - \cdots - k_m\theta^{-d-ln}].$$

Since $\theta > 1$, the sequence ε_n tends to zero as $n \to \infty$. But $\varepsilon_n \in \mathbb{Z}$, so there exists K such that $\varepsilon_n = 0$, $n \geq K$, which proves the claim.

Let $\alpha_1 = \theta^i$, and $\alpha_2, \ldots, \alpha_m$ be the other zeros of $P(X)$. Since $P(X)$ is irreducible, all α_j are distinct and every linear recurrent sequence satisfying (2) is a linear combination of geometric progressions $\{\alpha_j^n\}_{n\geq 0}$. Thus, for some complex numbers $c_1, \ldots, c_m$

$$A_n = \sum_{j=1}^{m} c_j\alpha_j^n, \quad n \geq K. \tag{3}$$

Moreover, all c_j are nonzero. (Indeed, if $c_\nu = 0$ and $c_\kappa \neq 0$, consider an automorphism of the splitting field of $P(X)$ which sends α_ν to α_κ. Applying this automorphism to (3) we get a contradiction.) Next we use the following standard result from linear algebra which is proved using the Vandermonde determinant.

Fact 1 *Let $\zeta_1, \ldots, \zeta_p$ be distinct complex numbers such that*

$$\sum_{j=1}^{p} B_j\zeta_j^n = 0, \quad n \geq K.$$

Then $B_j = 0$, $j \leq p$.

Applying this to (1) and (3) we obtain that $c_1 = \theta^a x$, $\theta^{-l} = \alpha_s$ for some $s \geq 2$, $\theta^{-d}y = c_s$, and $c_j = 0$ for $j \neq 1, s$. Since all c_j must be nonzero, we get that $m = 2$. Thus, $P(X)$ is quadratic with zeros θ^i and θ^{-l}. Then $\theta^{i-l} = -k_2 \in \mathbb{Z}$. Let us show that $i = l$. In fact, $i \geq l$, since $\theta > 1$, and $i - l \neq 1$, since θ is not an integer. If $i - l \geq 2$, we get that θ is the $(i-l)$-th root of an integer, so it is either an integer, which we forbid, or it has at least one conjugate of modulus θ which contradicts the assumption that θ is Perron. Thus, $i = l$ and $k_2 = -1$.

In order to prove that θ itself is a quadratic Pisot unit, we consider the polynomial $P(X^i)$. It has θ as a zero, so all the conjugates of θ must be its zeros as well. But the polynomial $P(X^i)$ has i zeros of modulus θ and i zeros of modulus θ^{-1}. Since θ is Perron, its conjugates must have modulus less than θ, so they all have modulus θ^{-1}. But the product of θ and all its conjugates is at least 1 in modulus, so there is exactly one conjugate. This conjugate is real, so it is either θ^{-1} or $-\theta^{-1}$. This means that θ is a quadratic Pisot unit. ■

The proof of Theorem 1 is thus complete. ■

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